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Nonlinear Parabolic Equations Involving Measure Data in Musielak-Orlicz-Sobolev Spaces

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Abstract

We prove the existence of solutions of nonlinear parabolic problems with measure data in Musielak-Orlicz-Sobolev spaces.

Keywords: Inhomogeneous Musielak-Orlicz-Sobolev spaces • Parabolic problems • Truncations

Classification: 46E35, 35K15, 35K20, 35K60

Introduction

Let Ω a bounded open subset of Rn and let Q be the cylinder Ω × (0, T) with some given T>0. We consider the following nonlinear parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} + A(u) = \mu in Q \\ u(x,t) = 0 \; and \; \Omega \; x \; (0,T) \\ u(x,0) = 0 \; in \; \Omega \end{cases} \qquad (1)$$

where A=−div (a(x, t, u, ∇u)) is an operator of Leray-Lions defined on D(A) ⊂ $W^{1,x}L_\phi(\Omega)$, ϕ is an appropriate Musielak-Orlicz function related to the growth of a(x, t, u, ∇u), and μ is a given Radon measure. Solution to problem (1) has been provided firstly by Boccardo-Gallouet, in the setting of classical spaces $L^p(0, T; W^{1,p})$. Meskine, in prove the existence of solution to problem (1) in the setting of inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_B$ for any B ∈ P_M, where P_M is a special class of N-functions and M the N-function. Let us point out that our result can be applied in the particular case when ϕ(x, t)=tp(x), in this case we use the notations $L^{p(x)}(\Omega)=L_\phi(\Omega)$ and $W^{m,p(x)}(\Omega)=W^mL_\phi(\Omega)$. These spaces are called Variable exponent Lebesgue and Sobolev spaces. For some classical and recent results on elliptic and parabolic problems in Orlicz-sobolev spaces and a Musielak-Orlicz-Sobolev spaces.

Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later.

Musielak-Orlicz-Sobolev spaces

Let Ω be an open subset of R^n.

A Musielak-Orlicz function ϕ is a real-valued function defined in Ω × R_+ such that:

a): ϕ(x, t) is an N-function i.e. convex, nondecreasing, continuous, ϕ(x, 0)=0, ϕ(x, t)>0 for all t>0 and

$$\underset{t-\to 0x\in}{limsup} \frac{\phi(x,t)}{t} = o$$

$$\underset{t-\to 0x\in}{lim\,int} \frac{\phi(x,t)}{t} = o$$

b): ϕ (., t) is a Lebesgue measurable function [1,2].

Now, let $\phi_x(t)=\phi(x, t)$ and let ϕ^{-1} be the non-negative reciprocal function with respect to t, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\phi_x^{-1}) = t$$

For any two Musielak-Orlicz functions ϕ and γ we introduce the following ordering:

c): if there exists two positives constants c and T such that for almost everywhere x ∈ Ω:

$$\varphi(x,t) \le \gamma(x,ct) \, for \, t \ge T$$

We write ϕ ≺ γ and we say that γ dominates ϕ globally if T=0 and near infinity if T>0.

d): if for every positive constant c and almost everywhere x ∈ Ω we have [3-5].

$$\lim_{t\to 0}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \, or \, \lim_{t\to\infty}(\sup_{x\in\varphi}\frac{\varphi(x,ct)}{\gamma(x,t)}) = 0$$

We write ϕ ≺≺ γ at 0 or near ∞ respectively, and we say that ϕ increases essentially more slowly than γ at 0 or near infinity respectively [6].

In the sequel the measurability of a function u: Ω 1⟶ R means the Lebesgue measurability. We define the functional

$$g_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx$$

Where u: Ω 1⟶ R is a measurable function.

The set

$$K_\varphi(\Omega) = \{u : \Omega \to R \, mesurable / g_{\varphi,}\Omega(u) < +\infty\}$$

is called the Musielak-Orlicz class (the generalized Orlicz class) [7,8].

The Musielak-Orlicz space (the generalized Orlicz spaces) Lϕ(Ω) is the vector space generated by Kϕ(Ω), that is, Lϕ(Ω) is the smallest linear space containing the set Kϕ(Ω). Equivelently:

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$$L_\varphi(\Omega)=\left\{u:\Omega\to R\, mesurable/\, g_\varphi,\Omega\ \ \left(\frac{|u(x)|}{\lambda}\right)<+\infty, for\, some\, \lambda>0\right\}$$

Let

$$\psi(x,s)=\sup_{t\ge 0}\{st-\varphi(x,t)\}$$

ψ is the Musielak-Orlicz function complementary to (or conjugate of) ϕ(x, t) in the sense of Young with respect to the variable S [9].

