Journal of Physical Mathematics
ISSN: 2090-0902
Krasnyuk, J Phys Math 2018, 9:2
DOI: 10.4172/2090-0902.1000277
Research Article
Open Access

Wave Packets of Mandelbrot Type in Boundary Problems of Quantum Mechanics

Igor B Krasnyuk*

Institute Physics and Technology, National Academy of Sciences of Ukraine, Ukraine

Abstract

An initial value boundary problem for the linear Schrodinger type equation with nonlinear functional boundary conditions is considered. It is shown that attractor of the problem contains periodic piecewise constant functions on a complex plane with infinite points of discontinuities on a period. It has been applied the method of reduction of the problem to a system of integro-difference equations. Further, it is proved that for large time integro-difference equations tends to a system of difference equations asymptotic of which is known. It has been done application to boundary problems of nonlinear optics, where impulse periodic distributions of light are interpreted as white and black solitons.

Keywords: The Schrodinger equation, The functional two points boundary conditions; Asymptotic periodic piecewise constant distributions of relaxation type

Introduction

In last years, in physics studied the nonlinear interaction of light which can mimic the physics at so called an event horizon. As shown in a study [1], this analogue arises when a weak probe wave is unable to pass through an intense soliton, despite propagating at a different velocity". These dynamics arises as a soliton-induced refractive index barrier. In all paper this barrier characterize the volume optic properties of a fibre with linear boundary conditions. In this paper, we consider the opposite problem when the optical medium is ideal or linear, but boundaries of the medium have the nonlinear optic properties, and describes, for example, the all-optical transistor [2]. It may be also a case when a bright soliton is passing through the soliton. In this case, the intensity of light depends on the fibre refractive index (that is take place the Kerr effect). Thus, the soliton creates a moving refractive index perturbations which passage through the another soliton [3-6]. This interaction between such surface solitons plays the mane role of distributions of the light in the linear medium with nonlinear surface interaction. Thus for the ideal medium the main role plays the fibre surface nonlinear refractive index. In a study [2], it has been mimicked two spectral modes of solitons when the mode-locked laser diode generate picosecond solitons. This generation will be described as a functional boundary conditions generated these solitons. The simulation is described by the hyperbolic type equations which models an evolution of amplitude of electric field A(x, t) [7]. These equations depends on the dispersion coefficients and on the nonlinear interaction coefficient of the fibre, depending on A(x, t), where x is a coordinate of a beam and t is the time. Of course, these equations must be follows from the Schrodinger type of equations. Thus phases and amplitudes of lights describes by systems of linear quantum equations. Boundary condition describes surface beams in an optical resonator with feedback. Thus the equations are linear without interaction potential, but the feedback is modeled by the nonlinear boundary conditions.

The problem of the coherent interaction of impulses in a nonlinear medium is well-known. As noted in a study [8]," an interaction may be utilized for the transmission of information, for frequency conversion, and for the description of processes which proceed in more intensive fields and at times shorter than to relaxation time" [9].

$$-i\hbar\frac{\partial\psi}{\partial t}=\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \tag{1}$$

where ψ:=ψ (x; t): $R^2 \to C$, where C is a complex space, is an unknown function, $\hbar$ is the Planck constant. Let us divide the two parts of equation on a value mv^2, where m is the mass of particles, v is their velocity and p=mv is an impulse. Further, we introduce $\bar{t}=t/\tau$, τ is a relaxation time of a wave function to some equilibrium, and we consider a dimensionless constant $h=\frac{\lambda}{v\tau}$. As a result, we obtain the dimensionless equation:

$$-i\hbar\frac{\partial\psi}{\partial t}=\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} \tag{2}$$

We can consider the functional two-point boundary conditions

$$\psi(0,t)=\theta\psi(l,t) \tag{3}$$

with the real or complex parameter θ. For these conditions there is a theorem of existent and uniqueness of solutions the problem [10-13]. But our aim is instead of the linear boundary conditions to consider nonlinear conditions

$$\psi(0,t,h)=\Phi\left[\psi(l,t,h)\right] \tag{4}$$

where Φ: C → C is nonlinear structural stable map, with additional initial conditions

$$\psi(x,0,h)=\psi_0(x,h),\ 0\le x\le 1 \tag{5}$$

where h>0 is a small parameter. We assume that conditions

$$\psi(0,0,h)=\Phi\left[\theta,\psi(l,0,h)\right],\ \psi(0,1,h)=\Phi\left[\theta\psi(l,1,h)\right] \tag{6}$$

and the similar conditions for second derivatives at points (0, 0) and (0, 0) are satisfied. This ensures the existence of solutions of $C^2[(0, l)\times[0, t_0)]$ - class. Of course, real and imaginary parts of the map Φ are from C^2 - class. This ensures that the map Φ: C → C is structural stable. It means that a corresponding map G: $R^2 \to R^2$ is structural stable. It means that

***Corresponding author:** Igor B Krasnyuk, Institute Physics and Technology, National Academy of Sciences of Ukraine, Ukraine, Tel: (057) 335 3530; E-mail: krasnjukigr@rambler.ru

Received April 02, 2018; **Accepted** June 27, 2018; **Published** July 04, 2018

Citation: Krasnyuk IB (2018) Wave Packets of Mandelbrot Type in Boundary Problems of Quantum Mechanics. J Phys Math 9: 277. doi: 10.4172/2090-0902.1000277

Citation: Krasnyuk IB (2018) Wave Packets of Mandelbrot Type in Boundary Problems of Quantum Mechanics. J Phys Math 9: 277. doi: 10.4172/2090-0902.1000277

that spectre σ(TG') ∩ {z: |z|=1}=ϕ, where ϕ is empty set.

