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Universality of Affine Semigroups on Supercyclicity of the Sequence Operators

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Abstract

In this paper, show that for all supercyclic strongly continuous sequence of operators semigroup acting on a complex F^j-space, every $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic. Furthermore, the set of supercyclic vectors of all $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is precisely the set of supercyclic vectors of the plenary semigroup.

Keywords: Hypercyclic semigroups • Hypercyclic operators • Supercyclic operators • Supercyclic semigroups

Introduction

Unless stated otherwise, all vector spaces in this article are over the field k, being either the field C of complex numbers or the field R of real numbers and all topological spaces are assumed to be Hausdorff. As usual, Z_+ is the set of non-negative integers, N is the set of positive integers and R_+ is the set of non-negative real numbers. The symbol L(X) stands for the space of a continuous linear sequence of operators on a topological vector space X, while **X** in the space of continuous linear functionals on **X**. As usual, for $T^j \in L(X)$ the dual sequence of operators $'T^j : X' \rightarrow X'$ is defined by the formula $\sum_j (T^j) f_j(x) = \sum_j f_j(T^j x)$ for x∈X and f_j∈X'. Recall that an affine map on a T^j vector space X is a map of the shapes $T^j x = u^j + S^j x$, where u^j is fixed vector in X and $S^j : X \rightarrow X$ is linear. Clearly, T^j are continuous if and only if S^j are continuous. The symbol $A^j(X)$ stands for the space of continuous affine maps on a topological vector space. F^j -space is a complete metrizable topological vector space. Recall that a family $\sum_j F^j = \sum_j \{T^j_a\}_{a \in A_j}$ of continuous maps from a topological space X to a topological space Y is called universal if there is x∈X for which { T^j_a x: a∈A_j} is dense in Y and such an x is called a universal element for F^j. Use the symbol $u(F^j)$ for the set of universal elements for F^j. If X is a topological space and $T^j : X \rightarrow X$ is a continuous map, then say that x∈X is universal for T^j if x is universal for the family { $(T^j)^n$: n∈Z_+ }Denote the sets of universal elements for T^j by $u\sum_j (T^j)$. Series families $\sum_j F^j = \sum_j \{T^j{}_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ of continuous maps from a topological space X to itself are called semigroups if $(T^j)_0$ = I and $T^j_{2(\varepsilon+1)} = T^j{}_{(1+\varepsilon)} T^j{}_{(1+\varepsilon)}$ for every (1+ε)∈R_+. Say that a semigroup Σ_j\{ $T^j_{(1+\varepsilon)}$ \}$_{(1+\varepsilon)\in R_+}$is strongly continuous if $(1+\varepsilon) \rightarrow T^j_{(1+\varepsilon)}x$ are continuous as a map from R_+ to X for every x∈X and say that $\sum_j \{T^j{}_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is jointly continuous if $((1+\varepsilon), x) \mapsto \mathrm{T}^j_{(1+\varepsilon)} x$ is continuous as a map from $R_+ \times$ X to X. If X is a topological vector space, semigroup $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ a linear semigroup if $\sum_j T^j_{(1+\varepsilon)} \in L(X)$ for every (1+ε)∈R_+ and $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is called an affine semigroup if $\sum_j T^j_{(1+\varepsilon)} \in A_j(X)$ for every (1+ε)∈R_+. Recall that $T^j \in L(X)$ is called hypercyclic if $u\sum_j (T^j) \neq \varnothing$ and elements of $u\sum_j (T^j)$ are called hypercyclic vectors. Universal linear semigroups $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are called hypercyclic and its universal elements are called hypercyclic vectors for $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$. If $\sum_j T^j_{(1+\varepsilon)} \in L(X)$, then universal elements of the family $\{z\sum_j (T^j)^n x : z \in \mathbb{k}, n \in \mathbb{Z}_+\}$ are called supercyclic vectors for T^j and T^j are called supercyclic if it has a supercyclic vector. Similarly, if $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ they are linear semigroup, then a universal element of the families $\{z\sum_j T^j_{(1+\varepsilon)} x : z \in \mathbb{k}, (1+\varepsilon) \in \mathbb{Z}_+\}$ are called supercyclic vector for $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ and the semigroup is called supercyclic if it has a supercyclic vector.

Hypercyclicity and supercyclicity have been studied during the last decades [1]. The concern is the relation between the supercyclicity of a linear semigroup and the supercyclicity of the individual members of the semigroup. The hypercyclicity version of the question was treated by Conejero, Müller, and Peris [2], who proved that for every strongly continuous hypercyclic linear semigroups $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on an F^j-space, all $T^j_{(1+\varepsilon)}$ with ε>-1 is hypercyclic and $u\sum_j \left(T^j_{(1+\varepsilon)}\right) = u\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$. Virtually the same proof works in the following much more general setting.

Theorem A

Let $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a hypercyclic jointly continuous linear semigroup on all topological vector space *X*. Then all $T^j_{(1+\varepsilon)}$ with ε>-1 is hypercyclic and $u\sum_j (T^j_{(1+\varepsilon)}) = u\sum_j (\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+})$.

The stronger condition of joint continuity coincides with the strong continuity in the case when *X* is an F^j -space due to a straightforward application of the Banach–Steinhaus theorem. It is based on a homotopy-type argument and goes through without any changes for semigroups of non-linear maps. Recall that a topological space X is called connected if it has no subsets different from ∅ and X, which are closed and open and it is called simply connected if for any continuous map $f_j : T \rightarrow X$, there is a continuous map $F^j : T\times[0,1] \rightarrow X$ and x_0∈Xsuch that $F^j(z,0)=f_j(z)$ and $F^j(z,1)=x_0$for any z∈T. Next, X is called locally path connected at χ∈X if for any neighborhood U of χ, there is a neighborhood V of χ such that for any y∈V, there is a continuous map f_j :[0,1]→X satisfying $f_j(0)=x$, $f_j(1)=y$, and $f_j([0,1]) \subseteq U$. Space X is called locally path-connected if it is locally path connected at every point. Just listing the conditions to run smoothly, get the following result.

