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Verification of Some Properties of the C-nilpotent Multiplier in Lie Algebras

Sadeghieh A, Araskhan M*

Yazd Branch, Islamic Azad University, Iran

Abstract

The purpose of this paper is to obtain some inequalities and certain bounds for the dimension of the c-nilpotent multiplier of finite dimensional nilpotent Lie algebras and their factor Lie algebras. Also, we give an inequality for the dimension of the c-nilpotent multiplier of L connected with dimension of the Lie algebras $\gamma_d(L)$ and $L/Z_{d-1}(L)$. Finally, we compare our results with the previously known result.

Keywords: C-nilpotent multiplier; Nilpotent lie algebra; Lie algebra

Introduction

All Lie algebras referred to in this article are (of finite or infinite dimension) over a fixed field F and the square brackets [,] denotes the Lie product. Let 0→R→F→L→0 be a free presentation of a Lie algebra L, where F is a free Lie algebra. Then we define the, c-nilpotent multiplier $c \geq 1$, to be

$$M^{(c)}(L)=\frac{\left(R\cap\gamma_{c+1}(F)\right)}{\gamma_{c+1}(R,F)},$$

where $\gamma_{c+1}(F)$ is the (c+1)-th term of the lower central series of *F*, $\gamma_1(R,F)=1$ and $\gamma_{c+1}(R,F)=[\gamma_c(R,F),F]$. This is analogous to the definition of the Baer-invariant of a group with respect to the variety of nilpotent groups of class at most c given by Baer [1-3] (for more information on the Baer invariant of groups). The Lie algebra $M^{(1)}(L)=\left(R\cap F^2\right)/[R,F]=M(L)$ is the most studied Schur multiplier of L [4,5]. It is readily verified that the Lie algebra $M^{(c)}(L)$ is abelian and independent of the choice of the free presentation of L [6]. The purpose of this paper is to obtain some inequalities for the dimension of the c-nilpotent multiplier of finite dimensional nilpotent Lie algebras and their factor Lie algebras (Corollary 2.3 and Corollary 2.5). Finally, we compare our results to upper bound given [6]. First, we show that for each ideal N in L, there is a close relationship between the $M^{(c)}(L)$ *and* $M^{(c)}(L/N)$

Lemma 1.1. Let L be Lie algebra with a free presentation 0→R→F→L→0. If S is an ideal in F with $N\cong S/R$, , then the following sequences are exact:

(i) $0 \to \dfrac{R\cap\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)} \to M^{(c)}(L) \to M^{(c)}(L/N) \to \dfrac{N\cap\gamma_{c+1}(L)}{\gamma_{c+1}(N,L)} \to 0,$

(ii) $0 \to (N)_{ab}\otimes^c\left(L/L^2\right)_{ab} \to M^{(c)}(L) \to M^{(c)}(L/N) \to N\cap\gamma_{c+1}(L) \to 0.$

under the condition that N is central, $N\otimes^c M=N\otimes\underbrace{M\otimes\ldots\otimes M}_{c-times}$ and $(N)_{ab}=N/N^2$.

Proof. We prove only part (ii). Since N is central, $[S,F]\subseteq R$ and

$\gamma_{c+1}\left(S,F^2+R\right)\subseteq\gamma_{c+1}(S,R)+\gamma_{c+1}\left(R,F^2\right)\subseteq\gamma_{c+1}(R,F)+\gamma_{c+1}(R,[F,F])\subseteq\gamma_{c+1}(R,F).$

Now, we have the following homomorphism

$$\alpha: S/R\otimes^c F/\left(F^2+R\right)\to\frac{R\cap\gamma_{c+1}(F)}{\gamma_{c+1}(R,F)}$$

such that Imα= $\gamma_{c+1}(S,F)/\gamma_{c+1}(R,F)$. Now the result holds by part (i).

The following corollary is an immediate consequence of Lemma 1.1, which gives some elementary results about dimension of the c-nilpotent multiplier of finite dimensional Lie algebras see corollary 2.2 of Salemkar et al. [6].

Corollary 1.2. Let N be an ideal of Lie algebra L. Then

(i) $\dim M^{(c)}(L)+\dim\left(N\cap\gamma_{c+1}(L)\right)=\dim M^{(c)}(L/N)+\dim\left(\dfrac{\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)}\right),$

Where F,S,R are defined in Lemma 1.1.