Further we assume that h>0 is a small parameter, and we consider the problem with accuracy $O(h^2)$, where $O(h^2) \to 0$) as $t \to +\infty$. We find solutions in the form:

$$\psi(\mathrm{x,t},h) = e^{iS(x,t)/h}\varphi(\mathrm{x,t},h) \tag{7}$$

where S(x, t) is a real phase, and φ(x, t, h) is real amplitude. (Below, where it will not cause misunderstandings, the parameter h will be omitted).

We assume that $C^0([0;l] \times [0+\infty)$ is the space of bounded continuous functions, and C^2 is the space of twice differentiable functions with the norm: $\|f\|c^2 = \sum_{k=0}^{2}\sup\|f^k\|$, where $\|f^0\|$ is the norm in C^0 ([0, l] × [0,+∞. The function $\psi \in C^2$ belongs to if its real and imaginary parts belong $C^2([0, l] \times [0,+\infty$. Then in C^2 - norm there is the following convergence:

$$\|S(x,t)\|c^2 \Rightarrow \|\Phi[p_1(t - x/p)]\|c^2 \tag{8}$$

where $p_1(\varsigma)$ is $2^N\!/p$ - periodic piecewise constant distribution with finite number Γ of points of discontinuities. Further

$$\|\varphi(x,t)\|c^2 \Rightarrow \|\Phi_2[p_1(t - x/p)]\|c^2 \tag{9}$$

where $p_1(\varsigma)$ is $2^N l/p$ - periodic piecewise constant distribution with finite number Γ of points of discontinuities. Here, $\Gamma = \varsigma^{-1}(D)$, where $D = \bigcup_{n\geq 0} G^{-n}(A^{\pm})$ and $A^{\pm}$ is a set of saddle points of codimension one, and $\varphi(\varsigma) = (S_0(\varsigma), \varphi_0(\varsigma))$ is an initial curve in R^2, which is determined by initial data of the boundary problem, and N is least common multiple of the map G: (S,φ) → (Φ_1(S,φ), Φ_2(S,φ)).

Method of Reduction of Problem to System of Integro-Difference Equations

Now we return to the problem. Indeed, substituting (7) in eqn. (2), we obtain that

$$\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2\right)\varphi + (-ih)\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}\varphi\Delta S\right) + \frac{(-ih)^2}{2}\Delta\varphi = 0 \tag{10}$$

We find solution with accuracy $O(h^2)$ so that

$$\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2\right)\varphi + (-ih)\left(\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}\varphi\Delta S\right) = 0 \tag{11}$$

Then we obtain the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2\right) = 0 \tag{12}$$

The boundary condition can be written in form:

$$\cos S\varphi_{|x=0} = \Re\Phi(S,\varphi)_{|x=l},\ \sin S\varphi_{|x=0} = \Phi(S,\varphi)_{|x=l} \tag{13}$$

Where $\Re$ and $\Im$ real and imaginary parts of a complex number. Let us denote F1:= $\Re\Phi$ and F1: = $\Im\ \Phi$. Then from eqn. (13) it follows that:

$$\varphi^2_{|x=0} = F_1^2(S,\varphi)_{|x=l} + F_2^2(S,\varphi)_{|x=l} \tag{14}$$

$$\tan S_{|x=0} = F_2^2(S,\varphi)_{|x-l} \big/ F_2^2(S,\varphi)_{|x=l} \tag{15}$$

We define the function $\Phi_2 := \sqrt{F_1^2 + F_1^2}$ and $\Phi_2 := \arctan F_2^2 / F_1^2$, then the boundary condition (4) can be written as

$$\varphi^2_{|x=0} = \Phi_2(S,\varphi)_{|x=l} \tag{16}$$

$$S_{|x=0} = \Phi_1(S,\varphi)_{|x=l} \tag{17}$$

For simplicity, we consider, initially, the case, when $\Phi_2 := \Phi_2(\varphi)$ and $\Phi_1 := \Phi_1(\varphi)$ Then general case can be treated similarly

Thus for the Hamilton-Jacobi equation we have the boundary condition

$S(0,t)=\Phi_1$ [S(l,t)] (18)

And, for the transport equation

$$\frac{\partial \varphi}{\partial t} + p\frac{\partial \varphi}{\partial x} + \frac{\partial^2 S}{\partial x^2}\varphi = 0 \tag{19}$$

We have the boundary condition

$$\varphi(0,t) = \Phi_2[\varphi(l,t)] \tag{20}$$

Here maps Φ_1, $\Phi_2 \in C^1(I \to I)$ are structural stable, where I is an open closed interval. The structural stable maps form an open dense subset [5].

In order to solve equations, we use the method of characteristics. To do this, we consider the Hamilton system of ordinary difference equations with hamiltonian $H(p) = \frac{1}{2}p^2$ as

$$\dot{x} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial x} \tag{21}$$

With the initial condition,

$$x(0) = x_0,\ \dot{p}(0) = -\frac{\partial S}{\partial x}(x_0) = p \tag{22}$$

For the given constant p the function x:=x(p,t) is the solution of equation

$$p - \frac{\partial S(x,t)}{\partial x} = 0 \tag{23}$$

Where p can be considered as additional coordinate in (x,p,t)-space.