Proposition 1.1

Let *X* be a topological space and $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a jointly continuous semigroup on *X* such that

(1) $\{\sum_j T^j_{(1+\varepsilon)}(u^j) : (1+\varepsilon) \in [0,(1+\varepsilon)]\}$ are nowhere dense in *X* for every ε>-1 and u^j∈X ;

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(2) for every ε>-1 and $x \in u\sum_j(\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+})$, there is $Y_{(1+\varepsilon),x} \subseteq X$ such that $Y_{(1+\varepsilon),x}$ is connected, locally path-connected, simply connected and

$$\{\sum_j T^j_{(1+\varepsilon)}x : (1+\varepsilon) \in [0,(1+\varepsilon)]\} \subseteq Y_{(1+\varepsilon),x} \subseteq u\sum_j(\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}).$$

Then $u\sum_j(T^j_{(1+\varepsilon)}) = u\sum_j(\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+})$ for every ε>-1.

The natural question of whether the supercyclicity version of Theorem A holds in [3]. They have produced the following example.

Example B: Let X be a Banach space over R, $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a hypercyclic linear semigroup on X and $(A_j)_{(1+\varepsilon)} \in L(\mathbb{R}^2)$ for (1+)∈R_+ be the linear sequence of operators with the matrices $A_j = \begin{pmatrix} \cos(1+\varepsilon) & \sin(1+\varepsilon) \\ -\sin(1+\varepsilon) & \cos(1+\varepsilon) \end{pmatrix}$

Then $\sum_j\{(A_j)_{(1+\varepsilon)} \oplus T^j_{(1+\varepsilon)}\}_{(\varepsilon+1)\in\mathbb{R}_+}$ is supercyclic linear semigroup on $\mathbb{R}^2 \times X$, while $\sum_j\left((A_j)\right)_{(1+\varepsilon)} \oplus T^j_{(1+\varepsilon)})$ are non-supercyclics whenever $\frac{(1+\varepsilon)}{\pi}$ is rational.

Example B shows that the natural supercyclicity version of Theorem A fails in the case k=R.

Proposition 1.2

Let X be a complex topological vector space and $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semigroup on X such that $T^j_{(1+\varepsilon)} - \lambda_j I$ has a dense range for every ε>-1 and every λ_j∈C. Then each $T^j_{(1+\varepsilon)}$ with ε>-1 is supercyclic. Furthermore, the set of supercyclic vectors for $T^j_{(1+\varepsilon)}$ does not depend on the choice of ε>-1 and simultaneity with the set of supercyclic vectors of the plenary semigroup.

Whatever one can obtain the same result directly by considering the induced action on subsets of the projective space and applying Proposition 1.1. show that in the case k=C, the supercyclicity version of Theorem A holds without any additional assumptions.

Theorem 1.2

Let X be a complex topological vector space and $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semigroup on X. Then all $T^j_{(1+\varepsilon)}$ with ε>-1 is supercyclic and the set of supercyclic vectors of $T^j_{(1+\varepsilon)}$ coincides with the set of supercyclic vectors of $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$.

It turns out that any supercyclic jointly continuous linear semigroup on a complex topological vector X either satisfies conditions of Proposition Cor has a closed invariant hyperplane Y. Reduces the following generalization of Theorem A to affine semigroups [4].

Theorem 1.3

Let X be a topological vector space and $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal jointly continuous affine semigroup on X. Then all $T^j_{(1+\varepsilon)}$ with ε>-1 is universal and $u\sum_j(T^j_{(1+\varepsilon)}) = u\sum_j\left(\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$.

A Dichotomy for Supercyclic Linear Semigroups

An analogue of the following result for supercyclic sequence of operators.

Proposition 2.1

Let X be a complex topological vector space and $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic strongly continuous linear semigroup on X. Then either $\sum_j(T^j_{(1+\varepsilon)} - \lambda_j I)(X)$ is dense in X for every ε>-1 and λ_j∈C or there is a closed hyperplane Hin X such that $\sum_j T^j_{(1+\varepsilon)}(H) \subseteq H$ for every (1+ε)∈R_+.

This section is devoted to the proof of Proposition 2.1. Recall that subsets B_j of vector space X are called balanced if $\lambda_j x \in B_j$ for every x∈B_j and λ_j∈k such that $|\lambda_j| \le 1$.

Proof: Assume that there is ε>-1 and λ_j∈k such that $\sum_j(T^j_{(1+\varepsilon)} - \lambda_j I)(X)$ are not dense in X. By Lemma 2.5, $H = \left(\sum_j(T^j_{(1+\varepsilon)_0} - \lambda_j I)(X)\right)$ are closed hyperplanes in X.

Lemma 2.2

Let K be a compact subset of an infinite dimensional topological vector space and X such that 0∉K. Then $\Lambda = \{\lambda_j x : \lambda_j \in \mathbb{k}, x \in K\}$ is a closed nowhere dense subset of X.

Proof: Closeness of Λ in *X* is a straightforward exercise. Assume that Λ is not anywhere dense. Since Λ is closed, it's interior Lis non-empty. Since Kis closed and $0 \notin K$, find a non-empty balanced open set U such that $U \cap K = \varnothing$. Obviously $\lambda_j x \in L$ whensoever $x \in L$ and $\lambda_j \in \mathbb{k}$, $\lambda_j \neq 0$. Since U is open and balanced property of Limplies that the open set $W^j = L \cap U$ is non-empty. Taking into account the definition of Λ, inclusion $L \subseteq \Lambda$, equality $U \cap K = \varnothing$ and the fact that Uis balanced, every x ∈ W^j be written as x=λ_jy, where y∈k and $\lambda_j \in \mathbb{D} = \{z \in \mathbb{K} : |z| \le 1\}$. Since both Kand $\mathbb{D}$ are compact, $Q = \{\lambda_j y : \lambda_j \in \mathbb{D}, y \in K\}$ is a compact subset of X. Since $X \subseteq Q$, W^j is a non-empty open set with compact closure. Such a set exists [5] only if X is finite dimensional. This contradiction completes the proof.

