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Semi-Simplicity of a Lie Algebra of Isometries

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Abstract

A spray S on the tangent bundle TM with a n dimensional differentiable manifold M defines an almost product structure Γ (Γ^2=I, I being the identity vector 1-form) and decomposes the TTM space into a direct sum of horizontal space (corresponding to the eigenvalue +1) and vertical space (for the eigenvalue -1). The Lie algebra of projectable vector fields $\overline{A_S}$ whose Lie derivative vanishes the spray S is of dimension at most n^2+n. The elements of the algebra $\overline{A_S}$ belonging to the horizontal nullity space of Nijenhuis tensor of Γ form a commutative ideal of $\overline{A_S}$. They are not the only ones for any spray S. If S is the canonical spray of a Riemannian manifold, the symplectic scalar 2-form Ω which is the generator of the spray S defines a Riemannian metric g upon the bundle vertical space of TM. The Lie algebra of infinitesimal isometries which is written $\overline{A_g}$ contained in $\overline{A_S}$ is of dimension at most $\frac{n(n+1)}{2}$. The commutative ideal of $\overline{A_S}$ is also that of $\overline{A_g}$. The Lie algebra $\overline{A_g}$ of dimension superior or equal to three is semi-simple if and only if the nullity horizontal space of the Γ Nijenhuis tensor is reduced to zero. In this case, $\overline{A_g}$ is identical to $\overline{A_S}$.
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Introduction

Frölicher-Nijenhuis and Rund H.'s works [1,2] have enabled Grifone J. [3,4]) to show a connection as a vector 1-form Γ such that JΓ=J and ΓJ=-J of which J is the natural tangent structure of the tangent bundle TM to a differentiable manifold M. Such Γ ıs an almost product structure (Γ^2=I), I being the identity vector 1-form. This formalism allows a more algebraic approach for the study of Lie algebras defined by Γ [5].

Given a paracompact differentiable manifold M of n ≥ 2 dimension of class C^∞, Γ is a connection [3]. In [5], we studied the associated Lie algebras to Γ, through their first Chevalley-Eilenberg cohomology spaces, namely, the Lie algebra of vector fields A_Γ on TM, whose Lie derivative applied to Γ is zero, and the horizontal nullity space N_R^h of the curvature R of Γ (R=$\frac{1}{8}$[Γ, Γ]). The Lie algebra A_Γ is formed by projectable vector fields, and contains two ideals A_Γ^h and A_Γ^v , where A_Γ^h consists of projectable vector fields of the horizontal nullity space of the curvature R and A_Γ^v an ideal of the vertical space.

In [6], Loos O. considered a spray S as a system of second order differential equations on M. If we denote by A_S the Lie algebra of vector fields on TM which commute with S, the projectable vector fields of A_S correspond to $A_S \cap \overline{\chi(M)}, \overline{\chi(M)}$ being the complete lift on the tangent bundle TM of the set of all vector fields χ(M) on M. We will denote the set $\overline{A_S}$ such that $\overline{A_S} = A_S \cap \overline{\chi(M)}$. The Lie algebra $\overline{A_S}$ corresponds to a Lie group G_S of transformations on M of which the tangent linear mappings preserve the spray S. The group G_S acts freely on the linear frame bundle of M. The dimension of $\overline{A_S}$ is at most equals to n^2+n.

In the following, a connection Γ is linear without torsion, in this case [3], Γ is written: Γ=[J, S] and [C, S]=S, where C being the Liouville field on TM. We prove that $\overline{A_S} = \overline{A_\Gamma}$ by considering $\overline{A_\Gamma} = A_\Gamma \cap \overline{\chi(M)}$. For a linear connection without torsion such that the rank of the horizontal nullity space of the projectable vector fields of curvature R is nonzero constant, the Lie algebra $\overline{A_\Gamma}$ contains a nonzero commutative ideal $\overline{A_\Gamma}^h$ formed by the horizontal elements of $\overline{A_\Gamma}$. All the commutative ideals of $\overline{A_\Gamma}$ do not always come from the horizontal nullity space of the curvature for any spray S. Let E be an energy function [3], that is to say, a mapping C^∞ on TM to $\mathbb{R}^+$, null on the null section, homogeneous of degree two (L_CE=2E), L_C being the Lie derivative with respect to C, such that the 2-form Ω=dd_JE is of maximum rank. The canonical spray S is defined [5] by

$$i_s dd_J E=-dE,$$

i_S being the inner product with respect to S. The 2-form Ω defines a Riemannian metric on the vertical bundle. In local coordinate system on an open set U of M, $(x^i, y^i)_{i,j\in\{1,\ldots,n\}}$, the coordinate system on TU , the function E is written

$$E = \frac{1}{2} g_{ij}(x^1,\ldots,x^n) y^i y^j$$

where $g_{ij}(x^1,\ldots,x^n)$ are positive functions such that the symmetric matrix $(g_{ij}(x^1,\ldots,x^n))$ is invertible . We denote by $\overline{A_g}$ the set of $\overline{X} \in \overline{\chi(M)}$ such that $L_{\overline{X}}\Omega = 0$, $\overline{A_\Gamma}^h$ is a commutative ideal of $\overline{A_g}$ and $\overline{A_\Gamma}$. We then have the following algebra inclusions: $\overline{A_\Gamma}^h \subset \overline{A_g} \subset \overline{A_\Gamma}$. The dimension of $\overline{A_g}$ is at most equal to $\frac{n(n+1)}{2}$. The $\overline{A_g}$ is also the Lie algebra of the Killing fields of projectable vectors which commute with the spray S. If the dimension of $\overline{A_g}$ is greater than or equal to three, the Lie algebra $\overline{A_g}$ is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of Γ is reduced to zero. In this context, the algebra $\overline{A_\Gamma}$ coincides with $\overline{A_g}$. It follows that for the flat Riemannian manifolds (R=0), the Lie algebra $\overline{A_\Gamma}$, the Killing algebra $\overline{A_g}$ also, contain commutative ideals. For Riemannian manifolds with nonzero constant sectional curvature, $\overline{A_\Gamma}$ is the isometry algebra $\left(\overline{A_\Gamma} = \overline{A_g}\right)$, and it is semi-simple.

This study also makes it possible to construct numerous examples of Lie algebra containing a commutative ideal. In particular, for a semi-simple Lie algebra, the derivative ideal coincides with algebra. This examples show that this property is not sufficient to be a semi-simple Lie algebra.

This paper is part of the continuation of the studies made in [5,7-9].

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Preliminaries

We will recall the bracket of two vector 1-form K and L on a manifold M [1]. If we denote χ(M) the set of vector fields on M,

[L,K] (X,Y)=[LX, KY]+[KX, LY]+LK[X, Y]+KL[X, Y]-L[KX, Y] (2.1)

-K[LX, Y]-L[X, KY]-K[X, LY]

for all X, Y$\in\chi$(M).

The bracket N_L=1/2 [L, L] is called the Nijenhuis tensor of L. The Lie derivative with respect to X applied to L is written:

[X, L]Y=[X, LY]-L[X, Y].

The exterior differentiation d_L is defined by [10] $d_L=[i_L, d]$. Let Γ be a connection. By

h=1/2(I+ Γ) and v=1/2(I-Γ),

h is the horizontal projector, projector of the subspace corresponding to the eigenvalue +1, v the vertical projector corresponding to the eigenvalue -1. The curvature of Γ is defined by

R=1/2 [h,h];

which is equal to

R=1/8 [Γ, Γ].

The Lie algebra A_Γ is written

A_Γ={X$\in\chi$(TM) such that [X, Γ]=0}.

The nullity space of the curvature R is written:

$$N_R=\{X\in\chi(TM)\, such\, that\, R(X,Y)=0,\forall Y\in\chi(TM)\}.$$

According to the results of [5], the elements of A_Γ are projectable vectors fields. The horizontal and vertical parts of A_Γ respectively A^h_Γ and A^v_Γ are ideals of A_Γ . If we denote by H° the set of horizontal and projectable vector fields, then we have

$$A^h_\Gamma=H^\circ\cap N_{R.} \quad (2.2)$$

Definition 1 [3]

We call strong torsion T of Γ the vector 1-form

T=i_St-1/2 [C, Γ],

where S indicates a spray, t=1/2 [J, Γ] called weak torsion of Γ, i_S the inner product with respect to a spray S.