Interpretation of the equation follow from the relation

$$\Phi\ = S_t + xS_x \tag{24}$$

Here, L= Φ_0 along a curve x:=ϑ_0(t, x, p) which is determined from a solution of equation (R31). Then L ≥ Φ_0 along each other curve and the curve ϑ_0 minimizes the functional $\int L(x(t),\dot{x}(t),t)dt$. In other words, the difference L - Φ_0 as a function of $\dot{x}$ has a minimal value 0 as $\dot{x}$ =p. It follows that

$$S_x = L_{\dot{x}},\ S_t = L - \dot{x}L_{\dot{x}} \tag{25}$$

Let us define $y = L_{\dot{x}}$ and $-H = L - y\dot{x}$. These values are called by spatial and temporal impulses in a space t, x, $\dot{x}$. The the differential form y dx – H dt is differential dS(x, t) if x:=p(x, t) [14]. The same result follows from the Euler equation

$$\frac{d}{dt}L_{\dot{x}} = L_x \tag{26}$$

Where $L = L_{\dot{x}} - H(\dot{x}), H[\dot{x}] = \frac{1}{2}\dot{x}$. Then a line $dx(t)/dt = p$ is extrenal of the functional. As shown above, this extrenal is the required minimum.

From the variational problem for a lagrangian L(t, x, $\dot{x}$) we can obtain a local hamiltonian L(t, x, y) and the corresponding local Hamilton-Jacobi equation. Then on characteristics dx(p, t)=p we have the equation

$$\frac{dS(x(t,p),t)}{dt} = \frac{\partial S(x,(t,p),t)}{\partial t} + \frac{\partial S(x(t,p),t)}{\partial x}\frac{dx(t,p)}{dt} = -H(p)\frac{dx(t,p)}{dt} \tag{27}$$

Citation: Krasnyuk IB (2018) Wave Packets of Mandelbrot Type in Boundary Problems of Quantum Mechanics. J Phys Math 9: 277. doi: 10.4172/2090-0902.1000277

By integration along characteristics dx/dt=p and with help of boundary conditions the problem can be reduced to the system of integro-difference equations:

$$s(x,t)=\Phi_1\left[S(x,t-l/p)\right]+\frac{px}{2}+\frac{p}{2}(l-x) \tag{28}$$

$$\varphi(x,t)=\varphi(0,t-x/l)+\int_{t-x/p}^{t}\frac{\partial^2 S}{\partial x^2}\left[p(s-t)+x,s\right]\varphi\left[\left(p(s-t)+x,s\right)\right]ds= \tag{29}$$

$$\Phi_2\left[\varphi(1,t-x/l)\right]+\int_{t-l/p}^{t-x/p}\frac{\partial^2 S}{\partial x^2}\left[\left(p(s-t)+l/p+x,s\right)\right]\varphi\left[\left(p(s-t)+l/p+x,s\right)\right]ds]$$

$$+\int_{t-x/p}^{t}\frac{\partial^2 S}{\partial x^2}\left[p(s-t)+x,s\right]\varphi\left[\left(p(s-t)+x,s\right)\right]ds$$

The first equation follow from the relation

$$S(x,t)=S(0,t-x/p)+\frac{p}{2}l=\Phi_1\left[S(l,t-x/p)\right]+\frac{pl}{2}= \tag{30}$$

$$\Phi_1\left[S(x,t-l/p)\right]+\frac{1}{2}(l-x)+\frac{1}{2}pl,$$

where we used the boundary conditions for the phase. Integro-difference eqn. (33) follows from the transport eqn. (19) which can be written as

$$\frac{d\varphi}{dt}=-\frac{1}{2}\frac{\partial^2 S}{\partial x^2}\varphi \tag{31}$$

along characteristics dx(t)/dt=p. Integrating this equation from a point (x, t) to a point (x, t - l/p) (Figure 1), and using functional boundary conditions, we obtain eqn. (33). Let us consider this system at a point x=l. Then we obtain

$$S(l,t)=\Phi_1\left[S(l,t-l/p)\right]+\frac{pl}{2} \tag{32}$$

$$\varphi(l,t)=\Phi_2\left[l,t-l/p\right]++\int_{t-l/p}^{t}\frac{\partial^2 S}{\partial x^2}\left[p(s-t)+l,s\right]\varphi\left[\left(p(s-t)+x,s\right)\right]ds \tag{33}$$

Here,

$$S(x,t)=S(0,t-x/p)+\frac{1}{2}px=\Phi_1\left[S(l,t-x/p)\right]+\frac{1}{2}px \tag{34}$$

Then,

$$\frac{\partial S}{\partial x}\left[0,t-x/p\right]=-\frac{1}{p}\Phi'_{1,S}\left[S(l,t-x/p)\right]\left[S'(l,t-x/p)\right]+\frac{1}{2}p \tag{35}$$