The following lemma is a particular case in [6].

Lemma 2.3

Let X be a complex topological vector space such that $2 \le \dim X < \infty$. Then X supports no supercyclic strongly continuous linear semigroups.

Lemma 2.4

Let *X* be an infinite dimensional topological vector space λ_j∈k, $(1+\varepsilon)_0$>0 and $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a strongly continuous linear semigroup such that $T^j_{(1+\varepsilon)_0} - \lambda_j I$. Then $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are not supercyclics.

Proof: Let $x \in X \setminus \{0\}$. It success to show that x is not a supercyclic vector for $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$.

First, consider the case λ_j=0, it is ε>-1 such that $0 \notin K = \{\sum_j T^j_{(1+\varepsilon)}x : (1+\varepsilon) \in [0,(\varepsilon+1)]\}$ and K is a compact subset of X. By Lemma 2.2, $\{z\sum_j T^j_{(1+\varepsilon)}x : z \in \mathbb{k},\ (\varepsilon+1) \in [0,(\varepsilon+1)\}$ is nowhere dense in X. Take n∈N such that $n(1+\varepsilon) \ge (1+\varepsilon)_0$. Since $T^j_{(1+\varepsilon)_0} = 0$ and n(1+ε)≥$(1+\varepsilon)_0$, have $\sum_j (T^j)^n_{(1+\varepsilon)} = \sum_j T^j_{(1+\varepsilon)_n} = 0$. Then $Y = \sum_j\left(T^j_{(1+\varepsilon)}(X)\right) \neq X$. Notably, Y is nowhere dense in X. Obviously, $\sum_j T^j_{(1+\varepsilon)}x \in Y$ whenever ε>-1. Hence $\{z(T^j)^n x : (1+\varepsilon) \in \mathbb{R}_+, z \in \mathbb{k}\}$ is contained in $A_j \cup Y$ and therefore is nowhere dense in X. Thus x is not a supercyclic vector for $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$.

Suppose that λ_j≠0. Then $T^j_{(1+\varepsilon)_0}x = \lambda_j^n x \neq 0$ for every n∈Z_+. Hence each of the compact sets $K_n = \{z(T^j)^n x : z \in \mathbb{C}, (1+\varepsilon)_0 n \le (1+\varepsilon) \le (1+\varepsilon)_0(n+1)\}$ with n∈Z_+ does not contain 0. The sets $\sum_j(A_j)_n = \{z\sum_j T^j_{(1+\varepsilon)}x : z \in \mathbb{C}, (1+\varepsilon)_0 n \le (1+\varepsilon) \le (1+\varepsilon)_0(n+1)\}$ are nowhere dense in X. On the other hand, for every $(1+\varepsilon) \in [(1+\varepsilon)_0 n, (1+\varepsilon)_0(n+1)]$, $\sum_j T^j_{(1+\varepsilon)+(1+\varepsilon)_0}x = \sum_j T^j_{(1+\varepsilon)}T^j_{(1+\varepsilon)_0} = \sum_j \lambda_j T^j_{(1+\varepsilon)}x$ and therefore $\sum_j(A_j)_n = \sum_j(A_j)_{n+1}$ for each n ∈ Z_+. Hence $\{z\sum_j T^j_{(1+\varepsilon)}x : (1+\varepsilon) \in \mathbb{R}_+, z \in \mathbb{k}\}$, which is clearly the union of $(A_j)_n$, coincides with $\sum_j\left((A_j)_1\right)_n$ and therefore is nowhere dense. Thus x is not a supercyclic vector for $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$

Lemma 2.5

Let X be a complex topological vector space and $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\notin\mathbb{R}_+}$ be a supercyclic strongly continuous linear semigroup on *X*. Let also $(\varepsilon+1)_0$ >0 and $\lambda_j \in$C Then space $Y=\left(\sum_j (T^j_{(1+\varepsilon)_0}-\lambda_j I)(X)\right)$ either coincides with X or is a closed hyperplane in *X*.

Proof: Using the semigroup property. Factoring Y out, arrive in a supercyclic strongly continuous linear semigroup $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ acting on X/Y, where $\sum_j S^j_{(1+\varepsilon)}(x+Y)=\sum_j T^j_{(1+\varepsilon)}x+Y$. Clearly, $\sum_j S^j_{(1+\varepsilon)_0}=\sum_j \lambda_j I$. If X/Y is infinite dimensional, arrive at a contradiction with Lemma 2.4. If X/Y is finite dimensional and $\dim X/Y \geq 2$, we obtain a contradiction with Lemma 2.3. Thus dim X/Y≤ 1, as required.

Lemma 2.6

Let X be a complex topological vector space and $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a strongly continuous linear semigroup on X. Assume also that there is a closed hyperplane Hin X such that $T^j_{(1+\varepsilon)}(H)\subseteq H$ for every (1+ε)∈R_+ and let f_j∈X' be such that H=ker f_j. Then there exists w ∈Csuch that $e^{w(1+\varepsilon)}\sum_j (T^j_{(1+\varepsilon)}f_j=\sum_j f_j$ for every (1+ε)∈R_+.