We recall a result of [3] on the decomposition of a connection

Γ=[J, S]+T.

The elements of A^h_Γ are well known [5]. In the following section, we will be interested in the ideal A^v_Γ .

Properties of the Vertical Ideal of A^v_Γ

A vertical vector field can be written in the form JX, J being the tangent structure on TM, and X a horizontal vector field.

Proposition 1 [5]

A vertical vector field JX is an element of A^v_Γ if and only if JX commutes with all horizontal and projectable vector fields.

Proposition 2

Let Γ be a zero weak torsion connection, X and Y two horizontal vector fields such that JX and JY are elements of A^v_Γ . So we have

J [X, Y]=0.

In particular,

[X, Y]=R(X, Y).

Proof: The nullity of the weak torsion of Γ allows to write

v [JX, Y]+v [X, JY]=J [X, Y], (3.1)

for all X, Y horizontal vector fields. If JX and JY are two elements of A^v_Γ , we get v [JX, Y]=0 and v [X, JY]=0, that is to say

J [X, Y]=0.

This means that the horizontal part of [X, Y] is zero, we find

[X, Y]=v [X, Y]=R (X, Y).

Proposition 3

Let Γ be a zero weak torsion connection, X a horizontal vector field. The two following conditions are equivalent:

a) X is a projectable vector field and [JX, h]=0;
b) [X, J]=0.

Proof: The nullity of the weak torsion of Γ is written:

[JX, h]Y+[X, J]Y=h[X, JY], (3.2)

for all horizontal vector fields X, Y. If X is projectable, the term h [X, JY] is zero; and if [JX, h]=0, we get [X, J]Y=0, for all horizontal vector fields Y. If Y is a vertical vector field, we have [X, J] Y=0, since X is projectable. This proves that the relation a) implies b). If [X, J]=0, the horizontal vector field X is projectable [11], so we have h [X, JY]=0; and the nullity of the weak torsion of Γ (3.2) results in [JX, h]=0. In the following section, we assume that the connection Γ has a null strong torsion, that is, Γ=[J, S] and [C, S]=S. Einstein's convention on the summation of indices will be adopted. In natural local coordinates on an open set U of M, $(x^i,y^i)_{i,j\in\{1,\dots,n\}}$, the coordinates on TU, a spray S is written:

$$S=y^i\frac{\partial}{\partial x^i}-2G^j(x^1,\dots,x^n,y^1,\dots,y^n)\frac{\partial}{\partial y^j}.$$

For a connection Γ =[J, S], the coefficients of Γ become $\Gamma^j_i=\frac{\partial G^j}{\partial y^i}$ and the horizontal projector is

$$h(\frac{\partial}{\partial x^i})=\frac{\partial}{\partial x^i}-\Gamma^j_i\frac{\partial}{\partial y^j}$$

$$h(\frac{\partial}{\partial y^j})=0.$$

The vertical projector is written:

$$v\left(\frac{\partial}{\partial x^i}\right)=\Gamma^j_i\frac{\partial}{\partial y^j}$$

$$h\left(\frac{\partial}{\partial y^j}\right)=\frac{\partial}{\partial y^j}.$$

The curvature R=1/2 [h, h] become

$$R=\frac{1}{2}R^k_{ij}dx^i\wedge dx^j\otimes\frac{\partial}{\partial y^k}$$

With $$R^k_{ij}=\frac{\partial\Gamma^k_i}{\partial x^j}-\frac{\partial\Gamma^k_j}{\partial x^i}+\Gamma^l_i\frac{\partial\Gamma^k_j}{\partial y^l}-\Gamma^l_j\frac{\partial\Gamma^k_i}{\partial y^l},i,j,k,l\in\{1,..,n\}.$$

We can associate with Γ a linear connection $\overset{\circ}{D}$ on TM, called Berwald connection [4] with curvature

$$\mathfrak{R}^\circ(X,Y)Z=D^\circ_{hX}D^\circ_{hY}JZ-D^\circ_{hY}D^\circ_{hX}JZ-D^\circ_{[hX,hY]}JZ,$$

for all X, Y and Z $\in\chi$(TM). The curvature R of Γ is linked to $\overset{\circ}{\mathfrak{R}}$ by the relation:

$$\overset{\circ}{\mathfrak{R}}(X,Y)Z=J[Z,R(X,Y)]-$$

$$[JZ,R(X,Y)]+R([JZ,X],Y)+R(X,[JZ,Y])$$

In particular,

$$\overset{\circ}{\mathfrak{R}}(X,Y)S=-R(X,Y).$$

Proposition 4

Let $JX\in A_\Gamma^v$ such that X be a projectable vector field, then we have

$$\overset{\circ}{\mathfrak{R}}(Y,Z)X=0,$$

for all Y, Z $\in\chi$(TM).

Proof: Let $JX\in A_\Gamma^v$. This means [JX,h]=0. According the expression of R=1/2 [h, h] and the identity of Jacobi, we get [JX,R]=0. In other words, for all Y, Z $\in\chi$(TM), we have

[JX, R(Y, Z)]-R([JX, Y], Z)-R(Y,[JX, Z])=0.

So the expression of $\overset{\circ}{\mathfrak{R}}$ becomes

$$\overset{\circ}{\mathfrak{R}}(Y,Z)X=J[X,R(Y,Z)].$$

The vector field X being projectable and R(Y, Z) vertical, the second member of the equality is null. So we find $\overset{\circ}{\mathfrak{R}}(Y,Z)X=0$.

Let $N_{\overset{\circ}{\mathfrak{R}}}$ be the nullity space of $\overset{\circ}{\mathfrak{R}}$, $Ker\overset{\circ}{\mathfrak{R}}$ the kernel of $\overset{\circ}{\mathfrak{R}}$:

$N_{\overset{\circ}{\mathfrak{R}}}$ ={X$\in\chi$(TM) such that $\overset{\circ}{\mathfrak{R}}$ (X, Y)Z=0, $\forall$ Y, Z $\in\chi$(TM)};

$Ker\overset{\circ}{\mathfrak{R}}$ ={X$\in\chi$(TM) such that $\overset{\circ}{\mathfrak{R}}$ (Y,Z)X=0, $\forall$ Y, Z $\in\chi$(TM)}.

Proposition 5

The horizontal ideal A_Γ^h of A_Γ becomes

$$A_\Gamma^h=N_{\overset{\circ}{\mathfrak{R}}}\cap H^\circ\subset Ker\overset{\circ}{\mathfrak{R}}\cap H^\circ.$$

Proof: According to the relation $\overset{\circ}{\mathfrak{R}}(X,Y)S=-R(X,Y)$ for all X, Y $\in\chi$(TM), we have

$$N_{\overset{\circ}{\mathfrak{R}}}\subset N_{R.}$$

If X $\in N_R$, according to the previous relationship between $\overset{\circ}{\mathfrak{R}}$ and R, we get

$$\overset{\circ}{\mathfrak{R}}(X,Y)Z=R([JZ,X],Y);$$

if X is projectable, [JZ, X] is vertical and R ([JZ, X], Y) is therefore zero. This results in

$$N_R\cap H^\circ\subset N_{\overset{0}{\mathfrak{R}}}\cap H^\circ.$$

In conformity with a result of [5], we have

$$N_{\mathfrak{R}^\circ}\cap H^\circ=N_R\cap H^\circ=A_\Gamma^h.$$

Bianchi identity on $\overset{\circ}{\mathfrak{R}}$ is written [4]:

$$\overset{\circ}{\mathfrak{R}}(X,Y)Z+\overset{\circ}{\mathfrak{R}}(Y,Z)X+\overset{\circ}{\mathfrak{R}}(Z,X)Y=0,$$

for all X, Y, Z $\in\chi$(TM). If X $\in N_{\overset{\circ}{\mathfrak{R}}}$, we find

$$\overset{\circ}{\mathfrak{R}}(Y,Z)X=0, \tag{3.3}$$

for all Y, Z $\in\chi$(TM).

Proposition 6

If the module $N_R\cap H^\circ$ on the ring of constant functions to the fibers is of nonzero and constant rank, there is locally a non-trivial vector field such that $X\in A_\Gamma^h$ and $JX\in A_\Gamma^v$.

