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Operad Theory of Algebra

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Introduction

Operad theory is a branch of mathematics that studies archetypal algebras that simulate various associativity levels, commutativity, and anticommutativity. By simulating computational trees inside the algebra, operads generalize the different associativity qualities already seen in algebras and coalgebras like Lie algebras or Poisson algebras. Operads are to algebras what groups are to groups, and vice versa. An operad can be thought of as a collection of operations, each with a fixed, finite number of inputs (arguments) and a single, composing result. They create an analog of universal algebra in category theory. Operads are a result of the investigation of iterated loop spaces by J. Michael Boardman, Rainer M. Vogt, and J. Peter May in algebraic topology [1].

Description

May coined the term "operad" as a combination of the words "operations" and "monad." Operads saw a significant resurgence in popularity in the early 1990s when Victor Ginzburg, Mikhail Kapranov, and Maxim Kontsevich realized that various duality events in rational homotopy theory could be described by Koszul duality of operads. Since then, operads have been used in a wide variety of contexts, including the Deligne conjecture, the deformation quantization of Poisson manifolds, and the study of graph homology by Maxim Kontsevich and Thomas Willwacher [2].

Non-Symmetric Operad

A non-symmetric operad (sometimes called an operad without permutations, or a non- Σ or plain operad) consists of the following:

- a sequence (P(n))n∈N of sets, whose elements are called *n-ary operations*,
- an element 1 in P(1) called the *identity*
- for all positive integers n, k1,…,kn, a *composition* function

Alternatively, a plain operad is a multicategory with one object.

Symmetric Operad

A Operad Theorya non-symmetric operad P as above, together with a right action of the symmetric group Σn on P(n), satisfying the above associative and identity axioms, as well as

- *equivariance*: given permutations $s_i \in \Sigma_{k_i}, t \in \Sigma_n$,

(where by abuse of notation, t on the right hand side of the first equivariance relation is the element of Σk1+⋯+kn that acts on the set {1,2,…,k1+⋯+kn} by breaking it into n blocks, the first of size k1, the second of size k2, through the nth block of size kn, and then permutes these n blocks by t).

The permutation actions in this definition are vital to most applications, including the original application to loop spaces [3].

Morphisms

A morphism of operads f: P⟶Q consists of a sequence which:

- preserves the identity: f(1)=1
- preserves composition: for every *n*-ary operation θ and operations θ1,…,θn,
- preserves the permutation actions: f(x∗s)=f(x)∗s.

We have so far considered only operads in the category of sets. It is actually possible to define operads in any symmetric monoidal category (or, for non-symmetric operads, any monoidal category).A common example would be given by the category of topological space, with the monoidal product given by the Cartesian product. In this case, a topological operad is given by a sequence of *spaces* (instead of sets) {P(n)}n≥0. The structure maps of the operad (the composition and the actions of the symmetric groups) must then be assumed to be continuous. The result is called a *topological operad*. Similarly, in the definition of a morphism, it would be necessary to assume that the maps involved are continuous. Other common settings to define operads include, for example, module over a ring, chain complexes, groupoids (or even the category of categories itself), coalgebras, etc [4].

Associativity Axiom

Associativity" means that *composition* of operations is associative (the function ∘ is associative), analogous to the axiom in category theory that f∘(g∘h)=(f∘g)∘h; it does *not* mean that the operations *themselves* are associative as operations. Compare with the associative operad, below.

Associativity in operad theory means that one can write expressions involving operations without ambiguity from the omitted compositions, just as associativity for operations allows one to write products without ambiguity from the omitted parentheses. For instance, suppose that θ is a binary operation, which is written as θ(a,b) or (ab). Note that θ may or may not be associative. Then what is commonly written ((ab)c) is unambiguously written operadically as θ∘(θ,1) . This sends (a,b,c) to (ab,c) (apply θ on the first two, and the identity on the third), and then the θ on the left "multiplies" ab by c.

"Little Something" Operads

A little discs operad or, little balls operad or, more specifically, the little *n*-discs operad is a topological operad defined in terms of configurations of disjoint *n*-dimensional discs inside a unit *n*-disc centered in the origin of R^n. The operadic composition for little 2-discs is illustrated in the figure.

Originally the little *n*-cubes operad or the little intervals operad (initially called little *n*-cubes PROPs) was defined by Michael Boardman and Rainer Vogt in a similar way, in terms of configurations of disjoint axis-aligned *n*-dimensional hypercubes (n-dimensional intervals) inside the unit hypercube.Later it was generalized by May to little convex bodies operad, and "little discs" is a case of "folklore" derived from the "little convex bodies" [5].

Conclusion

Another class of examples of operads are those capturing the structures of algebraic structures, such as associative algebras, commutative algebras and Lie algebras. Each of these can be exhibited as a finitely presented operad, in each of these three generated by binary operations.Here is an operad for

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which each P(n) is given by the symmetric group Sn. The composite $\sigma\circ(\tau 1,\ldots,\tau n)$ permutes its inputs in blocks according to σ, and within blocks according to the appropriate τi. Similarly, there is a non-Σ operad for which each P(n) is given by the Artin braid group Bn. Moreover, this non-Σ operad has the structure of a braided operad, which generalizes the notion of an operad from symmetric to braid groups.

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