Proof: Since H=ker f_j is invariant for every $T^j_{(1+\varepsilon)}$, there is a unique function Q_j:R_+→C such that $\sum_j T^j_{(1+\varepsilon)}f_i=\sum_j Q_j(1+\varepsilon)f_i$ for every (1+ε)∈R_+. Pick u^j ∈X such that f_j (u^j)=1. Then $\sum_j \left(T^j_{(1+\varepsilon)}f_i\right)(u^j)=\sum_j f_j\left(T^j_{(1+\varepsilon)}u^j\right)=\sum_j Q_j(1+\varepsilon)$ for every (1+ε)∈R_+. Since $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is strongly continuous, Q_j is continuous. The semigroup property for $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ implies the semigroup property for the dual sequence of operators: (T^j_0)=1 and $\sum_j T^j_{2(1+\varepsilon)}=\sum_j T^j_{(1+\varepsilon)}T^j_{(1+\varepsilon)}$ for every (1+ε)∈R_+. Together with equality $\sum_j T^j_{(1+\varepsilon)}f_j=\sum_j Q_j(1+\varepsilon)f_j$, it implies that Q_j(0)=1 and $Q_j\sum_j\left(2(1+\varepsilon)\right)=\sum_j Q_j(1+\varepsilon)Q_j(1+\varepsilon)$ for every (1+ε)∈R_+. The latter and the continuity of Q_j means that there is w∈C such that $Q_j\sum_j(1+\varepsilon)=e^{-w(1+\varepsilon)}$ for each (1+ε)∈R_+. Thus $e^{w(1+\varepsilon)}\sum_j\left(T^j_{(1+\varepsilon)}\right)f_j=\sum_j f_j$ for (1+ε)∈R_+, as required.

Supercyclicity Subtended Universality of Affine Maps

Start with the following general lemma.

Lemma 3.1

Let X=1 be a topological vector space, u^j∈X, $f_j\in X'\setminus\{0\}$,f(u^j)=1 and H=ker f_j. Assume also that $\sum_j\{T^j_a\}_{a\in A_j}$ is a family of a continuous linear sequence of operators on X such that $\sum_j T^j_a f_j=\sum_j f_j$ for each a∈A_j. Then the families $F^j=\{zT^j_a : z\in\Bbbk, a\in A_j\}$ are universals if and only if the families $G^j=\{R_a\}_{a\in A_j}$ of affine maps R_a:HH, $R_a x=\left(T^j_a u^j-u^j\right)+T^j_a x$ are universals on H. Moreover, x∈X is universal for F^j if and only if x = λ_j (u^j+w), where λ_j ∈k\{0} and w is universal for G^j. Next, if A_j = Z_+ and $T^j_a=(T^j_1)^a$ for every a∈Z_+, then $R_a=R^a_1$ for every a∈Z_+. Finally, if A_j = R_+ and $\sum_j\{T^j_a\}_{a\in\mathbb{R}_+}$ is strongly continuous linear semigroup, then $\sum_j\{T^j_a\}_{a\in\mathbb{R}_+}$ is strongly continuous affine semigroup.

Proof: Since $T^j_a(H)\subseteq H$ for every *a*, vectors from H cannot be universal for F^j. Clearly, they also do not have the form $\lambda_j(u^j+w)$ with $\lambda_j\in\Bbbk\setminus\{0\}$ and w∈H.

Let $x_0\in X\setminus H$. Then $f_j(x_0)\neq 0$ and hence $x=\dfrac{x_0}{f_j(x_0)}\in u^j+H$. Since $T^j_a(u^j+H)\subseteq u^j+H$ for every a∈A_j, $0=\{T^j_a x : a\in A_j\}\subseteq u^j+H$. It is straightforward to see that x_0 is universal for F^j if and only if 0 is dense in u^j+H. That is, x_0 is universal for F^j if and only if x is universal for the families $\{Q_a\}_{a\in A_j}$, where each $Q_a: u^j+H\to u^j+H$ is the restriction of T^j_a to the invariant subset u^j+H. Clearly, the translation map $\phi: H\to u^j+H$ $\phi(y)=u^j+y$ is a homeomorphism and $R_a=\phi^{-1}Q_a\phi$ for every a∈A_j. It follows that x_0 is universal for F^j if and only if $\phi^{-1}x=x-u^j$ is universal for G^j. Denoting w=x-u^j, if and only if $x_0=f_j(x_0)(u^j+w)$ with $w\in U(G^j)$.

Since Q_a are the restrictions of T^j_a to the invariant subset u^j+H and R_a are similar to Q_a in the same manner independent on a, {R_a} inherits all the semigroup or continuity properties from{ T^j_a } { T^j_a }. The proof is complete.

Lemma 3.2

Let X be a topological vector space, u^j∈X, $f_j\in X'\setminus\{0\}$, and H=ker f_j. Then T^j∈L(X) satisfying $\sum_j T^j f_j=\sum_j f_j$ is supercyclic if and only if the map R:H→H, $Rx=\left(T^j u^j-u^j\right)+T^j x$ is universal. Moreover, x∈X is a supercyclic vector for T^j if and only if x=λ_j (u^j+w), where λ_j∈k \{0} and w∈U(R).

Lemma 3.3

Let X be a topological vector space, u^j∈X, $f_j\in X'\setminus\{0\}$, f_j(u^j)=1 and H=ker f_j. Then a strongly continuous linear semigroup $\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on X satisfying $\sum_j T^{j'}_{(1+\varepsilon)}\left(T^j_{(1+\varepsilon)}\right)f_j=\sum_j f_j$ for (1+ε)∈R_+ is supercyclic if and only if the strongly continuous affine semigroup $\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on H defined by $R_{(1+\varepsilon)}x=(T^j_{(1+\varepsilon)}u^j-u^j)+T^j_{(1+\varepsilon)}x$ are universals. Furthermore, x∈X is a supercyclic vector for $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ if and only if x=λ_j (u^j+w), where λ_j∈k \{0} and $w\in U\left(\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$.

Universality of Affine Semigroups

The proof of the following lemma is a routine verification.