Proof: We will solve the equation

$$X\in N_R\cap H^\circ \text{ and } [JX,h]=0. \tag{3.4}$$

This leads to solving, according to the proposition 3, the equations

$$X\in N_R\cap H^\circ \text{ and } [X,J]=0.$$

In other words, X is a horizontal and projectable vector field of the nullity space belonging to the Lie algebra A_J. The elements of A_J are well known [11]. We can write in local coordinates:

$$X^i(x)\frac{\partial}{\partial x^i}-X^i(x)\Gamma_i^j\frac{\partial}{\partial y^j}=X^i(x)\frac{\partial}{\partial x^i}+y^l\frac{\partial X^j(x)}{\partial x^l}\frac{\partial}{\partial y^j}.$$

For a linear connection, we have $\Gamma_i^j(x,y)=y^l\Gamma_{il}^j(x)$, the system of equations to be solved becomes

$$\frac{\partial X^j(x)}{\partial x^l}=-X^i(x)\,\Gamma_{il}^j, \tag{3.5}$$

with i, j, l $\in$\{1,...,n\}. The compatibility condition of the equation (3.5), according to the Frobenius theorem, is

$$X^l\left(\frac{\partial\Gamma_{li}^j}{\partial x^k}-\frac{\partial\Gamma_{lk}^j}{\partial x^i}+\Gamma_{li}^s\Gamma_{sk}^j-\Gamma_{lk}^s\Gamma_{si}^j\right)=0, i,j,k,l,s\in\{1,...,n\}.$$

This condition is nothing but $X\in Ker\overset{\circ}{\mathfrak{R}}\cap H^\circ$ and according to the proposition 5, we have $H^\circ\cap N_R\subset Ker\overset{\circ}{\mathfrak{R}}\cap H^\circ$. If the rank $H^\circ\cap N_R$ is nonzero constant, $H^\circ\cap N_R$ defines a foliation. On a submanifold defined by the distribution $H^\circ\cap N_R$, the equation (3.5) admits a non-trivial solution.

Proposition 7

Let B be the set defined by

$$B=\left\{X\in\chi(TM)\,such\,that\,X\in A_\Gamma^h, et\,JX\in A_\Gamma^v\right\}.$$

Then B+JB is a commutative subalgebra of $A_\Gamma^h\times A_\Gamma^v$ and a commutative ideal of $A_\Gamma\cap A_J$.

Proof: On the one hand, let X and Y be two elements of B, according to the proposition 2, we have

[X, Y]=0

On the other hand, X and Y being projectable, we find

[JX, JY]=0.

It is obvious that B is a subalgebra of A_Γ^h, and JB a subalgebra of A_Γ^v. Consequently, B+JB is a commutative subalgebra of $A_\Gamma^h\times A_\Gamma^v$ and of A_Γ.

If Y $\in A_\Gamma\cap A_J$, for $X\in A_\Gamma^h$, we have $[X,Y]\in A_\Gamma^h$ since A_Γ^h is an ideal of A_Γ. Now let us prove $J[X,Y]\in A_\Gamma^v$, that is, [J[X, Y], h]=0. According to the proposition 3, this leads to prove that [[X, Y], J]=0 and [X, Y] is a horizontal vector field, which is immediate by the identity of Jacobi and $X\in A_\Gamma^h$, horizontal ideal of A_Γ.

Particular Lie Algebras from Γ and S

The Lie algebra A_J is well known [11], and it is written

$$A_J=\overline{\chi(M)}+J\overline{\chi(M)}$$

in which $\overline{\chi(M)}$ denotes the complete lift on the tangent bundle TM of χ(M). We denote A_S ={X $\in\chi$(TM) such that [X, S]=0}.

Proposition 8

The vertical vectors fields of A_S are trivial.

Proof: A vertical vector field is of the form JX, where X $\in\chi$(TM). JX is an element of A_S if [JX, S]=0. In conformity with the formula J [JX, S]=JX for all X $\in\chi$(TM), the previous relation implies JX=0 for all JX $\in A_S$.

Proposition 9

The set of projectable vector fields of A_S coincides with $A_s\cap\overline{\chi(M)}$ and that $A_s\cap\overline{\chi(M)}=A_\Gamma\cap\overline{\chi(M)}$.

Proof: Let X be a projectable vectors field, the identity of Jacobi makes it possible to write

[J, [X, S]]+[X, [S, J]]+[S, [J, X]]=0.

If [X, S]=0, the above relation becomes

[[J, S], X]=[[J, X], S]. (4.1)

But we have Γ=[J, S], Γh=h, Γv=-v and h+v=I, we find

[Γ, X]Y=[ΓY, X] -Γ [Y, X]=2v[Y, X],

for all Y horizontal vector field of Γ . On the other hand, we have

[S, [J, X]] Y=[S, [J, X]Y]-[J, X] [S, Y].

As [J, X] [S, Y] and [J, X] Y are vertical since X is projectable, by applying J to the relation (4.1) we find

[J, X] Y=0, (4.2)

for all Y horizontal vectors field. If Y is vertical, it is immediate to note that

[J, X] Y=0. (4.3)

The relations (4.2) and (4.3) prove that [J, X]=0. In other words, X $\in$ A_J. But $A_J=\overline{\chi(M)}+J\,\overline{\chi(M)}$, and taking into account the proposition 8, therefore, the set of projectable vector fields of A_S coincides with $A_S\cap\overline{\chi(M)}$.

With [J, X]=0, the relation (4.1) proves that X $\in$ A_Γ. That is to say

$$A_S\cap\overline{\chi(M)}\subset A_\Gamma\cap\overline{\chi(M)}\,.$$

Let us now prove that $A_\Gamma\cap\overline{\chi(M)}\subset A_S\cap\overline{\chi(M)}$.

Let be $\overline{X}\in A_\Gamma\cap\overline{\chi(M)}$. This results in $[\overline{X},h]=0$ with S=hS. If we apply to S the above relation, we get

$$[\overline{X},hS]-h[\overline{X},S]=0=v[\overline{X},S]\,. \quad (4.4)$$

From the relation $[\overline{X},J]=0$, , we find $J[\overline{X},S]=0$. In other words, taking into account that h is semi-basic

$$h[\overline{X},S]=0, \quad (4.5)$$

as v+h=I, the two relations (4.4) and (4.5) result in

$$[\overline{X},S]=0.$$

Which proves that

$$A_\Gamma\cap\overline{\chi(M)}\subset A_S\cap\overline{\chi(M)}.$$

Remark 1: The proposition 9 is contained in [12] in another way.

In the following, we assume $\overline{A_s}=A_s\cap\overline{\chi(M)}$ and

$\overline{A_\Gamma}=A_\Gamma\cap\overline{\chi(M)},\overline{A_\Gamma}^h$ the horizontal part of $\overline{A_\Gamma}$.

Proposition 10

The horizontal part of $\overline{A_\Gamma}$ is such that

$$\overline{A_\Gamma}^h=\{X\in A_\Gamma^h \; and \; JX\in A_\Gamma^v\}.$$

and $\overline{A_\Gamma}^h$ is a commutative ideal of the Lie algebra $\overline{A_\Gamma}$.

Proof: This is to prove that the horizontal part of $\overline{A_\Gamma}$ is only the B of the proposition 7, which is immediate according to the proposition 3.

Theorem 1

For all differentiable manifold of n dimension with a linear connection without torsion such that the rank of the nullity space of the projectable vector fields of the curvature is nonzero constant, the Lie algebra $\overline{A_\Gamma}$ contains a nonzero commutative ideal formed by the horizontal part of $\overline{A_\Gamma}$.

Proof: This is the consequence of previous studies. The existence of the nonzero horizontal part of $\overline{A_\Gamma}$ is given by the proposition 6.