Hence,

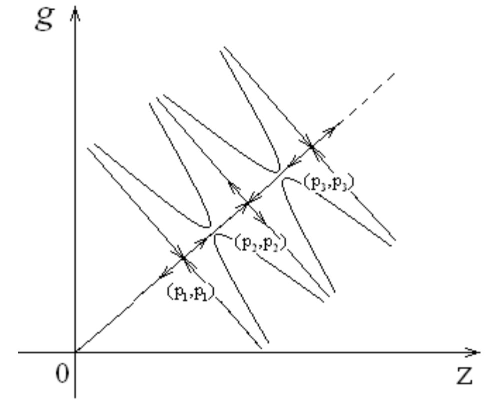

Figure 1: Typical distributions of trajectories for a hyperbolic map.

$$\frac{\partial S}{\partial x}\left[0,t-x/p\right]=-\frac{1}{p}\Phi''_{1,S,S}\left[S(l,t-x/p)\right]\left[S'(l,t-x/p)^2\right]+ \tag{36}$$

$$\frac{1}{p^2}\Phi'_{1,S}\left[S(l,t-x/p)\right]\left[S''(l,t-x/p)\right]\frac{1}{2}p$$

From eqn. (36) we obtain that

$$\frac{\partial S}{\partial x}\left[0,t-x/p\right]=-\frac{1}{p^2}\Phi''_{1,S,S}\left[S(l,t-l/p)\right]\left[S'(l,t-l/p)^2\right]+ \tag{37}$$

$$\frac{1}{p^2}\Phi'_{1,S}\left[S(l,t-l/p)\right]\left[S''(l,t-l/p)\right]\frac{1}{2}p$$

Here, a function S(l, t can be found from difference eqn. (32). It is known [5] that, in typical cases, symptotic solutions of difference equations tends to elements of attractor of corresponding dynamic system which are piecewise constant periodic functions $p_1(t)$ such that $p_1(t) \in A^+_1$, where A^+_1 is a set of attractive fixed points of the map $\Phi_{1,\mu}:=\Phi_1+\frac{pl}{2}$, for almost all points t ∈ R^+. Let $S=a+\varepsilon\tilde{S}$, where a $\in A^+_1$, and ε>0 is a small parameter. Then, with accuracy O(ε), we have

$$\tilde{S}(l,t)=\Phi'_1(a)\tilde{S}(l,t-l/p) \tag{38}$$

Where $\left|\Phi'_1(a)\right|=\lambda_1\le 1$. Then solutions of equation are $\tilde{S}'(l,t)=e^{k_1 t}$, where $k_1=\frac{p}{l}\ln\lambda_1$, so that $S''(l,t)=k_1^2e^{k_1 t}$ Since, for parametrisation, x(p, s, t)=p(s - t) + x, t(s)=s, we have

$$\frac{\partial^2 S}{\partial x^2}\left[p(s-t)+x,s\right]=S''(0,t-x/p) \tag{39}$$

Integro-difference equation can be written as

$$\varphi(l,t)=\Phi_2\left[\varphi(l,t-l/p)\right]-\frac{1}{2p^2}S''(0,t-l/p)\int_{t-l/p}^{t}\varphi\left[p(s-t)+x,s\right]ds= \tag{40}$$

$$\Phi_2\left[\varphi(1,t-l/p)\right]-\frac{1}{2p^2}k_1^2e^{k_1(t-l/p)}\int_{t-l/p}^{t}\varphi\left[\left(p(s-t)+x,s\right)\right]ds$$

Let $\varphi=b+\varepsilon\tilde{\varphi}$, where b $\in A^+_2$, and A^+_2 is a set of attractive fixed points of the map Φ_2. Let us also define $y(t)=e^{k_1(t-l/p)}$. Then eqns. (39) and (42) can be reduced, with accuracy $O(\varepsilon^2)$, to the system of equations:

$$\tilde{S}(l,t)=\lambda_1\tilde{S}(l,t-l/p) \tag{41}$$

$$\tilde{\varphi}(l,t)=\lambda_2\tilde{\varphi}\left[(l,t-l/p)\right]--\frac{1}{2p^3}lk_2^2by(t-l/p) \tag{42}$$

$$y(t)=e^{kl/p}y(t-l/p) \tag{43}$$

Where $\lambda_3=lk_2^2be^{kl/p}/2p^3$. The eigenvalues of these equations satisfies to the cubic equation

$$(\lambda_1-\chi)(\lambda_2-\chi)(\lambda_3-\chi)=0 \tag{44}$$

Since k<0, all trajectories are attracted by a plane S,φ. But on this plane we obtain two independent differential equations for the phase and amplitude. asymptotic solutions of the boundary problem are determined by asymptotic solutions of the system of difference equations.

Let as define P a set of fixed points of the map

$$G:(s,\varphi,y)\to\left(\Phi_1[S]+\frac{pl}{2},\Phi_1[S]-\frac{1}{2p^3}k_2^2by\right),\ y(t)=e^{kl/p}y(t-l/p) \tag{45}$$

Hence, for k<0 all trajectories in R^3 are attracted by a plane (S,φ) ⊂ R^2. It means that asymptotic behaviour of trajectories is determined

Citation: Krasnyuk IB (2018) Wave Packets of Mandelbrot Type in Boundary Problems of Quantum Mechanics. J Phys Math 9: 277. doi: 10.4172/2090-0902.1000277

by solutions of two difference equations. Let us a map G:=(Φ_1, Φ_2): R^2 → R^2 has two attractive fixed points (a^+_1, a^+_3) and one saddle point $a^\pm_2$ of codimension 1 as shown on Figure 1. Then from [15-17] it follows that that S(l, t) ⇒ p_1(t) and φ(l, t)) p_2(t) as t → ∞, where $p_1(t) \in A^+_1$ and $p_2(t) \in A+_2$. Here,

A^+_1, A^+_2 are sets of attractive fixed points of maps (φ_1,φ_2), correspondingly. The functions p_1(t), p_2(t) are periodic with periods 2^N is least common multiple of periods of attractive circles of the map G.