Lemma 4.1

Let X be a topological vector space, $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a collection of continuous affine maps on X, $\sum_j\{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a collection of the continuous linear sequence of operators on X and (1+ε)→$w_{(1+\varepsilon)}$ be a map from R_+ to X such that $\sum_j T^j_{(1+\varepsilon)}x=w_{(1+\varepsilon)}+\sum_j S^j_{(1+\varepsilon)}x$ for every (1+ε)∈R_+ and x∈X. Then $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are affines semigroup if and only if $\sum_j\{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are linear semigroups,

W_0 =0 and $w_{2(1+\varepsilon)}=w_{(1+\varepsilon)}+\sum_j S^j_{(1+\varepsilon)}w_{(1+\varepsilon)}$ for every (1+ε)∈R_+. (1)

Furthermore, the semigroup $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is strongly continuous if and only if $\sum_j\{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is strongly continuous and the map (1+ε)→$w_{(1+\varepsilon)}$ is continuous. Finally, the semigroup $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is jointly continuous if and only if $\sum_j\{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is jointly continuous and the map (1+ε)→$w_{(1+\varepsilon)}$ is continuous.

Lemma 4.2

Let X be a topological vector space and $\sum_j\{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal strongly continuous affine semigroup on X. Then $\sum_j\left(I-T^j_{(1+\varepsilon)}\right)(X)$ is dense in X for every ε>-1.

Proof: Suppose the contrary. Then there is ε>-1 such that Y_0 ≠ X, where $Y_0 = \sum_j \left(I - T^j_{(1+\varepsilon)}\right)(X)$. Let Y be a translation of Y_0, containing $0 : Y = Y_0 - u^j_0$ with $u^j_0 \in Y_0$. Factoring out the closed linear subspace Y, arrive in the universal strongly continuous affine semigroup $\sum_j \left\{F^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on X/Y, where $F^j_{(1+\varepsilon)}(x+Y) = T^j_{(1+\varepsilon)}x + Y$ for every (1+ε)∈R_+ and x∈X. By definition of Y, the linear part of $F^j_{(1+\varepsilon)}$ is I. Let β+ε∈X/Y be a universal vector for $\sum_j \left\{F^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. By Lemma 4.1, there is a strongly continuous linear semigroup $\left\{G^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on X/Y and a continuous map (1+ε)→$\gamma_{(1+\varepsilon)}$ from R_+ to X/Y such that $\gamma_0 = 0, F^j_{(1+\varepsilon)}\beta = G_{(1+\varepsilon)}\beta + \gamma_{(\varepsilon+1)}$, and $\gamma_{r+(1+\varepsilon)} = \gamma_r + G_r\gamma_{(1+\varepsilon)} = \gamma_{(1+\varepsilon)} + G_{(1+\varepsilon)}\gamma_r$ for every β∈X/Y and r, (1+ε)∈R_+, obtain that $F^j_{(1+\varepsilon)+n(1+\varepsilon)}(\beta+\varepsilon) = F^j_{(1+\varepsilon)}(\beta+\varepsilon) + n\gamma_{(1+\varepsilon)}$ for every n∈Z_+ and (ε+1)∈R_+. It follows that

$$\left\{\sum_j F^j_{(1+\varepsilon)}(\beta+\varepsilon) : (1+\varepsilon)\in\mathbb{R}_+\right\} = K + \mathbb{Z}_+\gamma_{(1+\varepsilon)}$$

where

$$K = \left\{\sum_j F^j_{(1+\varepsilon)}(\beta+\varepsilon) : (1+\varepsilon)\in[0,(1+\varepsilon)]\right\}.$$

Since (β+ε) is universal for $\sum_j \left\{F^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$, hence, $0 = K + \mathbb{Z}_+\gamma_{(1+\varepsilon)}$ is dense in X/Y. Since 0 is closed as a sum of a compact set and a closed set,0=X/Y. On the other hand, 0 it does not contain $-(1+\varepsilon)\gamma_{(1+\varepsilon)}$ any sufficiently large >-1. This contradiction completes the proof.

Lemma 4.3

Let X be a topological vector space, x∈X, ε>-1 and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal affine semigroup on X. Assume also that $\sum_j T^j_{(1+\varepsilon)}x = \sum_j S^j_{(1+\varepsilon)}x + w_{(1+\varepsilon)}$, where $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ strongly continuous linear semigroup on X and $(1+\varepsilon)\mapsto \mathrm{w}_{(1+\varepsilon)}$ is a continuous map from R_+ to X. Then $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is hypercyclic. Furthermore, $u(\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}) \cap (\mathrm{w}_{(1+\varepsilon)} + \sum_j \left(I - \sum_j S^j_{(1+\varepsilon)}\right)(X)) \neq \varnothing$ for every ε > -1.

Proof: Let $x \in u(\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$ and fixε ε > -1. By Lemma 4.2, $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)} \varepsilon > -1 \sum_j (T^j_{(1+\varepsilon)} - 1)(X)$ are dense in X. Hence $0 = \left\{\sum_j (T^j_{(1+\varepsilon)} - 1)T^j_{(1+\varepsilon)}x : (1+\varepsilon)\in\right\}$ are dense in X. Using the semigroup property of $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ and $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ together with (1), get

$$\sum_j \left(T^j_{(1+\varepsilon)} - I\right)T^j_{(1+\varepsilon)}x = \sum_j S^j_{(1+\varepsilon)}S^j_{(1+\varepsilon)}x + \sum_j S^j_{(1+\varepsilon)}w_{(1+\varepsilon)} + w_{(1+\varepsilon)} - \sum_j S^j_{(1+\varepsilon)}x - w_{(1+\varepsilon)}$$

$$= \sum_j S^j_{(1+\varepsilon)}S^j_{(1+\varepsilon)}x + \sum\sum_j S^j_{(1+\varepsilon)}S^j_{(1+\varepsilon)}x + \sum_j S^j_{(1+\varepsilon)}w_{(1+\varepsilon)} - \sum_j S^j_{(1+\varepsilon)}x = \sum_j S^j_{(1+\varepsilon)}\left(w_{(1+\varepsilon)} - \left(I - S^j_{(1+\varepsilon)}\right)x\right)$$

for every $(1+\varepsilon)\in\mathbb{R}_+$, then, 0 is exactly the St-orbit of $w_{(1+\varepsilon)} - \sum_j (I - S^j_{(1+\varepsilon)})x$. Since o is dense in X, $w_{(1+\varepsilon)} - \sum_j (I - S^j_{(1+\varepsilon)})x \in w_{(1+\varepsilon)} + \sum_j (I - S^j_{(1+\varepsilon)})(\mathrm{X})$ is hypercyclic vector for $\left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ and therefore $u\sum_j\left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right) \cap (\mathrm{w}_{(1+\varepsilon)} + \sum_j \left(I - \sum_j S^j_{(1+\varepsilon)}\right)(X)) \neq \varnothing$.