On the space Lϕ(Ω) we define the Luxemburg norm:

$$\|u\|_{\varphi,\Omega}=\inf\{\lambda>0/\int_\Omega \varphi(x,\frac{|u(x)|}{\lambda})dx,\le 1\}$$

and the so-called Orlicz norm:

$$\|u\|_{\varphi,\Omega}=\sup_{\|u\|_\psi\le 1}\int_\Omega |u(x)v(x)|\,dx$$

where ψ is the Musielak-Orlicz function complementary to ϕ. These two norms are equivalent [10].

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $E_\varphi(\Omega)^*=L_\varphi(\Omega)$.

The following conditions are equivalent:

$$e):E_\varphi(\Omega)=K_\varphi(\Omega)$$

$$f):K_\varphi(\Omega)=L_\varphi(\Omega)$$

$$g):\varphi\, has\, the\, \Delta_2\ property$$

We recall that φ has the Δ_2 property if there exists k>0 independent of x ∈ Ω and a nonnegative function h, integrable in Ω such that φ(x, 2t) ≤ kφ(x, t) + h(x) for large values of t, or for all values of t, according to whether Ω has finite measure or not Let us define the modular convergence: we say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant k>0 such that [11].

$$\lim_{n\to\infty} g_\varphi,\Omega(\frac{u_n-u}{k})=0$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega)=\{u\in L_\varphi(\Omega):\forall|\alpha|\le m\quad D^\alpha u\in L_\varphi(\Omega)\}$$

Where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with nonnegative integers α_i; $|\alpha|=|\alpha_1|+|\alpha_2|+\ldots+|\alpha n|$ and $D^\alpha u$ denote the distributional derivatives.

The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space [11].

Now, the functional

$$g_\varphi,\Omega(u)=\sum_{|\alpha|\le m} g_{\varphi,\Omega}(D^\alpha u)$$

for $u \in W^m L_\varphi(\Omega)$ is a convex modular. And is a norm on $W^m L_\varphi(\Omega)$.

$$\|u\|^m_{\varphi,\Omega}=\inf\{\lambda>0:\bar{g}_{\varphi,\Omega}(\frac{u}{\lambda})\le 1\}$$

The pair $\left\langle W^m L_\varphi(\Omega),\|u\|^m_{\varphi,\Omega}\right\rangle$ is a Banach space if ϕ satisfies the following condition:

There exist a constant c>0 such that $\inf_{x\in\Omega}\varphi(x,1)\ge c$ by Elmahi, A [12].

The space $W^m L_\varphi(\Omega)$ will always be identified to a $\sigma(\Pi L_\varphi,\Pi E_\psi)$ closed subspace of the product $\Pi_{|\alpha|\le m} L_\varphi(\Omega)=\Pi L_\varphi$.

Let $W_\circ^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi,\Pi E_\psi)$ closure of D (Ω) in $W^m L_\varphi(\Omega)$.

Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$ and let $W_0^m E_\varphi(\Omega)$ be the (norm) closure of D(Ω) in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m}L_\psi(\Omega)=\{f\in D^1(\Omega);f=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha f_\alpha with f_\alpha\in L_\psi(\Omega)\}$$

$$W^{-m}E_\psi(\Omega)=\{f\in D^1(\Omega);f=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha f_\alpha with\, f_\alpha\in E_\psi(\Omega)\}$$

As we did for $L_\varphi(\Omega)$, we say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$if there exists a constant k>0 such that

$$\lim_{n\to\infty}\overline{g}_{\varphi\S}\left(\frac{un-u}{\grave{U}\ \ k}\right)=0$$

For two complementary Musielak-Orlicz functions φ and ψ the following inequalities hold:

h): the young inequality:

$$t.s\le\varphi(x,t)+\psi(x,s)\, for\, t,s\ge 0,x\in\Omega$$

i): the holder inequality:

$$\left|\int_\Omega u(x)v(x)dx\right|\le\|u\|_{\phi,\S}\,\||y\||_{\psi,\Omega}$$

For all u ∈ $u\in L_\varphi(\Omega)\, and\, v\in L_\psi(\Omega)$

Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω an bounded open subset of R^n and let Q=Ω × 10, T [with some given T>0. Let φ be a Musielak function. For each α ∈ N^n, denote by Dα the distributional derivative on Q of order α with respect to the variable x ∈ R^n. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows [12,13].

$$W^{1,x}L_\varphi(Q)=\{u\in L_\varphi(Q):\forall|\alpha|\le 1 D_x^\alpha\ u\in L_\varphi(Q)\}$$

And

$$W^{1,x}E_\varphi(Q)=\{u\in E_\varphi(Q):\forall|\alpha|\le 1 D_x^\alpha\ u\in E_\varphi(Q)\}$$