Automorphisms

We denote by G_S the set of diffeomorphisms ϕ of M such that the induced diffeomorphisms ϕ_* on TM preserve a spray S, G_S forms a group. From the study of the equations of $\overline{A_s}$ and G_S on a local coordinate system of M, using the classical theorem of Palais, Loos O. [6] deduces the following result:

Theorem 2 [6]

a) G_S acts freely on the linear frame bundle of M. In particular, the dimension of $\overline{A_s}$ is less than or equals to n^2+n.

b) There exists a unique Lie group structure on G_S such that G_S is a Lie transformation group of M with Lie algebra the set of complete vector fields in A_s .

c) If the dimension of $\overline{A_s}$ =n^2+n and M a manifold simply connected, then (M, S) is isomorphic to $(\mathbb{R}^n,Z_\lambda)$ for an unique $\lambda\in\mathbb{R};Z_\lambda$ is given by the equations $\frac{dy^i}{dt}=\lambda y^i, i=1,...,n.$

Examples

Example 1

Let be $M=\mathbb{R}^3$ and the connection Γ=[J, S] with

$$S=y^1\frac{\partial}{\partial x^1}+y^2\frac{\partial}{\partial x^2}+y^3\frac{\partial}{\partial x^3}-2e^{x^3}(y^1)^2\frac{\partial}{\partial y^1}-2e^{x^3}(y^2)^2\frac{\partial}{\partial y^2},$$

then the coefficients of Γ being $\Gamma_i^j=\frac{\partial G^j}{\partial y^i}$ for i, j $\in$ {1, 2, 3}, the nonzero coefficients of the connection are $\Gamma_1^1=2e^{x^3}y^1,\Gamma_2^2=2e^{x^3}y^2$.

The horizontal space is generated by

$$\frac{\partial}{\partial x^1}-2e^{x^3}y^1\frac{\partial}{\partial y^1},\frac{\partial}{\partial x^2}-2e^{x^3}y^2\frac{\partial}{\partial y^2},\frac{\partial}{\partial x^3}.$$

We notice that $\overline{A_\Gamma}^h$ ={0}. The Lie algebra $\overline{A_\Gamma}$ is generated by

$$\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},x^1\frac{\partial}{\partial x^1}+x^2\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x^3}+y^1\frac{\partial}{\partial y^1}+y^2\frac{\partial}{\partial y^2} \text{ on } \mathbb{R}.$$

We note that all the subalgebras of $\left\{\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\right\}$ are commutative ideals of $\overline{A_\Gamma}$.

None of these nonzero ideals are horizontal.

Example 2

Let E be the energy function such that

$E=\frac{1}{2}(e^{x^1}(y^1)^2+e^{x^2}(y^2)^2+e^{2x^1}(y^3)^2)$. The canonical spray is written:

$$S=y^1\frac{\partial}{\partial x^1}+y^2\frac{\partial}{\partial x^2}+y^3\frac{\partial}{\partial x^3}-\frac{1}{2}((y^1)^2-2e^{x^1}(y^3)^2)\frac{\partial}{\partial y^1}-\frac{1}{2}(y^2)^2\frac{\partial}{\partial y^2}-2y^1y^3\frac{\partial}{\partial y^3}.$$

The nonzero coefficients of the connection are

$$\Gamma_1^1=\frac{y^1}{2},\Gamma_3^1=-e^{x^1}y^3,\Gamma_2^2=\frac{y^2}{2},\Gamma_1^3=y^3,\Gamma_3^3=y^1.$$

The horizontal vector fields are generated by

$$\frac{\partial}{\partial x^1}-\frac{y^1}{2}\frac{\partial}{\partial y^1}-y^3\frac{\partial}{\partial y^3},\frac{\partial}{\partial x^2}-\frac{y^2}{2}\frac{\partial}{\partial y^2},\frac{\partial}{\partial x^3}+e^{x^1}y^3\frac{\partial}{\partial y^1}-y^1\frac{\partial}{\partial y^3}.$$

The horizontal nullity space of the curvature is generated by $\frac{\partial}{\partial x^2}-\frac{y^2}{2}\frac{\partial}{\partial y^2}$. The elements of $\overline{A_\Gamma}$ are generated by

$$\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3},e^{\frac{-x^2}{2}}\left(\frac{\partial}{\partial x^2}-\frac{y^2}{2}\frac{\partial}{\partial y^2}\right),2\frac{\partial}{\partial x^1}-x^3\frac{\partial}{\partial x^3}-y^3\frac{\partial}{\partial y^3}$$

The commutative ideal is generated by $e^{\frac{-x^2}{2}}(\frac{\partial}{\partial x^2}-\frac{y^2}{2}\frac{\partial}{\partial y^2})$.

We note that for any linear connection (see the example 1), the commutative ideals do not always come from the horizontal nullity space of the curvature.

The example 2 shows that the commutative ideal is of the horizontal nullity space of the curvature. The question is to know if it is true for all canonical connection coming from an energy function in the language of [3]. This is the object of our study.

Connections on Riemannian Manifolds

In this paragraph, we are content to state the well-known results on the Riemannian manifolds which will be used.

Definition 2

We call the Riemannian manifold, the couple (M, E):

- M is a differentiable manifold;
- E a function of $\mathcal{TM}=TM-\{0\}$ into $\mathbb{R}^+$, with E(0)=0, $\mathcal{C}^\infty$ on TM, $\mathcal{C}^2$ on the null section, and homogeneous of degree two, such that dd_JE has a maximum rank.

We notice that if E is of class $\mathcal{C}^1$ on the null section, we will have a Finsler manifold. Imposing E of class $\mathcal{C}^2$ on the null section implies that the function E is of class $\mathcal{C}^\infty$ on TM.

The application E is called energy function, $\Omega=dd_JE$ its fundamental form defines a spray S [13]:

$i_S dd_J E=-dE$ and $\Gamma=[J, S]$ is called canonical connection. (7.1)

The fundamental form Ω allows to define a metric g on the tangent bundle by

g (X, Y)=Ω (X, FY), (7.2)

X and Y being two vector fields on TM, F the almost complex structure associated with Γ, (FJ=h and Fh=-J where $h=\frac{I+\Gamma}{2}$. There is, [4], one and only one metric lift D of the canonical connection such that:

1. $J\mathbb{T}(hX,hY)=0,$
2. $\mathbb{T}(JX,JY)=0\left(\mathbb{T}(X,Y)=D_XY-D_YX-[X,Y]\right),$,
3. DJ=0,
4. DC=v,
5. DΓ=0,
6. Dg=0,
7. DF=0.

The linear connection D is called Cartan connection. For a Riemannian manifold, the Cartan connection D and the Berwald connection $\overset{\circ}{D}$ are identical. We have

$D_{JX}JY=[J, JY]\, X$, $D_{hX}JY=[h, JY]\, X$.

To the linear connection D, we associate a curvature

$$\Re(X,Y)Z=D_{hX}D_{hY}JZ-D_{hY}D_{hX}JZ-D_{[hX,hY]}JZ, \quad (7.3)$$

for all X, Y, Z ∈ χ(TM). In particular,

$$\Re(X,Y)S=-R(X,Y). \quad (7.4)$$

In natural local coordinates on an open set U of M, $(x^i, y^i)\in TU$ ∈ i, j ∈ {1,..., n} , the energy function is written [3] p.330

$$E=\frac{1}{2}g_{ij}(x^1,...,x^n)y^iy^j$$

where $g_{ij}(x^1, . . . , x^n)$ are symmetric positive functions such that the matrix $(g_{ij}(x^1, . . . , x^n))$ is invertible . And the relation $i_Sdd_JE=-dE$ gives the spray S,

$$S=y^i\frac{\partial}{\partial x^i}=2G^i(x^1,...,x^n,y^1,...,y^n)\frac{\partial}{\partial y^i}.$$

with $G_k=\frac{1}{2}y^iy^j\gamma_{ikj}$

where $\gamma_{ikj}=\frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right).$

By $\gamma_{ij}^k=g^{kl}\gamma_{ilj}$,

we have $G^k=\frac{1}{2}y^iy^j\gamma_{ij}^k.$

The relation (7.1) is equivalent to the following relation

$d_hE=0.$ (7.5)