Lemma 4.4

Let X be a topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be an affine semigroup on X. Then for every $(1+\varepsilon)_1$,..., $(1+\varepsilon)_n$∈R_+ and every z_1,...,z_n∈k satisfying z_1+... z_n=1, the map $\sum_j S^j = z_1\sum_j T^j_{(1+\varepsilon)_1} + \ldots + z_n\sum_j T^j_{(1+\varepsilon)_n}$ commutes with every $\sum_j T^j_{(1+\varepsilon)}$.

Proof: To verify that for every affine map A_j: X→X and every x_1,...,x_n∈X,

$\sum_j A_j(z_1x_1 + \ldots + z_nx_n) = z_1\sum_j A_jx_1 + \ldots + z_n\sum_j A_jx_n$ provided z_j∈k and z_1+... z_n=1.

Let (1+ε)∈R_+. Therefore,

$$\sum_j T^j_{(1+\varepsilon)}S^j = z_1\sum_j T^j_{(1+\varepsilon)}T^j_{(1+\varepsilon)_1}x + \ldots + z_n\sum_j T^j_{(1+\varepsilon)}T^j_{(1+\varepsilon)_n}.$$

Since $\sum_j T^j_{(1+\varepsilon)}$ commute with each other, get

$$\sum_j T^j_{(1+\varepsilon)}S^j = z_1\sum_j T^j_{(1+\varepsilon)}T^j_{(1+\varepsilon)_1}x + \ldots + z_n\sum_j T^j_{(1+\varepsilon)}T^j_{(1+\varepsilon)_n} = \sum_j S^jT^j_{(1+\varepsilon)}$$

Lemma 4.5

Let X be a topological vector space, $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be universals strongly continuous affine semigroup on X and $x \in u\sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$. Then $\Lambda(x) \subseteq u\sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$, where

$$\Lambda(x) = \left\{z_1\sum_j T^j_{(1+\varepsilon)_1}x + \cdots + z_n\sum_j T^j_{(1+\varepsilon)_n}x : n\in\mathbb{N}, (1+\varepsilon)_j \in \mathbb{R}_+, z_j \in \mathbb{K}, z_1 + \cdots + z_n = 1\right\}. \quad (2)$$

Proof: Let n∈N,$(1+\varepsilon)_1$,..., $(1+\varepsilon)_n$∈R_+, z_1+...z_n∈k and z_1+...... z_n=1. Have to show that $x \in u\sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$ where $A_j = z_1\sum_j T^j_{(1+\varepsilon)_1} + \ldots + z_n\sum_j T^j_{(1+\varepsilon)_n}$. A commute with all $T^j_{(1+\varepsilon)}$. Since $x \in u\sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$ it suffices to verify that A_j(X) are dense in X. By Lemma 4.1, write $\sum_j T^j_{(1+\varepsilon)}y = \sum_j S^j_{(1+\varepsilon)}y + w_{(1+\varepsilon)}$ for every y∈X, where $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is strongly continuous linear semigroup on X and (1+ε)→$w_{(1+\varepsilon)}$ is a continuous map from R_+ to X. By Lemma 4.3, $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are hypercyclics. Thus $B_j = z_1\sum_j S^j_{(1+\varepsilon)1} + \ldots + z_n\sum_j S^j_{(1+\varepsilon)n}$ has dense range. Since A_j(X) is translation, B_j(X),A_j(X) is also dense in X, which completes the proof.

Proof of Theorem 1.3

Let X be a topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal jointly continuous affine semigroup on X. By Theorem A, there is a hypercyclic continuous linear operator on X. Since no such thing exists on a finite-dimensional topological vector space [7], X is infinite-dimensional. Since any compact subspace of an infinite-dimensional topological vector space is nowhere dense [4], condition (1) of Proposition 1.1 is satisfied. Let $x \in u\sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$. By Lemma 4.5, the set $\Lambda(x)$ defined in (4.2) consists entirely of universal vectors for $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. Obviously, $\left\{\sum_j T^j_{(1+\varepsilon)}x : (1+\varepsilon)\in\mathbb{R}_+\right\} \subseteq \Lambda(x)$. By its definition, $\Lambda(x)$ is an affine subspace of X. $\Lambda(x)$ satisfies all requirements for the set $\mathrm{Y}_{(1+\varepsilon)}, x$ (for every ε > -1) from condition (2) in Proposition 1.1. By Proposition 1.1, $u\sum_j(T^j_{(1+\varepsilon)}) = u\sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$ for every ε > -1, as required.