Example 1

For example, we consider a quadratic map Φ: z → z^2+μ, where μ, z ∈ C and C is a complex space. Then from the boundary conditions follows the relations:

$$\cos S\varphi\big|x=0=\cos 2S\varphi^2+\mu\big|x=l \tag{46}$$

$$\sin S\varphi\big|x=0=\sin 2S\varphi^2+\mu\big|x=l \tag{47}$$

Where $\mu_1=\Re\mu \ and\ \mu_2=\Im\mu$. Next, from eqns. (46) and (47), we obtain that

$$\tan S_{|x=0}=\frac{\sin 2S\varphi^2+\mu_2}{\cos 2S\varphi^2+\mu_1}\quad as=l \tag{48}$$

$$y_{|x=0}=y^2+2\mu_1\cos 2Sy+2\mu_1\sin 2Sy+\mu_1^2+\mu_2^2 \tag{49}$$

Where y:=φ^2. Now linearising relation (48), (52) at a point S=0, we obtain that

$$\tan S_{|x=0}=\frac{2Sy+\mu_2}{y+\mu_1}\ as\ x=l \tag{50}$$

$$y_{|x=0}=y^2+2\mu_1 y+4\mu_2 Sy+\mu_1^2+\mu_2^2\ as\ x=l \tag{51}$$

We assume that μ_2=0. then

$$\tan S_{|x=0}=\frac{2Sy+\mu_2}{y+\mu_1}\ as\ x=l \tag{52}$$

$$y_{|x=0}=y^2+2\mu_1 y+\mu_1^2\ as\ x=l \tag{53}$$

If $\tilde{y}=y+\mu_1$, then

$$\tilde{y}_{|x=0}=\tilde{y}^2+\mu\ as\ x=l \tag{54}$$

Where μ=μ^2_1-μ_1. Hence,

$$\varphi^2(0,t)\Rightarrow\tilde{y}-\mu_1, \tag{55}$$

As t → ∞, and we obtain the difference equation

$$\tilde{y}(t)=\tilde{y}^2(t-l/p)+\mu\ as\ x=l \tag{56}$$

If μ> 1⁄4, then the map $\Theta_1:u\to u^2$ has no fixed points, so that for each u ∈ R we have $\Theta_1^i(u)\to\infty$ as t → ∞. If μ⁄le1⁄4, then the map Θ_1 has fixed points

$$\beta_{0,1}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\mu} \tag{57}$$

Where β_0 is a attractive point. In this case $\Theta_1(u)\to\infty$ for each u ∈ R J, where J=[-β_0,β_0]. If -2 ≤ μ ≤ 1⁄4, then $\Theta(J)\subset J$. If μ ≤-2, then each point u ∈ J Ω (Θ_1), where Ω (Θ_1) is a set of non wandering point (in our case this set of attractive fixed point). For μ ≤ -2, the set Ω (Θ_1) is a cantor set on J. A solution u(t) is bounded on [0,∞)if and only if u(t)∈J for each t ∈ [-1⁄p,0).

We assume that -2 <μ<1⁄4. Then interval Iε:=[-β+ε,β_0-ε]is invariant and $\bar{\Theta}_1$ (u) is closure of a set. Hence, for -3⁄4<μ<1⁄4 on J is unique attractive fixed point β1. Then each solution u(t) tends to β1 as t → ∞, where $\beta_1=\frac{1}{2}-\sqrt{\frac{1}{4}-\mu_1^2-\mu_1}$ This point exists if $\mu_1\ge-\frac{1}{2}+\frac{\sqrt{2}}{2}\ or\ \mu_1\le-\frac{1}{2}-\frac{\sqrt{2}}{2}$.

Next, eigenvalues of the Jacobi matrix at a fixed point A=(, 0) are $\chi_1=2\beta_1+2/mu_1$ and $\chi_2=\frac{2\beta_1}{\beta_1+\mu_1}$. If $|\chi_1|\le1\ and\ |\chi_2|\le1$, then a point A:=$A^+$ is attractive. Then

$$S(x,t)=S(0,t-x/p)+\frac{p}{2}x\Rightarrow p_1(t-x/p)+\frac{p}{2}x \tag{58}$$

Where $p_1(\zeta)\Rightarrow0$ as $\zeta\to\infty$. Similarly, since integral term tend to zero as e^{kt}, where k<0, as shown above, we have

$$\varphi^2(x,t)\Rightarrow\varphi^2(0,t-l/p)+\frac{p}{2}x+\frac{p}{2}(l-x/p)p_2^2(t-x/p)+\frac{p}{2}x+\frac{p}{2}(l-x) \tag{59}$$

where $p_2(\zeta)\Rightarrow\beta_1\ as\ \zeta\to\infty$.