Properties of the Curvature $\Re$ of a Riemannian Manifold

From the properties stated above, we obtain the following classic results: for all X, Y, Z, T ∈ χ(TM)

$$g\left(\Re(X,Y)Z,JT\right)=-g\left(\Re(X,Y)T,JZ\right) \quad (8.1)$$

$$\Re(X,Y)Z+\Re(Y,Z)X+\Re(Z,X)Y=0; \quad (8.2)$$

$$g\left(\Re(T,X)Y,JZ\right)+g\left(\Re(T,Y)Z,JX\right)+g\left(\Re(T,Z)X,JY\right)=0; \quad (8.3)$$

$$g\left(\Re(X,Y)Z,JT\right)=g\left(\Re(Z,T)X,JY\right). \quad (8.4)$$

Properties of the Horizontal Nullity Space of Curvatures

From the properties of the curvatures given above, we have:

$$JX\perp ImR^\circ \Leftrightarrow X\in KerR^\circ\left(R^\circ=i_SR\right),X\in\chi(TM); \quad (9.1)$$

$$JX\perp ImR\Leftrightarrow\Re(S,X)Y=0,X\in\chi(TM),\forall Y\in\chi(TM); \quad (9.2)$$

$$X\in N_R\Leftrightarrow JX\perp\Re(S,Y)Z,X\in\chi(TM),\forall Y,Z\in\chi(TM); \quad (9.3)$$

$$X\in N_\Re\Leftrightarrow JX\perp\Re(Y,Z)T,X\in\chi(TM),\forall Y,Z,T\in\chi(TM); \quad (9.4)$$

$$X\in N_\Re\Leftrightarrow\Re(Y,Z)X=0,\forall Y,Z\in\chi(TM). \quad (9.5)$$

Proposition 11

On a Riemannian manifold, the horizontal ideal A_Γ^h of the Lie algebra A_Γ satisfies

$$A_\Gamma^h=N_R\cap H^\circ=Ker\Re\cap H^\circ=N_\Re\cap H^\circ=Ker\,R^\circ\cap H^\circ.$$

Proof: According to the proposition 5, the horizontal ideal A_Γ of A_Γ is written

$$A_{\Gamma}^{h} = N_R \cap H^{\circ} = N_{\Re} \cap H^{\circ} \subset Ker\Re \cap H^{\circ}.$$

From the relation (8.4), we get $Ker\Re \cap H^{\circ} \subset N_{\Re} \cap H^{\circ}$. It remains to prove that

$$N_{\Re} \cap H^{\circ} = KerR^{\circ} \cap H^{\circ}.$$

It is immediate to notice that

$$N_{\Re} \cap H^{\circ} \subset KerR^{\circ} \cap H^{\circ}.$$

Let be X $\in$ KerR^o ∩ H^o. According to the relation (9.1), we have for all Y $\in\chi$(TM),

g (R^o (Y), JX)=0.

As Dg=0, we get

g ([J, R^o (Y)]Z, JX)=-g(R^o (Y), [J, JX]Z).

The vector field X being projectable, [J, JX]=0, we have

g ([J, R^o (Y)]Z, JX)=0. (9.6)

From the relation (8.2), namely: namely: $\Re(S,Y)Z+\Re(Y,Z)S+\Re(Z,S)Y=0$, we conclude

g ([J, R^o (Y)]Z- [J, R^o (Z)]Y, JX)=g(3R(Y, Z)+R^o ([JZ, Y]-R^o ([JY, Z]), JX).

According to the relation (9.6) and that JX is orthogonal to Im R^o, we find

$$g\left(R(Y,Z),JX\right)=0,$$

that is, JX⊥ ImR. Taking into account D_{JT} g=0, and X a projectable vectors field, we have

$$g\left(\Re(Y,Z)T,JX\right)=0,$$

that is, $KerR^{\circ} \cap H^{\circ} \subset Ker\Re \cap H^{\circ}$

Proposition 12

On a Riemannian manifold, the three following relations are equivalent:

i. the horizontal nullity of the curvature R is reduced to zero

ii. the horizontal nullity space of the projectable vector fields of R is reduced to zero

iii. the dimension of the image space of the curvature R is equal to n-1

Proof: If the horizontal nullity space of the projectable vector fields is not reduced to zero, there will be a nonzero horizontal and projectable vector field X, according to the proposition 11, such that $X \in N_{\Re}$, and according to the relation (9.4), JX is orthogonal to Im$\Re$, therefore orthogonal to ImR according to the relation (7.4). The image space of the curvature R is both orthogonal to JX and to the Liouville field C=JS with JX $\wedge$ C ≠ 0. This is only possible if the dimension of the image space of R is strictly less than n-1.. So the relation (iii) implies (ii). We notice by (7.5) that d_RE=0. That is, the image space of R is in the kernel of dE. If the dimension of the image space of the curvature R is strictly less than n-1, there will be a horizontal vector field X such that JX $\in$ KerdE and JX ⊥ ImR. According to the relation (9.2), we have $\Re(S,X)Y=0, \forall Y \in \chi(TM)$. By developing this equality and taking into account the relation (9.1), we have R(X, Y)=R^o[JY, X] ∀Y $\in\chi$(TM).. This is only possible if X=S or X$\in$ N_R. Since, JX ⊥C, therefore, we have X$\in$ N_R.

This last calculation proves at the same time that the horizontal nullity space of R is generated by projectable vectors field belonging to N_R. Consequently, we have (i)=⇒ (iii) and (ii)=⇒ (i).

Conformal Infinitesimal Transformations of a Riemannian Manifold

Definition 3

Let ω be a p-differential form defined on a manifold M. A conformal infinitesimal transformation of ω is a vector field X defined on M, such that for all elements (t, x) of the domain D(X) of the flow Φ of X, we have

$$\Phi_t^{*}\omega_{(x)} = \rho(t,x)\omega_{(x)},$$

where ρ is a differentiable numeric function, with positive values, defined on D(X).

We have the following result:

Proposition 13

For a vector field X to be a conformal infinitesimal transformation of p-differential form ω, it is necessary and sufficient that we have,

$L_X\omega=\lambda\omega$,

where L_X is the Lie derivative with respect to X and λ a differentiable numerical function defined on M. When this is the case, the function λ is linked to the function ρ which appears in the definition 3, by the relation

$$\rho(t,x) = \exp\left(\int_0^t \lambda \circ \Phi_s(x)ds\right).$$

In the following, we are interested in conformal transformations of the fundamental form Ω of a Riemannian manifold.

Proposition 14

If X is a conformal infinitesimal transformation of Ω, i.e.

$L_X\Omega=\lambda\Omega$,then λ is a constant function.

Proof: Let be X a vector field belonging to χ(TM) such that

$L_X\Omega=\lambda\Omega$,

with $\lambda \in \mathcal{F}(TM)$ the set of differentiable functions on TM. Since Ω=dd_JE, we have

$\lambda dd_J E=dL_X d_J E$.

The expression λdd_JE is an exact 2-form, this implies

$d\lambda\wedge dd_J E=0$. (10.1)

as dd_JE is of maximum rank and the dimension of the manifold M is assumed to be greater than or equal to two, the relation (10.1) results in

$d\lambda=0$.

In other words, the function λ is a constant.

In the following, we will denote A_{gc} the set of conformal infinitesimal transformations of Ω and $\overline{A_{gc}} = A_{gc} \cap \overline{\chi(M)}$.

Proposition 15

$\overline{X} \in \overline{A_{gc}}$ if and only if

$$L_{\overline{X}}E = \lambda E,$$

where λ is a constant given by the relation $L_{\overline{X}}\Omega = \lambda\Omega$.

Proof: Let $\overline{X} \in \overline{A_{gc}}$, we have

$$L_{\overline{X}}dd_J E = \lambda dd_J E,$$

where λ is a constant according to the proposition 14. As the two derivations $L_{\overline{X}}$ and d_J commute, we obtain

$$dd_J L_{\overline{X}}E = \lambda dd_J E.$$

By derivating by i_C, the two 2-forms above, C being Liouville field, we find

$$d_J L_{\overline{X}}E = \lambda d_J E.$$

The application of i_S to the above equality gives

$$L_{\overline{X}}E=\lambda E \quad .$$

Conversely, if $L_{\overline{X}}E=\lambda E$ with $\lambda\in\mathbb{R}$, it is immediate to note that by continuing the previous calculation, we have $L_{\overline{X}}dd_JE=\lambda dd_JE.\cdot$

Proposition 16

$\overline{A_{gc}}\subset\overline{A_\Gamma}$.

Proof: Let $\overline{X}\in\overline{A_{gc}}$. Since $i_S dd_J E=-dE$, we can write successively

$$i_sL_{\overline{X}}dd_JE=\lambda\, i_sdd_JE=-\lambda dE$$

and $L_{\overline{X}}i_sdd_JE=-L_{\overline{X}}dE=-\lambda dE$.