Proof of Theorem 1.2

Let X be a complex topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semigroup on X in [8]. To prove that all $\sum_j T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic and the sets of supercyclic vectors of $T^j_{(1+\varepsilon)}$ simultaneity with the set of supercyclic vectors of $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. If $T^j_{(1+\varepsilon)} - \lambda_j I$ has a dense range for every ε > -1 and every λ_j∈C, then Proposition C provides the required result. Else, by Proposition 2.1, there is a closed hyperplane H in X invariant for all $T^j_{(1+\varepsilon)}$. By Lemma 2.6, there is f_j∈X and (β+ε)∈C such that H=kerf_j and $\sum_j e^{(1+\varepsilon)(\beta+\varepsilon)}(T^j_{(1+\varepsilon)})f_j = f_j$ for every (1+ε)∈R_+. Obviously $\left\{e^{(1+\varepsilon)(\beta+\varepsilon)}\sum_j T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is a jointly continuous supercyclic linear semigroup on X with the same sets S^j of supercyclic vectors as the original semigroup $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$. Fix $u^j \in X$ satisfying $f_j(u^j)=1$. Now fix ε>-1 and $v^j \in S^j$. Have to show that v^j is supercyclic for $\sum_j T^j_{(1+\varepsilon)}$. By Lemma 3.3, applied to the semigroup $\{e^{(1+\varepsilon)(\beta+\varepsilon)}\sum_j T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$, write $v^j = \lambda_j(u^j + y)$, where $\lambda_j \in \mathbb{k}\setminus\{0\}$ and y is a universal vector for the jointly continuous affine semigroup $\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on H defined by the formula $R_{(1+\varepsilon)}x = w_{(1+\varepsilon)} + e^{(1+\varepsilon)(\beta+\varepsilon)}T^j_{(1+\varepsilon)}x$ with $w_{(1+\varepsilon)} + e^{(1+\varepsilon)(\beta+\varepsilon)}\sum_j (T^j_{(1+\varepsilon)} - \mathrm{I})u^j$. By Theorem 1.3, y is universal for $R_{(1+\varepsilon)}$. By Lemma 3.2, $v^j = \lambda_j(u^j + y)$ is a supercyclic vector for $e^{(1+\varepsilon)(\beta+\varepsilon)}\sum_j T^j_{(1+\varepsilon)}$ and hence v^j is a supercyclic vector for $T^j_{(1+\varepsilon)}$. The proof is complete.

Remarks

By Lemma 4.3, the universality of a strongly continuous affine semigroup implies hypercyclicity of the underlying linear semigroup. The following example shows that the converse is not true [4].

Example 6.1

Consider the backward weighted shift $T^j \in L(l_2)$ with the weight sequence $\{e^{-2n}\}_{n\in\mathbb{N}}$. That is, $T^j e_0 = 0$ and $T^j e_n = e^{-2n}e_{n-1}$ for n∈N, where $\{e^n\}_{n\in\mathbb{Z}_+}$ is the standard basis of l_2. Then the jointly continuous linear semigroups $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ with $\sum_j S^j_{(1+\varepsilon)} = e^{(1+\varepsilon)\ln\sum_j(I+T^j)}$ are hypercyclics. Furthermore, there exists a continuous map (1+ε)→$w_{(1+\varepsilon)}$ from R_+ to l_2 such that $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is jointly continuous non-universal affine semigroup, where $\sum_j T^j_{(1+\varepsilon)}x = w_{(1+\varepsilon)} + \sum_j S^j_{(1+\varepsilon)}x$ for x∈l_2.

Proof: Since T^j being compacts weighted backward shift, is quasinilpotent, the sequence of operators $\ln(I+T^j)$ is well defined and bounded and $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is a jointly continuous linear semigroup. Moreover, $S^j_1 = I + T^j$ are hypercyclics [9] as a sum of the identical sequence of operators and a backward weighted shift. Hence $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are hypercyclics.

Let $u^j \in l_2$, $u^j_n = (n+1)^{-1}$ for n∈Z_+. For each (1+ε)∈R_+, let $w_{(1+\varepsilon)} = v^j_{(1+\varepsilon)}(T^j)u_j$, where $v^j_{(1+\varepsilon)}(z) = \sum_{n=1}^{\infty}\frac{(1+\varepsilon)\varepsilon\ldots(\varepsilon-n+2)}{n!}z^{n-1}$.

Since T^j are quasinil potents, $v^j_{(1+\varepsilon)}(T^j)$ are well defined bounded linear sequence of operators and the map $(1+\varepsilon)\mapsto v^j_{(1+\varepsilon)}(T^j)$ are a sequence of operators-norm continuous. Hence (1+ε)→$w_{(1+\varepsilon)}$ is continuous as a map from R_+ to l_2, to verify that $w_0=0$, $w_1=u^j$ and $w_{2(1+\varepsilon)} = S^j_{(1+\varepsilon)}w_{(1+\varepsilon)} + w_{(1+\varepsilon)}$ for every ε>-1. By Lemma 4.1, $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is a jointly continuous affine semigroup, where $T^j_{(1+\varepsilon)}x = w_{(1+\varepsilon)} + S^j_{(1+\varepsilon)}x$. It remains to show that $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is non-universal. Assume the contrary. Since $w_1=u^j$ and $S^j_1 = I + T^j$, Lemma 4.3 implies that the coset $\sum_j (u^j + T^j(l_2))$ must contain a hypercyclic vector for $I+T^j$. This, however, is not the case as shown in [10].

Remark 6.2

Let *X* be a topological vector space and $S^j \in L(X)$ be hypercyclic. If $u^j \in (I - S^j)(X)$, then the affine map $\sum_j T^j x = \sum_j (u^j + S^j)x$ is universal. Actually, let w∈X be such that $u^j = w - S^j w$. It is easy to show that $\sum_j (T^j)^n x = w + \sum_j (S^j)^n (x-w)$ for every x∈X and n∈N. Thus x is universal for T^j if and only if x-w is universal for S^j.

If additionally X is separable metrizable and Baire, then a standard Baire category type argument shows that the set of $u^j \in X$ for which the affine map $\sum_j T^j x = \sum_j (u^j + S^j x)$ is universal is a dense G_δ-subset of X. Example 6.1 shows that this set can differ from X.

Recall that a locally convex topological vector space X is called barrelled if every closed convex balanced subset B_j of X satisfying $X = U_{n=1}^{\infty} n(B_j)$ contain a neighborhood of 0. The joint continuity of a linear semigroup follows from the strong continuity if the underlying space X is an F^j-space. The same is true for wider classes of topological vector spaces. For the case, it is sufficient X to be a Baire topological vector space or a barreled locally convex topological vector space. Thus the following observation holds true.

Remark 6.3

The joint continuity condition in Theorems A, 1.2 and 1.3 can be replaced by the strong continuity, provided X is Baire or X is locally convex and barrelled.