The last space is a subspace of the first one and both are Banach spaces under the norm

$$|u|=\sum_{|\alpha|\le m}\left\|D_x^\alpha u\right\|_{\varphi,Q}$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has (N+1) copies. We shall also consider the weak topologies $\sigma(\Pi L_\varphi,\Pi E_\psi)$ and $\sigma(\Pi L_\varphi,\Pi L_\psi)$. If u ∈ $W^{1,x}L_\varphi(Q)$ then the function : t −→ u(t)=u(t, .) is defined on (0, T) with values in $W^1L_\varphi(\Omega)$. If, further, u ∈ $W^{1,x}L_\varphi(Q)$ then this function is a $W^1L_\varphi(\Omega)$-valued and is strongly measurable. Furthermore the following imbed-ding holds: $W^{1,x}L_\varphi(Q)\subset L^1(0,T;W^1L_\varphi(\Omega))$. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, if u ∈ $W^{1,x}L_\varphi(Q)$, we cannot conclude that the function u(t) is measurable on (0, T). However, the scalar function t 1⟶ /u (t)/ϕ, Ω is in $L_1(0, T)$. The space $W^{1,x}L_\varphi(Q)$ is defined as the (norm) closure in $W^{1,x}L_\varphi(Q)$ of D(Q). We can easily show as in that when Ω a Lipschitz domain then each element u of the closure of D(Q) with respect of the weak topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit, in $W^{1,x}L_\varphi(Q)$, of some subsequence $(u_i)\subset D(Q)$ for the modular convergence; i.e., there exists λ>0 such that for all |α| ≤ 1,

$$\int_Q\varphi\left((x,\left(\frac{D_x^\alpha u_i-D_x^\alpha u}{\lambda}\right)\right)dx\,dt\to 0\, as\, i\to\infty.$$

This implies that (u_i) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology σ(ΠLM, ΠLψ). Consequently

$$\overline{D(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{D(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)}$$

This space will be denoted by Elmahi A and Meskine D [14].

$$W_0^{1,x} L_\psi(Q). Furthermore, W_0^{1,x} E_\varphi(Q) = W_0^{1,x} L_\varphi(Q) \cap \Pi E_\varphi$$

We have the following complementary system

$$\begin{pmatrix} W_0^{1,X} L_\varphi(Q) & F \\ W_0^{1,X} E_\varphi(Q) & F_0 \end{pmatrix}$$

F being the dual space of $W^{1,x}E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W^{1,x} L_\varphi(Q)^\perp$, and will be denoted by $F = W^{-1,x} L_\psi(Q)$ and it is shown that

$$W^{-1,x} L_\psi(Q) = \left\{ f \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_\varphi(Q) = \right\}$$

This space will be equipped with the usual quotient norm [14].

$$\|f\| \sum_{|\alpha|\le 1} \|f_\alpha\|_{\psi,Q}$$

Where the inf is taken on all possible decompositions

$$f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q)$$

The space F_0 is then given by

$$F_0 = \left\{ \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in E_\psi(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$.

In order to deal with the time derivative, we introduce a time mollification of a function $u \in L_\phi(Q)$. Thus we define, for all μ>0 and all (x, t) ∈ Q

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s) exp(\mu(s-t))ds$$

Where $\tilde{u}(x, s) = (x, s)\, X_{(0,T)}(s)$ is the zero extension of u.

Proposition1:

If $u \in L\varphi(Q)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = (\mu - u_\mu)$ and if $u \in L\varphi$

$$\int_Q \varphi(x, u_\mu) dxdt \le \int_Q \varphi(x,u) dxdt$$

Proof: Since $(x, t, s) \mapsto u(x, s) exp(\mu(s - t))$ is measurable in $\Omega \times [0, T] \times [0, T]$, we deduce that u is measurable by Fubini's theorem. By Jensen's integral inequality we have, since

$$\int_{-\infty}^{0} \exp(\mu s) ds = 1$$

$$\varphi\left(x, \int_{-\infty}^{t} \mu \tilde{u}(x,s) exp(\mu(s-t))ds\right) = \varphi\left(x, \int_{-\infty}^{0} \mu exp(\mu s) \tilde{u}(x, s+t) ds\right)$$

$$\le \int_{-\infty}^{0} \mu exp(\mu s) \varphi\left(x, \tilde{u}(x, s+t)\right) ds$$