From the formula $[i_s,L_{\overline{X}}]=i_{[S,\overline{X}]}$, we get

$$i_{[S,\overline{X}]}dd_JE=0 .$$

As dd_JE is a symplectic form, we find

$$[S,\overline{X}]=0\cdot$$

This means that $\overline{X}\in\overline{A_s}$. For a Riemannian manifold, $\overline{A_s}$ coincides with $\overline{A_\Gamma}$. Thus,

$$\overline{A_{gc}}\subset\overline{A_\Gamma} \cdot$$

Proposition 17

Let F be the almost complex structure associated to Γ . So by noting $\overline{A_F}=A_F\cap\overline{\chi(M)}$,
we have

$$\overline{A_\Gamma}\subset\overline{A_F} .$$

Proof: Let $\overline{X}\in\overline{A_\Gamma}$. According to the definition of F, namely,

FJ=h and Fh=-J,

we can write successively for all Y $\in\chi$(TM)

$$[\overline{X},F]JY=[\overline{X},FJY]-F[\overline{X},JY]=[\overline{X},hY]-FJ[\overline{X},Y] \quad (10.2)$$

$$=h[\overline{X},Y]-h[\overline{X},Y]=0$$

Similarly, we have

$$[\overline{X},F]hY=[\overline{X},FhY]-F[\overline{X},hY]=-[\overline{X},JY]-Fh[\overline{X},Y] \quad (10.3)$$

$$=-J[\overline{X},Y]+J[\overline{X},Y]=0 \quad ,$$

for all Y $\in\chi$(TM), the two relations (10.2) and (10.3) lead to

$$[\overline{X},F]=0 .$$

In other words,

$$\overline{A_\Gamma}\subset\overline{A_F} .$$

Proposition 18

Let $\overline{X}\in\overline{\chi(M)},\lambda\in\mathbb{R}$ .So

$L_{\overline{X}}\Omega=\lambda\Omega$ is equivalent to $L_{\overline{X}}g=\lambda g$.

Proof: a) Suppose $L_{\overline{X}}\Omega=\lambda\Omega$, we have

$$(L_{\overline{X}}g)(Y,Z)=L_{\overline{X}}g(Y,Z)-g(L_{\overline{X}}Y,Z)-g(Y,L_{\overline{X}}Z),$$

for all Y, Z$\in\chi$(TM). Given the relation g(Y, Z)=Ω(Y, FZ), the second member of the above equality becomes

$$(L_{\overline{X}}g)(Y,Z)=L_{\overline{X}}\Omega(Y,FZ)-\Omega(L_{\overline{X}}Y,FZ)-\Omega(Y,FL_{\overline{X}}Z).$$

By taking into account the propositions 16 and 17, we obtain

$$(L_{\overline{X}}g)(Y,Z)=L_{\overline{X}}\Omega(Y,FZ),$$

for all Y, Z $\in\chi$(TM). If $L_{\overline{X}}\Omega=\lambda\Omega$, we find

$$L_{\overline{X}}g=\lambda g .$$

b) Suppose $L_{\overline{X}}g=\lambda g$. We have

$$(L_{\overline{X}}\Omega)(Y,Z)=L_{\overline{X}}\Omega(Y,Z)-\Omega(L_{\overline{X}}Y,Z)-\Omega(Y,L_{\overline{X}}Z), \quad (10.4)$$

for all Y, Z $\in\chi$(TM). The expression of Ω in function of g is [3]

Ω (Y, Z)=g (Y, JZ)+g (JY, Z), (10.5)

for all Y, Z $\in\chi$(TM).. The expression (10.4) becomes

$$(L_{\overline{X}}\Omega)(Y,Z)=L_{\overline{X}}(g(Y,JZ)+g(JY,Z))-g(L_{\overline{X}}Y,JZ)-g(JL_{\overline{X}}Y,Z)$$
$$-g(Y,JL_{\overline{X}}Z)-g(JY,L_{\overline{X}}Z).$$

As $\overline{X}$, belongs to A_J, we get

$$(L_{\overline{X}}\Omega)(Y,Z)=(L_{\overline{X}}g)(Y,JZ)+(L_{\overline{X}}g)(JY,Z).$$

If $L_{\overline{X}}g=\lambda g$, taking into account the relation (10.5) we find

$$L_{\overline{X}}\Omega=\lambda\Omega .$$

Automorphisms of a Riemannian Manifold

Definition 4

A vector field X on a Riemannian manifold (M, E) is said to be infinitesimal automorphism of the symplectic form Ω if

$L_X\Omega=0.$

We notice the canonical spray S of (M, E) is an infinitesimal automorphism of the symplectic form Ω.

The set of infinitesimal automorphisms of Ω forms a Lie algebra. We will denote this Lie algebra by A_g.

Proposition 19

By supposing $\overline{A_g}=A_g\cap\overline{\chi(M)}$, we have

a) $X\in\overline{A_g}$ if and only if X is a projectable vector field such that X $\in A_g$ and $L_XE=0$;

b) $\overline{A_g}\subset\overline{A_\Gamma}$; the horizontal elements of $\overline{A_\Gamma}$ form a commutative ideal of $\overline{A_g}$.

c) The elements of $\overline{A_g}$ are Killing fields of projectable vectors of the metric g belonging to $\overline{A_\Gamma}$. The dimension of $\overline{A_g}$ is at most equal to $\frac{n(n+1)}{2}$.

Proof: It is clear that if $X\in\overline{A_g}$ then $L_XE=0$. This is a particular case of the proposition 15 for λ=0.

Let X be a projectable vector field such that $L_X\Omega=0$ and $L_XE=0$, we will prove that $X\in\overline{A_g}$. The fundamental form Ω is written $\Omega=dd_JE$, as $L_X\Omega=0$, we have

$i_sL_X\Omega=0.$

Besides, according to the relation (7.1) and the hypothesis $L_XE=0$

$L_Xi_s\Omega=-dL_XE=0$. According to the formula $i_{[s,x]}=i_sL_x-L_xi_s$, we find

$i_{[S,X]}\Omega=0$.

Since the fundamental form Ω is symplectic, we get [S, X]=0. The vector field X being projectable by hypothesis, according to the proposition 9, $X \in A_s \cap \chi(M)$.

-The result of b) is a consequence of the proposition 16 because $\overline{A_g}$ is a particular case of $\overline{A_{g_c}}$ for λ=0. Since, we have by (7.5), the horizontal elements of $\overline{A_\Gamma}$ belong to $\overline{A_g}$ and form a commutative ideal of $\overline{A_g}$, according to the theorem 1.

- Finally, the property c) is given by the proposition 18 by taking λ=0.

In addition, g defines a metric on the vertical bundle which is of n dimension. The flow Φ of a vector field $\overline{X} \in \overline{A_g}$ is a local isometry for g. Consequently, Φ preserves the orthogonal bases of g on the vertical bundle. With the method used in theorem 3.3 of [14] vol.1 p.238, the dimension of $\overline{A_g}$ is at most equal to $\frac{n(n+1)}{2}$.

Theorem 3

Let (M, E) be a Riemannian manifold with a nonzero constant sectional curvature. Then $\overline{A_g}$ coincides with $\overline{A_\Gamma}$.

Proof: If the sectional curvature K is constant, we have [2]

$$R^\circ = 2EKJ - Kd_J E \otimes C$$

where $R^\circ = i_S R$. For all $\overline{X} \in \overline{A_\Gamma}$, we have $L_{\overline{X}} R^\circ = 0$. If K is not zero, we get

$$2L_{\overline{X}} EJ = (L_{\overline{X}} dE) \otimes C,$$

this is only possible if $L_{\overline{X}} E = 0$, that is, according to the proposition 19 a), $\overline{X} \in \overline{A_g}$. In other words,

$$\overline{A_\Gamma} \subset \overline{A_g}.$$

The proposition 19 b) gives

$$\overline{A_g} \subset \overline{A_\Gamma}.$$

Riemannian Manifolds with Regular Curvature

Definition 5

We will say that a connection Γ has a regular curvature at a point z of TM if the vector space generated by the image of the curvature R_z is of n-1 dimension.