If μ^2_1-μ_1<3⁄4, a point β_1 is repelling, but instead appears attractive circle of period 2 which contains the points $\beta_{2,3}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\mu_1^2+\mu_1}$ so that $\Omega(\Theta_1):=(\beta_1,\beta_2\beta_3)$. Then trajectories $\Theta_1^i(u)\Rightarrow(\beta_2,\beta_3)$, for almost all u ∈ J, as i → ∞.

A set $I^-:=\bigcup_{i\ge0}\Theta_1^{-i}(\beta_1)$ is finite. If u ∈ J I^-, Then trajectories tends to the points (β_2, β_3). A limit function $\Theta^*[u]:=\{\beta_2\ or\ \beta_3\}$ if u ∈ J, and $\Theta^*[u]:=[\mu,\Theta^*[/mu]]$ if u∈I^-.

In this case, each solution on the interval [-1⁄p, 0) tends to a 2^{N1}⁄p - periodic distribution, where N_1=1 (Figure 1).

Next, if $\chi_{2,3}=|2\beta_{2,3}+2\mu_1|$<1 then on a plane (S,φ) points (0,$\beta_{2,3}$ are attractive and a point (0, β_1 is a saddle type point of codimension 1. Then each initial curve γ(t) such that that t ∈ [-p⁄l, 0) can be represented as $\gamma:=\gamma_1^+\bigcup\gamma_2^+$, where iterations $G^i[\gamma_1^+]$ tends to a point (0,β_2 2, and iterations $G^i[\gamma_2^+]$ tends to a point (0,β_2 as i → ∞. Here G is produced by system of difference equations. Iterations G^i(t'), where $t'\in\gamma\bigcap W^u$ determines points of discontinuities of limit p(t):=(S*(t), y*(t)) ∈ (Γ,Γ), where W^u(.) is an unstable manifold of the saddle point.

Asymptotic of System of Difference Equations

Thus asymptotic behaviour of difference equations is known for the so-called hyperbolic maps [18]. Indeed, if the map G: R^2 → R^2, which is produced by these equations, has a finite number of fixed points A, then functions S(l, t) and φ(l, t) tends to asymptotic 2^N1⁄p periodic piecewise constant functions p_1(t) ∈ A and p_2(t) ∈ A for almost all points t ∈ (-1⁄p,1), excluding finite or infinite points of discontinuities (Figure 2). It is possible if the map G is hyperbolic. It means that the spectrum of the differential T(G) has no real points with values equal in modulus 1. If also stable manifolds $W^s(A^+)$ intersect unstable manifolds $W^u(A^-)$ and $W^u(A^+)$ transversally, and an initial curve (S_0(-1⁄p, 0), φ(-1⁄p; 0)) interact an unstable manifolds $W^u(A^\pm)$, then the map G is structural stable and hyperbolic.

Here, A+ is a set of attractive points the map G, A^- is a set of repelling points, and A^- is a set of saddle type points.

Let us define a set of non-wandering points Ω(G)=Per (G)=Fix (G^N), where Per (G) is a set of periodic points, Fix (G) S is a set

Citation: Krasnyuk IB (2018) Wave Packets of Mandelbrot Type in Boundary Problems of Quantum Mechanics. J Phys Math 9: 277. doi: 10.4172/2090-0902.1000277

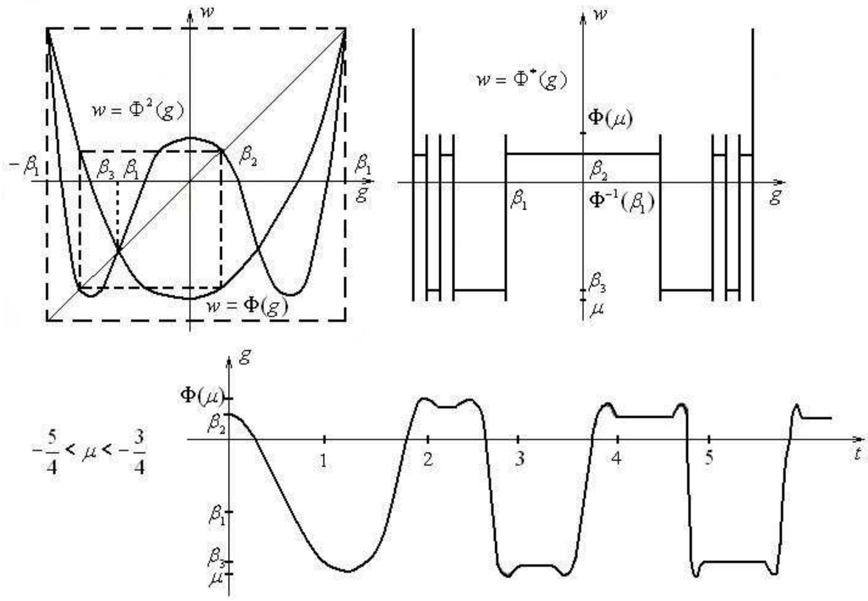

Figure 2: Limit solution of relaxation type.