Which implies

$$\int_Q \varphi\left(x, u_\mu(x,t)\right) dxdt \le \int_{\Omega\times R} \left(\int_{-\infty}^{0} \mu exp(\mu s) \varphi\left(x, \tilde{u}(x, s+t) ds\right) \right) dxdt$$

$$\le \int_{-\infty}^{0} \mu exp(\mu s) \left(\int_{\Omega\times R} \varphi\left(x, \tilde{u}(x, s+t)\right) dxdt \right) ds \le \int_{-\infty}^{0} \mu exp(\mu s) \left(\int_Q \varphi(x, u(x,t)) dxdt \right) ds$$

$$= \int_Q \varphi(x,u) dxdt$$

Furthermore

$$\frac{\partial u_\mu}{\partial t} = lim_{\delta\to 0} \frac{1}{\delta}(exp(-\mu\delta) - 1) u_\mu(x,t) + lim_{\delta\to 0} \frac{1}{\delta} \int_t^{t+\delta} u(x,s) exp(\mu(s-(t+\delta)) ds = -\mu u_\mu + \mu u$$

Proposition 2. Assume that $(u_n)_n$ is a bounded sequence in $W_0^{1,x} L\varphi(Q)$ *Such that* $\frac{\partial u_\mu}{\partial t}$ *is* bounded in $W^{-1,x} L_\psi(Q) + L^1(Q)$, then u_n relatively compact $L^1(Q)$.

Proof. It is easily by using Corollary 1 of 2.

Results

Let $P\varphi$ be a subset of Musielak-Orlicz functions defined by:

$$P_\varphi = \left\{ \phi : \Omega \times R_+ \to R_+ \text{ is an Musielak} - \text{Orlicz function, such that } \phi \ll \varphi \text{ and } \int_0^1 \phi \circ H^{-1}\left(x, 1/t^{1-1/n}\right) dt < \infty \text{ for a.e } x \in \Omega \right\}$$

Where $(x, r) = \varphi(x, r)/r$

We assume that

$P_\varphi \ne \emptyset$ (2)

$$Let\ A : D(A) \subset W_0^{1,x} L\varphi(Q) \to W^{1,x} L\psi(Q)\ be\ a\ mapping\ given\ by$$

A (u)=−div a(x, t, u,∇u) where $a: Q \times R^n \times R^n$ be Caratheodory function satisfying for a.e $(x, t) \in \Omega$ and all $s \in R$, $\xi, \eta \in R^n$ with: $\xi \ne \eta$

$$|a(x,t,s,\xi)| \le \beta\varphi(x,|\xi|)/|\xi| \ (3)$$

$$(a(x,t,s,\xi) - a(x,s,\eta))(\xi - \eta) > 0 (4)$$

$$a(x,t,s,\xi)\xi \ge \alpha\varphi(x,|\xi|)(5)$$

Where α, β>0. Furthermore, assume that there exists $D \in P_\varphi$ such that

$D \circ H^{-1}$ is a Musielak-Orlicz Function. (6)

Set Tk(s)=(−k, min(k, s)), ∀s ∈ R, for all k ≥ 0

Denote by M_b the set of all bounded Radon measure defined on Q and by $T_0^{1,\varphi}(Q)$ as the set of measurable functions

$Q \to R$ *such that* $T_k(u) \in W_0^{1,x} L\varphi(Q) \cap D(A)$ assume that $f \in M_b(\Omega)$ and consider the following nonlinear parabolic problem with Dirichlet boundary

$$\frac{\partial u}{\partial t} + A(u) = f \ in\ Q \quad (7)$$

Theorem 1. Assume that (2)-(6) hold and $f \in M_b(Q)$. Then there exists at least one weak solution of the problem

$$\{u \in T_0^{1,\varphi}(Q) \cap W_0^{1,x} L_\phi(Q), \forall \phi \in P_\varphi \ - \int_Q u \frac{\partial v}{\partial t} + \int_\Omega a(x,t,u,\nabla u) \nabla v dx = \langle f, v \rangle, \forall v \in D(Q)$$

Proof: The proof will be given in two steps.

Step 1: A priori estimates.

Consider now the following approximate equations:

$$\{u_n \in W_0^{1,x} L_\varphi(Q), u_n(x,0) = 0;\ \frac{\partial u_n}{\partial t} - diva\left(x,t,u_n,\nabla u_n\right) = f_n \ (8)$$

Where f_n is a smooth function which converges to f in the distributional sense and $\|f_n\|_{L^1(Q)} \le \|f\|_{M_b(Q)}$ By Theorem 2 of 3, there exists atleast one solution of U_n of (8), For k>0, by taking $T_k(u_n)$ as test function in (8), one has

$$\int_\Omega a\left(x,t,T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \le Ck$$

In view of (5), we get

$$\int_\Omega \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right)dx \le Ck$$

Take a C^2 (R), and no decreasing function β_k such that for and 2

$$\beta_k(S) = S \mid S \mid \le \frac{k}{2}\, and\ \beta_k(S) = k\ sign\ if \mid k \mid \ge S$$

$$\frac{\partial\beta_k(u_n)}{\partial t} - div\left(a\left(x,t,u_n,\nabla u_n\right)\beta_k{}'\left(u_n\right)\right) + a\left(x,t,u_n,\nabla u_n\right)\nabla u_n \beta_k{}''\left(u_n\right) = f_n\beta_k{}'\left(u_n\right)\ in\ D'(Q)$$

Which implies easily that $\frac{\partial\beta_k(u_n)}{\partial t}$ is bounded in $W^{-1,x}L\psi\ (Q) + L^1(Q)$. Thanks to Proposition 2, we deduce that $\beta(u_n)$ is compact in $L^1(Q)$.