Proposition 20

Let (M, E) be a Riemannian manifold such that

$\Re(X,Y) \neq 0$ for all horizontal vector fields X, Y linearly independent, then (M, E) has a regular curvature.

Proof: The dimension of ImR is less than or equal to n-1, since the canonical field C=JS is orthogonal to ImR.

If the dimension of ImR is less than or equal to n-2, there will be a horizontal vector field Z such that JZ ⊥ ImR and S ∧ Z ≠ 0, and according to the relation (9.2), we would have

$$\Re(S,Z) = 0$$

this contradicts the hypothesis.

Remark 2: The converse of the proposition 20 is not true in general.

Theorem 4

Let (M, E) be a Riemannian manifold, Γ a canonical connection of (M, E) with regular curvature. So

$$\overline{A_\Gamma} = \overline{A_{gc}}.$$

Proof: The proposition 16 gives

$$\overline{A_{g_c}} \subset \overline{A_\Gamma}.$$

It remains to prove that $\overline{A_\Gamma}$ is contained in $\overline{A_{g_c}}$; and for that, it is simply sufficient to prove $L_{\overline{X}} E = \lambda E$ for all $\overline{X} \in \overline{A_\Gamma}$, $\lambda \in \mathbb{R}$ according to the proposition 15.

Let $\overline{X} \in \overline{A_\Gamma}$, as A_Γ is identical to A_h, we have

$$d_{[\overline{X},h]} E = 0.$$

According to the formula $[L_{\overline{X}}, d_h] = d_{[\overline{X},h]}$ and by (7.5), we obtain

$$d_h L_{\overline{X}} E = 0. \tag{12.1}$$

That implies

$$d_h d_h L_{\overline{X}} E = 0 \text{ or } d_R L_{\overline{X}} E = 0.$$

Let H be the horizontal space of Γ, $H \oplus \operatorname{Im} R$ is contained in the kernel of dE. By hypothesis, the curvature R is regular, so H ⊕ ImR is of 2n-1 dimension, and the kernel of dE. Therefore, H ⊕ ImR is a completely integrable distribution. According to Frobenius theorem, at each point z of TM, it passes a maximum integral manifold which is the solution of the equation (12.1). We notice that $L_{\overline{X}} E$ is homogeneous of degree two and that the function $L_{\overline{X}} E$ is continuous and null on the null section.

This imposes the initial conditions for the solutions of the equation (12.1). Hence the result

$$L_{\overline{X}} E = \lambda E,$$

With λ is a constant.

Remark 3: A Riemannian manifold with a constant and nonzero sectional curvature has a regular curvature, but the value of λ is zero. The theorem 3 excludes the case of λ ≠ 0.

Proposition 21

Let (M, E) be a Riemannian manifold with regular curvature, $R^\circ = i_S R$, we have the following relation:

$$[\overline{X}, R^\circ(\overline{Y})] = R^\circ[\overline{X}, \overline{Y}] = -[\overline{Y}, R^\circ(\overline{X})],$$

for all $\overline{X}, \overline{Y} \in \overline{A_\Gamma}$.

Proof: For all $\overline{X}, \overline{Y} \in \overline{A_\Gamma}$, we have

$$[\overline{X}, R^\circ] = 0.$$

That is, $[\overline{X}, R^\circ(\overline{Y})] - R^\circ([\overline{X}, \overline{Y}]) = 0$, or

$$[\overline{X}, R^\circ(\overline{Y})] = R^\circ([\overline{X}, \overline{Y}]) = -[\overline{Y}, R^\circ(\overline{X})].$$

Theorem 5

The Lie algebra $\overline{A_g}$ of the Killing fields contained in $\overline{A_\Gamma}$ of dimension greater than or equal to three is semi-simple if and only if the Riemannian manifold is with regular curvature.

Proof: If the Riemannian manifold is not with regular curvature, there is the horizontal nullity space of the projectable vector fields of the curvature R which provides the commutative ideal of $\overline{A_\Gamma}$, according to the propositions 19 b) and 12, which is also that of $\overline{A_g}$. Then $\overline{A_g}$ is not semi- simple.

It is assumed that the Riemannian manifold has a regular curvature. Let $\overline{X_1}, \overline{X_2} \in \overline{A_g}$ such that $\overline{X_1}$ and $\overline{X_2}$ are linearly independent. As the mapping R° is injective on the projectable vector fields according to the proposition 11 and the definition 5. $R^\circ(\overline{X_1}), R^\circ(\overline{X_2})$ are also linearly independent.

According to the proposition 21,

$\left[\overline{X}_1, R^\circ(\overline{X_2})\right] = R^\circ\left[\overline{X}_1, \overline{X}_2\right] = -\left[\overline{X}_2, R^\circ\left(\overline{X}_1\right)\right].$

As $\overline{X_1}, \overline{X_2}$ are elements of $\overline{A_g}$, we have the following system of equations

$$\begin{cases} L_{\overline{X_1}} g(R^\circ(\overline{X_2}), JY_1) = g(R^\circ\left[\overline{X}_1, \overline{X}_2\right], JY_1) + g\left(R^\circ\left(\overline{X}_2\right), J\left[\overline{X}_1, Y_1\right]\right), \forall Y_1 \in \chi(TM), \\ L_{\overline{X_2}} g(R^\circ(\overline{X_1}), JY_2) = g(R^\circ\left[\overline{X}_2, \overline{X}_1\right], JY_2) + g\left(R^\circ\left(\overline{X}_1\right), J\left[\overline{X}_2, Y_2\right]\right), \forall Y_2 \in \chi(TM). \end{cases}$$

Taking $JY_1 \perp R^\circ(\overline{X_2})$ and $JY_2 \perp R^\circ(\overline{X_1})$, we have

$$\begin{cases} g(R^\circ\left[\overline{X}_1, \overline{X}_2\right], JY_1) = -g(R^\circ(\overline{X_2}), J[\overline{X_1}, Y_1]), \\ g(R^\circ\left[\overline{X}_1, \overline{X}_2\right], JY_2) = g(R^\circ(\overline{X_1}), J[\overline{X_2}, Y_2]). \end{cases} \quad (12.2)$$

If we have $\overline{X}_1, \overline{X}_2, \overline{X}_3 \in \overline{A_g}$ linearly independent, we have $R^\circ(\overline{X}_1), R^\circ(\overline{X}_2), R^\circ(\overline{X}_3)$ linearly independent, and we have a system of six equations as (12.2). For $\overline{X_1}, ..., \overline{X_p} \in \overline{A_g}$, we have a system of 2p equations as (12.2). Such a system of equations do not allow us to have a commutative ideal of $\overline{A_g}$ other than zero.

We conclude that if $\overline{A_g}$ is of dimension superior or equals to three for a Riemannian manifold with regular curvature, $\overline{A_g}$ is semi-simple.

Corollary 1

On a Riemannian manifold (M, E) of n ≥ 2 dimension, the Lie algebra $\overline{A_g}$ of infinitesimal isometries contained in $\overline{A_\Gamma}$ of dimension superior or equals to three is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of Γ is reduced to zero. In this case, the Lie algebra $\overline{A_\Gamma}$ coincides with $\overline{A_g}$.

Proof: The reasoning used in the theorem 5 is based on the existence of two linearly independent elements of $\overline{A_g}$. So the dimension of $\overline{A_g}$ is assumed to be superior or equals to two. We know that there is no semi-simple Lie algebra of one or two dimension. Hence, the condition of the dimension of $\overline{A_g}$ superior or equals to three.

We notice the interlocking of the algebras $\overline{A_\Gamma}^h \subset \overline{A_g} \subset \overline{A_\Gamma}$. So if the dimension of $\overline{A_g}$ is superior or equals to three, for a Riemannian manifold with regular curvature, according to the theorem 5, $\overline{A_g}$ is semi-simple. In this case, according to the theorem 4, $\overline{A_\Gamma} = \overline{A_{g_c}}$ and $\overline{A_g}$ is an ideal of $\overline{A_{g_c}}$. Consequently $\overline{A_g} = \overline{A_\Gamma}$, otherwise we have a contradiction.