of fixed points, and N=1; 2;:::. Let $W=\bigcup_{a\in\Omega(G)}W^s(a)$ where $W^s(a)=\left\{u\in W:\lim_{m\to\infty}G^{mN}(u)=a\right\}$ is a stable manifold of a fixed point a of the map G. Particularly, for any point u 2 W there is the finite limit

$$\lim_{j\to+\infty}G^{Nj}(u):=G^*(u) \tag{60}$$

Let u(t)=(S(t), φ(t)) ∈ $C^0(R^+ \to R2)$. We define a set of initial functions

$$\hat{H}=\left\{h(t)\in C^0\left(\left[-l/p,0\right],R^2\right):h(0)=G\left[h(0)\right]\right\} \tag{61}$$

Then for each function $h(t)\in\hat{H}$ there is periodic piecewise constant function p*[h(.)]:$R^+\Omega(G)$ such that

$$P^*[h(t)]=G^i[G^*(h(t-i))]=G^*[G^i(h(t-il=p))]$$

Where t∈[i,i+l/p),i=0,1,…The function p*[h(t)] is constant if and only if h(t)∈ $\hat{H}'$, where $\hat{H}'=\bigcup_{a\in Fix(G)}\hat{H}_a$, and

$$\hat{H}_a=\left\{h(t)\in\hat{H}:h(t')\in W^s(a),t'\in\left[-l/p,0\right)\right\} \tag{63}$$

and $\hat{H}_a\neq\phi$ where ϕ is empty set, if and only if a ∈ Fix (G)g.

Then each solution u(t of the system of difference equations with an initial function $u(t)_{[-l/p,0)}\in\hat{H}a$ tends to a constant a if t → +∞. Each solution u(t) of the system of difference equations with an initial function $u(t)_{[-l/p,0)}\in\hat{H}/\hat{H}'$ is asymptotic periodic function, and

$$\lim_{j\to+\infty}\left\|u(t'+Nj)-p^*\left[h(t')\right]\right\|_{R^2}=0 \tag{64}$$

where t' ∈ R^+.

$$u(t)_{[-l/p,0)}\in\hat{H}_a$$

Further, for any ε>0 and each solution u(t of the system of the difference equation such that

$$\lim_{j\to+\infty}\sup\left\|u(t+Nj)-p^*\left[u(t')\right]\right\|_{R^2}=0 \tag{65}$$

It must be noted that

$$\lim_{j\to+\infty}\sup\left\|u(t+Nj)-p^*\left[u(t')\right]\right\|_{R^2}\neq0 \tag{66}$$

Thus asymptotic solution has the form

$$\psi(x,t)=e^{ip_1(t-x/p)/h}p_2(t-x/p)+O\left(h^2\right) \tag{67}$$

where $O(h^2)$ → 0 as h^2 → 0. For example, such type solutions describe distributions of order parameter has been obtained for the Ginzburg-Landau equation with some boundary conditions for the phases and amplitudes of order parameter.

Thus it has been proved that the initial boundary value problem can be reduced to a system of integro-difference equations. These system can be, in its turn, reduced to a system of non-autonomic difference equations in R^3, where non-autonomic perturbations are produced by the phase S(x, t) on characteristics dx/dt=p. These perturbations tends to zero as t → +∞. As a result, the difference equations in R^3 are reduced to a difference equations in R^2 at (S,φ) - plane. Solutions of these equations (S,φ) are piecewise constant asymptotic periodic functions (p_1, p_2) with finite 'points' of discontinuities on a period. It should be noted that for difference equations we get asymptotic distributions from some class of initial data from C^2(I, I), but for integro-difference equations it has been proved only that if solutions of integro-difference equations exists, then these solutions tends to functions (p_1, p_2) for a special class of initial data, which belongs to a small neighbourhood of limit functions (p_1, p_2). It should be noted also that if asymptotic invariant distributions of difference equations are asymptotic invariant solutions of integro-difference equations, becomes functions (p_1, p_2) are piecewise constant asymptotic periodic functions. It should be noted also that if a limit function p2(t-x/p) is piecewise constant, the a function $\frac{(-ih)^2}{2}\Delta\varphi(t-x/p)$ in eqn. (10) tends to zero as t → +∞. It means that obtained solutions of the initial boundary value problem are accurate.

Physical sense of the boundary conditions

The boundary conditions describes the changing of phases between input S(0, t) and output phase in the electronic device the multiplier of the phase N and the amplifier J of a frequency of the input signal at a point x=l. Additionally, there is the resistor R. The device is designed so that we can change independently the phase and amplitude. The map Φ^{-1}_1 describes actions of the amplifier J and the multiplier on the input phases independently on the values of the amplitude of input signal [14,15].

Physical Applications for Mandelbrot and Julia Sets

In this section, we use the paper [19] where it the dynamics of a kicked particle which moves in a double-well potential. In this paper, it is shown that the stroboscopic dynamics of the particle can be reduced to the research of dynamics of orbits of the known complex logistic map. The logistic map is homeomorphic to the quadratic map $Q_c(z):=z^2+c$ where C ∈ C is a complex parameter. But this map has well-known Julia and Mandelbrot sets. Indeed, the map $Q_c(z)$: C → C is connected or totally disconnected [20,21]. The Mandelbrot set represent is a 'delimiter' between c - values with connected Julia and totally disconnected Julia sets [22]. The Mandelbrot set is connected [23]. They also discussed also the non-solved hypotheses that the Mandelbrot set is locally connected [20]. Next, by Brunner and Hubbard it has been studied the Mandelbrot set for cubic. There are two critical orbits for cubics [21].