Then as in (20) and by the proof of Theorem 3 of 1, we deduce that there exists [15].

$u \in L^\infty(0, T; L^1(\Omega))$ such that: $u_n \longrightarrow u$ almost everywhere in Q and (almost everywhere in and

$$T_k\left(u_n\right) \rightharpoonup T_u\ weakly\ in\ W_0^{1,x}L_\varphi(Q)\ for\ \sigma\left(\Pi L_\varphi, \Pi E_\psi\right) \quad (9)$$

Now, let $\phi \in P\varphi$. By a slight adaptation of the context of Lemma 2.1. of 4, it follows that

$$\int_Q \phi\left(x, \left|\nabla\left(u_n\right)\right|\right)dx \le C, \forall n \quad (10)$$

We shall show that $a\left(x,t,\left(u_n\right),\nabla\left(u_n\right)\right)\nabla\left(u_n\right)$ is bounded in $\left(L_\psi(Q)\right)^n$

Let $\omega \in (E\varphi\ (Q))^{\|}\ \omega\|_{(\varphi)}$ By (5) and Young inequality, one has

$$\int_Q a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\omega dx$$

$$\le \beta\int_Q \psi\left(x, \frac{\varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right)}{\left|\nabla T_k\left(u_n\right)\right|}\right)dx + \beta\int_Q \varphi(x,|\omega|)dx$$

$$\le \beta\int_Q \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right)dx + \beta$$

This last inequality is deduced from the fact that $\psi(x, \varphi(x, u)/u) \le \varphi(x, u)$, for all u>0 and

$\int_Q \varphi(x,|\omega)dx \le 1.\ in\ view\ of\ (10)$ In view of (10), [15,16].

$$\int_\Omega a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\omega dx \le Ck + \beta$$

Which implies that $(a(x, t, T_k(u_n), \nabla T_k(u_n))_n$ is a bounded sequence in $(L_\psi(Q))n$.

Step 2. Almost everywhere convergence of the gradient and passage to the limit. Since $T_k(u) \in W^{1,x}L_\varphi(Q)$, then there exists a sequence $(\alpha_j^k) \subset D(Q)$ such that $(\alpha_j^k) \longrightarrow T_k(u)$ for the modular convergence in $W^{1,x}L_\varphi(Q)$. For the remaining of this article, χ_s and $\chi_{j,s}$ will denoted respectively the characteristic functions of the sets

$$Q_s = \{(x,t) \in Q; \left|\nabla T_k(u(x,t))\right| \le s\}\ \text{and}\ Q_{j,s} = \{(x,t) \in Q; \left|\nabla T_k\left(v_j(x,t)\right)\right| \le s\}$$

For the sake of simplicity, we will write only ε (n, j, μ, s) to mean all quantities (possibly different) such that

$$\lim_{n\to\infty}\lim_{j\to\infty}\lim_{\mu\to\infty}\lim_{s\to\infty}\varepsilon(n,j,\mu,s) = 0$$

For every μ>0, we define

$$w_\mu(x,t) = \mu\int_{-\infty}^{t} exp(\mu(s-t))w(x,t)\chi_{[0,T]}(s)ds$$

the time regularized of any function $w \in W_0^{1,x}L_\varphi(Q)$

Taking now $T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)$ as test function in (8), we obtain

$$\langle\frac{\partial u_n}{\partial t}, T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\rangle + \int_Q a\left(x,t,u_n,\nabla\left(u_n\right)\right)\nabla T_\eta\left(u_n - T_k\left(v_j\right)\right)dx \le C\eta$$

The first term of the left hand side of the last equality reads as

$$\langle\frac{\partial u_n}{\partial t}, T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\rangle = \langle\frac{\partial u_n}{\partial t} - \frac{\partial T_k\left(\alpha_j^k\right)_\mu}{\partial t}, T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\rangle + \langle\frac{\partial T_k\left(\alpha_j^k\right)_\mu}{\partial t}, T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\rangle$$