Remark 4: If the Riemannian manifold is flat (R=0), the horizontal elements of $\overline{A_\Gamma}$ constitute a nonzero commutative ideal of $\overline{A_g}$. If the Riemannian manifold has a nonzero constant sectional curvature, according to the theorem 3, $\overline{A_\Gamma} = \overline{A_g}$. It is obvious that this manifold has a regular curvature.

Remark 5: The result of the theorem 5 is not true in general if we only impose the fields of projectable vectors to be those of Killing, that is to say to belong to A_g without being elements of $\overline{\chi(M)}$.

The elements of A_g even projectable form in general an algebra of infinite dimension.

Examples

Example 3

We assume $M = \mathbb{R}^2$, $E = \frac{1}{2}(e^{x^2}(y^1)^2 + (y^2)^2)$ The spray S is written

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - y^1 y^2 \frac{\partial}{\partial y^1} + \frac{1}{2} e^{x^2} (y^1)^2 \frac{\partial}{\partial y^2}.$$

The nonzero coefficients of the connection are:

$$\Gamma_1^1 = \frac{y^2}{2}, \Gamma_2^1 = \frac{y^1}{2}, \Gamma_1^2 = -\frac{y^1}{2} e^{x^2}.$$

The horizontal vector fields are generated by

$$\frac{\partial}{\partial x^1} - \frac{y^2}{2}\frac{\partial}{\partial y^1} + \frac{y^1}{2} e^{x^2} \frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^2} - \frac{y^1}{2}\frac{\partial}{\partial y^1}.$$

The curvature R is not zero. The elements of $\overline{A_\Gamma}$ are

$$e_1 = \frac{\partial}{\partial x^1}, e_2 = -\frac{x^1}{2}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - \frac{y^1}{2}\frac{\partial}{\partial y^1}$$

$$e_3 = \left(\frac{1}{4}\left(x^1\right)^2 - e^{-x^2}\right)\frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + \left(\frac{x^1 y^1}{2} + y^2 e^{-x^2}\right)\frac{\partial}{\partial y^1} - y^1 \frac{\partial}{\partial y^2}.$$

The Lie algebra $\overline{A_\Gamma}$ coincides with $\overline{A_g}$. It is semi-simple.

Example 4

We assume $M = \mathbb{R}^3$, $E = \frac{1}{2}\left(e^{x^3}\left(y^1\right)^2 + e^{x^1}\left(y^2\right)^2 + e^{x^2}\left(y^3\right)^2\right)$.

The spray is written:

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - \frac{\left(2y^1y^3 - \left(y^2\right)^2 e^{x^1 - x^3}\right)}{2}\frac{\partial}{\partial y^1}$$

$$-\frac{\left(2y^1y^2 - \left(y^3\right)^2 e^{x^2 - x^1}\right)}{2}\frac{\partial}{\partial y^2} - \frac{\left(2y^3y^2 - \left(y^1\right)^2 e^{x^3 - x^2}\right)}{2}\frac{\partial}{\partial y^3}.$$

The coefficients of the connection are:

$$\Gamma_1^1 = \frac{y^3}{2}, \Gamma_2^1 = -\frac{y^2 e^{x^1 - x^3}}{2}, \Gamma_3^1 = \frac{y^1}{2}, \Gamma_1^2 = \frac{y^2}{2}, \Gamma_2^2 = \frac{y^1}{2}, \Gamma_3^2 = -\frac{y^3 e^{x^2 - x^1}}{2},.$$

$$\Gamma_1^3 = -\frac{y^1 e^{x^3 - x^2}}{2}, \Gamma_2^3 = \frac{y^3}{2}, \Gamma_3^3 = \frac{y^2}{2}.$$

The Γ connection has a regular curvature. We note that $\overline{A_\Gamma}$ is generated by $\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}$ on $\mathbb{R}$, and $\overline{A_g} = \{0\}$.

According to this example, we note that the Lie algebra of Killing fields A_g even projectable is of infinite dimension.

Example 5

We take $M = \mathbb{R}^3$, $E = \frac{1}{2}\left(e^{x^1}\left(y^1\right)^2 + e^{x^2}\left(y^2\right)^2 + e^{x^3}\left(y^3\right)^2\right)$.

The spray S is written

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} - \frac{\left(y^1\right)^2}{2}\frac{\partial}{\partial y^1} - \frac{\left(y^2\right)^2}{2}\frac{\partial}{\partial y^2} - \frac{\left(y^3\right)^2}{2}\frac{\partial}{\partial y^3}.$$

The non-zero coefficients of the connection are:

$$\Gamma_1^1 = \frac{y^1}{2}, \Gamma_2^2 = \frac{y^2}{2}, \Gamma_3^3 = \frac{y^3}{2}.$$

Horizontal fields are generated by

$$\frac{\partial}{\partial x^1} - \frac{y^1}{2}\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2} - \frac{y^2}{2}\frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^3} - \frac{y^3}{2}\frac{\partial}{\partial y^3}.$$

The curvature R is zero.

The elements of $\overline{A_\Gamma}$ are generated by:

$$g_1=\frac{\partial}{\partial x^1},g_2=e^{\frac{x^2-x^1}{2}}\left(\frac{\partial}{\partial x^1}-\frac{\left(y^1-y^2\right)}{2}\frac{\partial}{\partial y^1}\right),g_3=e^{-\frac{x^1}{2}}\left(\frac{\partial}{\partial x^1}-\frac{y^1}{2}\frac{\partial}{\partial y^1}\right),$$

$$g_4=e^{\frac{x^3-x^1}{2}}\left(\frac{\partial}{\partial x^1}-\frac{\left(y^1-y^3\right)}{2}\frac{\partial}{\partial y^1}\right),g_5=e^{\frac{x^1-x^2}{2}}\left(\frac{\partial}{\partial x^2}-\frac{\left(y^2-y^1\right)}{2}\frac{\partial}{\partial y^2}\right),g_6=\frac{\partial}{\partial x^2},$$

$$g_7=e^{-\frac{x^2}{2}}\left(\frac{\partial}{\partial x^2}-\frac{y^2}{2}\frac{\partial}{\partial y^2}\right),g_8=e^{\frac{x^3-x^2}{2}}\left(\frac{\partial}{\partial x^2}-\frac{\left(y^2-y^3\right)}{2}\frac{\partial}{\partial y^2}\right),$$

$$g_9=e^{\frac{x^1-x^3}{2}}\left(\frac{\partial}{\partial x^3}-\frac{\left(y^3-y^1\right)}{2}\frac{\partial}{\partial y^3}\right),$$

$$g_{10}=e^{\frac{x^2-x^3}{2}}\left(\frac{\partial}{\partial x^3}-\frac{\left(y^3-y^2\right)}{2}\frac{\partial}{\partial y^3}\right),g_{11}=\frac{\partial}{\partial x^3},g_{12}=e^{-\frac{x^3}{2}}\left(\frac{\partial}{\partial x^3}-\frac{y^3}{2}\frac{\partial}{\partial y^3}\right).$$

The horizontal vectors fields which form the commutative ideal are $\{g_3,g_7,g_{12}\}$. The elements in $\overline{A_g}$, according to the proposition 19 (Table 1), are

$$e_1=g_2-g_5,e_2=g_3,e_3=g_4-g_9,e_4=g_7,e_5=g_8-g_{10},e_6=g_{12}.$$

Table 1. Multiplication table of $\overline{A_g}$

[.,.]	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	$\frac{e_4}{2}$	$\frac{e_5}{2}$	$\frac{-e_2}{2}$	$\frac{-e_3}{2}$	0
e_2	$\frac{-e_4}{2}$	0	$\frac{-e_6}{2}$	0	0	0
e_3	$\frac{-e_5}{2}$	$\frac{e_6}{2}$	0	0	$\frac{e_1}{2}$	$\frac{-e_2}{2}$
e_4	$\frac{e_2}{2}$	0	0	0	$\frac{-e_6}{2}$	0
e_5	$\frac{e_3}{2}$	0	$\frac{-e_1}{2}$	$\frac{e_6}{2}$	0	$\frac{-e_4}{2}$
e_6	0	0	$\frac{e_2}{2}$	0	$\frac{e_4}{2}$	0

\We see that the derivative ideal from $\overline{A_g}$ coincides with $\overline{A_g}$ .The commutative ideal is generated by $\{e_2, e_4, e_6\}$.

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