For general topological vector spaces however strong continuity of a linear semigroup does not imply joint continuity. Furthermore, the following example shows that Theorem A fails in general if the joint continuity condition is replaced by strong continuity. Recall that the Fréchet space $L^2_{loc}(\mathbb{R}_+)$ consists of the scalar-valued functions R_+, square-integrable on [0,(1+ε)] for each ε>-1. Its dual space can be naturally interpreted as the space $L^2_{fin}(\mathbb{R}_+)$ of square-integrable scalar-valued functions R_+ with bounded support. The duality between $L^2_{loc}(\mathbb{R}_+)$ and $L^2_{fin}(\mathbb{R}_+)$ is provided by the natural dual pairing $\sum_j f_j, g_j = \int_0^\infty \sum_j f_j(t)g_j(t)dt$. Clearly, the linear semigroup $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ of backward shifts $\sum_j S^j_{(1+\varepsilon)}f_j(x) = \sum_j f_j(x+(1+\varepsilon))$ is strongly continuous and therefore jointly continuous on the Fréchet space $L^2_{loc}(\mathbb{R}_+)$. It follows that the same semigroup is strongly continuous on $L^2_{\sigma,loc}(\mathbb{R}_+)$ being $L^2_{loc}(\mathbb{R}_+)$ endowed with the weak topology.

Example 6.4

Let $X = L^2_{\sigma,loc}(\mathbb{R}_+)$ and $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be the above strongly continuous semigroup on X. Then there are $f_j \in X$ hypercyclics for $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ such that f_j are not-hypercyclics for S^j_1.

Proof: Let H be the hyperplane in $L^2[0,1]$ consisting of the functions with zero Lebesgue integral. Fix norm-dense countable subsets A_j of H. One can easily construct $f_j \in L^2_{loc}(\mathbb{R}_+)$ such that for every n∈N, the function $(f_j)_n : [0,1] \to K, (f_j)_n(1+\varepsilon) = f_j(n+(1+\varepsilon))$ belongs to A_j; for every n∈N and $h_1,\ldots,h_n \in A_j$, there is $m \in \mathbb{N}$ such that $h_j = (f_j)_{m+j}$ for $1 \le j \le n$.

For $(1+\varepsilon) \in \mathbb{R}_+$, let $\chi_{(1+\varepsilon)} \in X' = L^2_{fin}(\mathbb{R}_+)$ be the indicator function of the interval $[(1+\varepsilon),(2+\varepsilon)] : \chi_{(1+\varepsilon)}(1+\varepsilon) = 1$ if $(1+\varepsilon) \le (1+\varepsilon)^2 + 1$ and $\chi_{(1+\varepsilon)}((1+\varepsilon)) = 0$ otherwise. By (a), $(S^j_1)^n f_j \in \ker\chi_0$ for every n∈N and therefore f_j are not hypercyclics vector for S^j_1.

It remains to show that f_j are hypercyclics vector for $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ acting on X. Using (a) and (b), we see that the Fréchet space topology closure of the orbits $\{S^j_{(1+\varepsilon)}f_j : (1+\varepsilon)\in\mathbb{R}_+\}$ is exactly the sets

$$o = \bigcup_{(1+\varepsilon)\in[0,1]}\bigcap_{n\in\mathbb{Z}_+}\ker\chi_{(1+\varepsilon)+n}$$

In order to show that f_j are hypercyclics for $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ acting on X, it suffices to verify that 0 is dense in $L^2_{\sigma,loc}(\mathbb{R}_+)$. Assume the contrary. Then there is a weakly open set W^j in $L^2_{loc}(\mathbb{R}_+)$, which does not intersect 0. That is, there are linearly independent $(Q_j)_1,\ldots,(Q_j)_m \in L^2_{fin}(\mathbb{R}_+)$ and

$(1+\varepsilon)_1,\ldots,(1+\varepsilon)_m \in \Bbbk$ such that

$$\max_{1\le j\le m}\left|\Sigma_j\left((1+\varepsilon)_j - \langle g, Q_j\rangle\right)\right| \ge 1 \ for\,all\, g^j \in O$$

Let k∈N be such that all Q_j vanishes on $[k,\infty)$. Pick any $0<(1+\varepsilon)_0<\ldots<(1+\varepsilon)_m<1$. Note that for every $1\le j\le m$, the restrictions of the functionals Q_j to $\bigcap_{n=0}^{k}\ker\chi_{(1+\varepsilon)_l}+n$ are not linearly independent, see [11]. Actually, otherwise can find $h_0\in\bigcap_{n=0}^{k}\ker\chi_{(1+\varepsilon)_l}+n$ such that $\langle h_0, Q_j\rangle=(1+\varepsilon)_j$ for $1\le j\le m$. It is easy to see that it is $h\in L^2_{loc}(\mathbb{R}_+)$ such that $h\big|_{[0,k]}=h_0\big|_{[0,k]}$, $h\big|_{[k+1,\infty]}$ and $\langle h,\chi_{(1+\varepsilon)_l+k-1}\rangle=\langle h,\chi_{(1+\varepsilon)_l+k}\rangle=0$. Then $\langle h, Q_j\rangle=(1+\varepsilon)_j$ for $1\le j\le m$ and $h\in\bigcap_{n=0}^{\infty}\ker\chi_{(1+\varepsilon)1+n}\subseteq O$. arrived at a contradiction.

The fact that Q_j is not linearly independents on $\bigcap_{n=0}^{k}\ker\chi_{(1+\varepsilon)1+n}$ implies that there is a non-zero $(g_j)_l\in span\{(Q_j)_1,\ldots,(Q_j)_m\}\bigcap span\{\chi_{(1+\varepsilon)l},\ldots,\chi_{(1+\varepsilon)l+k}\}$. Since $\chi_{(1+\varepsilon)_l+r}$ they are all linearly independent, $(g_j)_0,\ldots,(g_j)m$ they are $m+1$ linearly independent vectors in the m dimensional space $span\{(Q_j)_1,\ldots,(Q_j)_m\}$. completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interest.

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