The second term of the last equality can be easily to see that is positive and the third term can be written as

$$\langle\frac{\partial T_k\left(\alpha_j^k\right)_\mu}{\partial t}, T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\rangle = \mu\int_Q\left(T_k\left(\alpha_j^k\right) - T_k\left(\alpha_j^k\right)_\mu\right)\left(T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\right)dxdt$$

thus by letting $n, j \to \infty$ and since $\left(\alpha_j^k\right) \to T_k(u)$ a.e. in Q and by using Lebesgue Theorem

$$\int_Q\left(T_k\left(\alpha_j^k\right) - T_k\left(\alpha_j^k\right)_\mu\right)\left(T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\right)dxdt = \int_Q\left(T_k(u) - T_k(u)_\mu\right)\left(T_\eta\left(u - T_k(u)_\mu\right)\right)dxdt + \varepsilon(n,j)$$

Consequently

$$\langle\frac{\partial u_n}{\partial t}, T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)\rangle \ge \varepsilon(n,j)$$

On the other hand,

$$\int_Q a\left(x,t,u_n,\nabla\left(u_n\right)\right).\nabla T_\eta\left(u_n - T_k\left(\alpha_j^k\right)_\mu\right)dxdt$$

$$= \int_{\{\left|T_k(u_n) - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right).\left(\nabla T_k\left(u_n\right) - \nabla T_k\left(\alpha_j^k\right)_\mu\chi_{j,s}\right)dxdt$$

$$+ \int_{\{k<|u_n|\}\cap\{\left|u_n - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} a\left(x,t,u_n,\nabla u_n\right)\ \nabla u_n\,dxdt$$

$$- \int_{\{k<|u_n|\}\cap\{\left|u_n - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} a\left(x,t,u_n,\nabla u_n\right).\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\{\left|\nabla T_k\left(\alpha_j^k\right)\right| > s\}}dxdt$$

which implies, by using the fact that [17].

$$\int_{\{k<|u_n|\}\cap\{\left|u_n - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} a\left(x,t,u_n,\nabla u_n\right).\nabla u_n\,dxdt \ge 0$$

$$\int_{\{\left|T_k(u_n) - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right).\left(\nabla T_k\left(u_n\right) - \nabla T_k\left(\alpha_j^k\right)_\mu\chi_{j,s}\right)dxdt \le C\eta$$

$$+ \int_{\{k<|u_n|\}\cap\{\left|u_n - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} a\left(x,t,u_n,\nabla u_n\right).\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\{\left|\nabla T_k\left(\alpha_j^k\right)\right| > s\}}dxdt$$

Since $a\left(x,t,T_{k+\eta}\left(u_n,\nabla T_{k+\eta}\left(u_n\right)\right)\right)$ is bounded in $\left(L_\psi(\Omega)\right)^n$ there exists some $h_{k+\eta} \in \left(L_\psi(\Omega)\right)^n$ such that

$$a\left(x,t,T_{k+\eta},\nabla T_{k+\eta}\left(u_n\right)\right) \rightharpoonup h_{k+\eta}$$

weakly in $\left(L_\psi(\Omega)\right)^n$ for $\sigma\left(\Pi L_\psi, \Pi E_\varphi\right)$

Consequently,

$$\int_{\{k<|u_n|\}\cap\{\left|u_n - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} a\left(x,t,u_n,\nabla u_n\right).\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\{\left|\nabla T_k\left(\alpha_j^k\right)\right| > s\}}dxdt =$$

$$= \int_{\{k<|u_n|\}\cap\{\left|u_n - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}} h_{k+\eta}\nabla T_k\left(\alpha_j^k\right)_\mu\chi_{\{\left|\nabla T_k\left(\alpha_j^k\right)\right| > s\}}dxdt + \varepsilon(n)$$

where we have used the fact that $\nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\{k<|u_n|\}\cap\{|u_n - T_k(\alpha_j^k)_\mu|<\eta\}}$ tends strongly to $\nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\{k<|u|\}\cap\{|u - T_k(\alpha_j^k)_\mu|<\eta\}}$ in $\left(E_\varphi(Q)\right)^n$. Letting $j \to \infty$ we obtain

$$\int_{\{k<|u_n|\}\cap\{|u_n - T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,u_n,\nabla u_n\right).\nabla T_k\left(\alpha_j^k\right)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dxdt$$

$$= \int_{\{k<|u|\}\cap\{|u - T_k(\alpha_j^k)_\mu|<\eta\}} h_{k+\eta} \nabla T_k(u)_\mu \chi_{\{|\nabla T_k(u)|>s\}} dxdt + \varepsilon(n,j)$$