For the map $\breve{Q}_c(z) := z^n + c, n = 2,3,4....,$ Roshon generalize the study of the case n=3,4. He proved that in a case n=4 the Mandelbrot set is connected. In a study [20] the cases n=2,3,4,5 has been considered. Let f:X → X any function, where X is a topological space. Let us define the Picard orbit by the recurrent formula:

$$u_n=f(u_n-1),\ n=1,2.... \tag{68}$$

Let M be the Mandelbrot set for the map $Q_c(z):=z^2 + c$ where c ∈ C such that the orbit of the point 0 is bounded so that

$$M:=\{c\in C: Q^k_c(0),\ k=0,1..., \text{ is bounded}\} \tag{69}$$

where Q^k_c (0) are iterations of the function with a condition $Q_c(0)=0$ which produce a critical orbit. Then there is known for the quadratic map that if f(c)>2 and z>c, then $f^n(z) \to \infty$ as $n \to \infty$. For example, the point c is not in the set M. As shown in a study [20], if c ∈ [-2,1⁄4), then $f^n(z)$<const. In this case, the Julia set is connected.

Conclusion

Thus an initial boundary problem for the linear Shrodinger equation with functional nonlinear two-points boundary conditions is considered. A structure of attractors of the problem has been constructed. It is shown that the attractor contains piecewise constant periodic functions with finite 'points' of discontinuities on a period. Thus phases and amplitudes of wave functions of the Shrdinger equation are piecewise constant periodic functions. The method of reduction of the problem to a system of integro-difference equations has been applied. It is shown that these equations have invariant piecewise constant periodic solution. It is proved that perturbations of such solutions are asymptotically stable.

References

1. Webb KE, Erkintalo M, Xu Y, Broderick NGR, Dudley JM, et al. (2014) Murdoch Nonlinear optics of fibre event horizons. Nature Communications 5: 4969.
2. Demircan A, Amiranashvili SH, Steinmeyer G (2011) Controlling light by light with an optical event horizon. Phys Rev Lett 106: 163901.
3. Philbin TG (2008) Fiber-optical analog of the event horizon. Science 319: 1367-1370.
4. Faccio, Laser D (2012) pulse analogues for gravity and analogue Hawking radiation. Contemp Phys 53: 97-112.
5. Sharkovsky AN, Maistrenko YL, Romanenko EY (1993) Difference Equations and Their Applications, Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands 250.
6. Belgiorno F (2010) Hawking radiation from ultrashort laser pulse filaments. Phys Rev Lett 105: 203901.
7. Dudley JM, Genty, GCoen S (2006) Supercontinuum generation in photonic crystal fiber. Rev Mod Phys 78: 1135-1184.
8. Zabolotskii AA (1998) Multiwave coherent interaction and nonlinear frequency shift. Zh Eksp Teor Fiz 94: 33-45.
9. Butylkin VS, Kaplan AE, Yu G, Khronopulo, Yakubovich EI (1977) Resonant Interaction of Light with Matter, Nauka.
10. Maimistrov AI (1999) Optical solitons. Soros Educational Journal 11: 97-102.
11. Buljan H, Segev M, Soljacic M, Efremidis NK, Christodoulides DN (2003) White-light Solitons. Optics Letters 28(14): 1239-1241.
12. Branner B, Hubbard J (1988) The iteration of cubic polynomials. Part i. The global topology of parameter space. Acta Math 66: 143-206.
13. Meskauskas T, Ivanauskas E (2002) Initial Value-Boundary Problems for Derivative Non-linear Schrodinger Equation. Justification of Two-Step Algorithm.Nonlinear Analysis: Modelling and Control 7: 69-104.
14. Young LG (1969) Lectures on the calculus of variation and optimal control theory. MIR.
15. Isaeva OB (2001) On possibility of realization of the phenomena of complex analytic dynamics for the physical systems, built of coupled elements, which demonstrate period-doublings. Izv VUZov PND (Applied Nonlinear Dynamics) 9: 129-146.
16. Romanenko, Yu E, Slusarchuk LM, Sharkovsky AN (1982) Asymptotic behaviour of bounded solutions for system of difference equations. Kiev, Institute of Mathematics NAN Ukraine, 75-86 (in Russia).
17. Krasnyuk IB (1984) On a generalization of the nonlinear hyperbolic boundary-value problems in the plane. Kiev, Institute of Mathematics NAN Ukraine, 38-42.
18. Isaeva OB, Kuznetsov SP, Ponomarenko VI (2011) Mandelbrot Set in Coupled Logistic Maps ad in Electronic Experiment. Phys Rev E 64: 055201-055209.
19. Beck C (1999) Physical meaning for Mandelbrot and Julia sets. Physica D 125: 171-182.
20. Rani M, Kumar V (2004) Journal of the Korea Society of Mathematical Education. Series D: Research in Mathematical Education 8: 279-291.
21. Roshon D (2000) A generalized Mandelbrot set for bicomplex numbers. Fractals 8: 355-368.
22. Peitgen H, Jurgens H, Saup D (1993) Fractals for the classroom. Part 2. Complex systems and Mandelbrot set, New-York-Verlag. MATHD D.
23. Shishimura M (1998) Hausdorff dimension of the boundary of the Mandelbrot sets. Ann of Math 174: 225-267.