Thanks to Proposition 1, one easily has $\int_{\{k<|u|\}\cap\{|u-T_k(u)_\mu|<\eta\}} h_{k+\eta}\nabla T_k(u)_\mu\chi_{\{|\nabla T_k(u)|>s\}}dxdt = \int_{\{k<|u|\}\cap\{|u-T_k(u)_\mu|<\eta\}} h_{k+\eta}\nabla T_k(u)\chi_{\{|\nabla T_k(u)|>s\}}dxdt + \varepsilon(\mu)$

Hence

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right).\left(\nabla T_k(u_n) - \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{j,s}\right) dxdt + C\eta + \varepsilon(n,j,\mu,s)$$

On the other hand, remark that

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right).\left(\nabla T_k(u_n) - \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{j,s}\right) dxdt$$

$$= \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right).\left(\nabla T_k(u_n) - \nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right) dxdt$$

$$+ \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right).\left(\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{j,s}\right) dxdt$$

The latest integral tends to 0 as n and j go to ∞. Indeed, we have [18].

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right).\left(\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{j,s}\right) dxdt$$

tends to

$$\int_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} h_k\left[\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{j,s}\right] dxdt$$

as $n \to \infty$, since $a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right) \rightharpoonup h_k$ weakly in $\left(L_\psi(\Omega)\right)^n$ for $\sigma(\Pi L_\psi, \Pi E_\varphi)$ while $\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{j,s} \in \left(E_\psi(Q)\right)^n$. It's obvious that

$$\int_{\{|T_k(u)-T_k(\alpha_j^k)_\mu|<\eta\}} h_k\left[\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k\left(\alpha_j^k\right)_\mu \chi_{j,s}\right] dxdt$$

goes to 0 as j, μ ⟶ ∞ by using Lebesgue theorem [18,19]. We deduce then $\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right).\left(\nabla T_k(u_n) - \nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)dxdt + C\eta + \varepsilon(n,j,\mu,s)$.

Let 0 < δ < 1. We have

$$\int_{Q_r} \left[\left[a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n),\nabla T_k(u)\right)\right] \times \left[\nabla T_k(u_n) - \nabla T_k(u)\right]\right]^\delta dxdt \le C\, meas\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}^\delta$$

$$+ C\left[\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}\cap Q_r} \left[a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n),\nabla T_k(u)\right)\right] \times \left[\nabla T_k(u_n) - \nabla T_k(u)\right]dxdt\right]^\delta \quad (11)$$

On the other hand, we have for every s ≥ r

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}\cap Q_r} \left[a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n),\nabla T_k(u)\right)\right] \times \left[\nabla T_k(u_n) - \nabla T_k(u)\right]dxdt$$

$$\le \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} \left[a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n),\nabla T_k(u)\chi_s\right)\right] \times \left[\nabla T_k(u_n) - \nabla T_k(u)\chi_s\right]dxdt$$

$$\le \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} \left[a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\right] \times \left[\nabla T_k(u_n) - \nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt$$

$$+ \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,T_k(u_n),\nabla T_k(u_n)\right)\left[\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k(u)\chi_s\right]dxdt$$

$$+ \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} \left[a\left(x,t,T_k(u_n),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right) - a\left(x,t,T_k(u_n),\nabla T_k(u)\chi_s\right)\right]\nabla T_k(u_n)dxdt$$

$$- \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}dxdt$$

$$+ \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k(u)\chi_s\right)\nabla T_k(u)\chi_s dxdt$$

$$\le I_1(n,j,\mu,s) + I_2(n,j,\mu,s) + I_3(n,j,\mu,s) + I_4(n,j,\mu,s) + I_5(n,j,\mu,s) \quad (12)$$

We shall go to limit as n, j, μ and s ⟶ ∞ in the last fifth integrals of the last side. Starting with I_1, we have [19].

$$I_1(n,j,\mu,s) \le C\eta + \varepsilon(n,j,\mu,s) - \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}dxdt$$

since

$$a\left(x,t,T_k(u_n),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\chi_{\{|T_k(u_n)-(\alpha_j^k)_\mu|<\eta\}} \to a\left(x,t,T_k(u),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\chi_{\{|T_k(u)-(\alpha_j^k)_\mu|<\eta\}}\ in\ \left(E_\psi(Q)\right)^n$$

while

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)\ weakly$$

We deduce then that

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u_n),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\left[\nabla T_k(u_n) - \nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt$$

$$= \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\left[\nabla T_k(u) - \nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt + \varepsilon(n)$$

which gives by letting $j \to \infty$ and using the modular convergence of $\nabla T_k\left(\alpha_j^k\right)$

$$\int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} a\left(x,t,T_k(u),\nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right)\left[\nabla T_k(u) - \nabla T_k\left(\alpha_j^k\right)\chi_{j,s}\right]dxdt + \varepsilon(n)$$

$$= \int_Q a\left(x,t,T_k(u),\nabla T_k(u)\chi_s\right)\left[\nabla T_k(u) - \nabla T_k(u)\chi_s\right]dxdt + \varepsilon(j) = \varepsilon(j)$$

Finally

$$I_1(n,j,\mu,s) \le C\eta + \varepsilon(n,j,\mu,s) + \varepsilon(n,j) = \varepsilon(n,j,\mu,s,\eta) \quad (13)$$

For what concerns I_2, by letting $n \to \infty$, we have

$$I_2(n,j,\mu,s) = \int_{\{|T_k(u_n)-T_k(\alpha_j^k)_\mu|<\eta\}} h_k\left[\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k(u)\chi_s\right]dxdt + \varepsilon(n)$$

since

$$a\left(x,t,T_k(u_n),\nabla T_k(u_n)\chi_{j,s}\right)\chi_{\{|T_k(u_n)-(\alpha_j^k)_\mu|<\eta\}} \rightharpoonup h_k for\ \sigma\left(\Pi L\psi, E_\varphi\right)$$

while

$$\chi_{\{|T_k(u_n)-(\alpha_j^k)_\mu|<\eta\}}\left[\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k(u)\chi_s\right] \to$$

$$\chi_{\{|T_k(u)-(\alpha_j^k)_\mu|<\eta\}}\left[\nabla T_k\left(\alpha_j^k\right)\chi_{j,s} - \nabla T_k(u)\chi_s\right] strongly\ in\ \left(E_\varphi(Q)\right)^n$$

By letting now $j \to \infty$, and using Lebesgue theorem, we deduce then that

$$I_2(n, j, \mu, s) = \varepsilon(n, j) \quad (14)$$

Similar tools as above, give

$$I_3(n, j, \mu, s) = \varepsilon(n, j)$$

$$I_4(n, j, \mu, s) = -\int_Q a\left(x, t, T_k(u), \nabla T_k(u)\right)\nabla T_k(u) + \varepsilon(n, j, \mu, s)$$

$$I_5(n, j, \mu, s) = \int_Q a\left(x, t, T_k(u), \nabla T_k(u)\right)\nabla T_k(u) + \varepsilon(n, j, \mu, s)$$

Combining (11),(12),(13),(14) and (15) we have

$$\int_{\Omega_r} \left[\left(a\left(x, t, T_k(u_n), \nabla T_k(u_n)\right) - a\left(x, t, T_k(u_n), \nabla T_k(u)\right)\right) \times \left(\nabla T_k(u_n) - \nabla T_k(u)\right)\right]^{\delta} dxdt$$

$$\leq C\left(meas\{\left|T_k(u_n) - T_k\left(\alpha_j^k\right)_\mu\right| < \eta\}\right)^{\delta} + C(\varepsilon(n, j, \mu, s, \eta))^{\delta}$$

and by passing to the limit sup over n, j, μ, s and η

$$\lim_{n\to\infty} \int_{Q_r} \left[\left[a\left(x, t, T_k(u_n), \nabla T_k(u_n)\right) - a\left(x, t, T_k(u_n), \nabla T_k(u)\right)\right] \times \left[\nabla T_k(u_n) - \nabla T_k(u)\right]\right)\right]^{\delta} dxdt = 0$$

and thus, there exists subsequence also denote by (u_n) such that

$$\nabla u_n \to \nabla u a.e.\ in\ Q_r$$

and since r is arbitrary, we have

$$\nabla u_n \to \nabla u\ a.e\ in\ Q$$

On the other hand, thanks to (3), (6) and (10), we deduce that

$$\int_Q D \circ H^{-1}\left(s, \frac{\left|a\left(x,t,u_n,\nabla u_n\right)\right|}{\beta}\right) dxdt \leq \int_\Omega D\left(x, \left|\nabla u_n\right|\right) dxdt \leq C$$

Hence

$$a\left(x, t, u_n, \nabla u_n\right) \rightharpoonup a(x, t, u, \nabla u)$$

$$\text{weakly for } \sigma\left(\Pi L_{D\circ H^{-1}}, \Pi E_{\overline{D\circ H^{-1}}}\right)$$

Going back to approximate equations 8 and using v ∈ D(Q) as the test function, one has

$$-\int_Q u_n \frac{\partial v}{\partial t} dxdt + \int_Q a\left(x, t, u_n, \nabla u_n\right)\nabla v dxdt = < f_n, v >$$

In which we can pass to the limit since we have [20,21].

$$u_n \to u\ strongly\ in\ \left(E_\gamma(Q)\right)^n\ for\ every\ \gamma \ll \phi \in P_\varphi$$

This completes the proof of Theorem 1.

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Conflict of Interest

None.

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