(ii) $\dim M^{(c)}(L)+\dim\left(N\cap\gamma_{c+1}(L)\right)\leq\dim M^{(c)}(L/N)+\dim\left((N)_{ab}\otimes^c\left(L/L^2\right)_{ab}\right)$

Suppose that L is generated by n elements. Let F be a free Lie algebra generated by n elements and $L\cong F/R$ Witt,s formula from Bahturin et al. [7] gives us

$$\text{Dim }\gamma_d(F)/\gamma_{d+1}(F)=\frac{1}{d}\sum_{m|d}\mu(m)n^{\frac{d}{m}}=l_n(d),$$

where μ(m) is the Mobius function, defined by $\mu(1)=1$, $\mu(k)=0$ if K is divisible by a square, and $\mu(p_1\ldots p_s)=(-1)^s$ if $p_1,\ldots,p_s$ are distinct prime numbers.

Lemma 1.3. Let L be an abelian Lie algebra of dimension n. Then dim $M^{(c)}(L)=l_n(c+1)$. In particular, dim $M(L)=\frac{1}{2}n(n-1)$.

Proof. Consider a free Lie algebra F freely generated by n elements. By Witt's formula, F/F^2 is an abelian Lie algebra of dimension n, and so it is isomorphic to L. Hence dim $M^{(c)}(L)=\dim\left(\gamma_{c+1}(F)/\gamma_{c+2}(F)\right)$, which gives the result.

Let $L=\gamma_1(L)\supset\gamma_2(L)\supset\ldots\supset\gamma_c(L)\supset\gamma_{c+1}(L)=0$ be the lower central series of nilpotent Lie algebra, L. L is said to have class c if c is the least integer for which $\gamma_{c+1}(L)=0$. Furthermore, if dim $\gamma_j(L)/\gamma_{j+1}(L)=1$ for j=2,3,…,c and dim $L/\gamma_2(L)=2$, then L is said to be of maximal class c. Additionally, let $0=Z_0(L)\subset Z_1(L)\subset Z_2(L)\subset\ldots\subset Z_c(L)=L$ be the upper central series of nilpotent Lie algebra L. If L is of maximal

***Corresponding author:** Araskhan M, Yazd Branch, Islamic Azad University, Iran, Tel: +982147911; E-mail: h.araskhan@yahoo.com

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class, then $Z_i(L)=\gamma_{c-i+1}(L)$ for $0\le i\le c$.

By the above notation we have the following corollary.

Corollary 1.4. Let L be a finite dimensional nilpotent Lie algebra of maximal class (c+1), then

$$\dim M^{(c)}(L)\le \dim M^{(c)}(L/Z(L))+2^c-1$$

Proof. Using Corollary 1.2(ii) with N= Z(L), we get

$$\dim M^{(c)}(L)+\dim Z(L)=\dim M^{(c)}(L)+1\le \dim M^{(c)}(L/Z(L))+(1)2^c.$$

Discussion and Results

2 Bounds on dim $M^{(c)}(L)$

Let L be a finite dimensional nilpotent Lie algebra of class d ≥ 2. First, we give an inequality for the dimension of the c-nilpotent multiplier of L connected with dimension of the Lie algebras $\gamma_d(L)$ and $L/Z_{d-1}(L)$ (Corollary 2.3) and some inequalities for the dimension of the c-nilpotent multiplier of finite dimensional nilpotent Lie algebras will be given. For this purpose, we need the following two lemmas.

Lemma 2.1. Let H and N be ideals of Lie algebra L and $N=N_0\supseteq N_1\supseteq\ldots$, a chain of ideals of N such that $[N_i,L]\subseteq N_{i+1}$ for all I = 1,2,…. Then

$$[N_i,[H,_j L]]\subseteq N_{i+j+1} \text{ for all i, j.}$$

Proof. We have

$$\begin{aligned}[N_i,[H,_{j+1}L]]&\subseteq[N_i,[[H,_j L],L]]\\ &\subseteq[[N_i,[H,_j L]],L]+[[N_i,L],[H,_j L]]\\ &\subseteq[N_{i+j+1},L]+[N_{i+1},[H,_j L]]\\ &\subseteq N_{i+j+2}+N_{i+j+2}=N_{i+j+2}.\end{aligned}$$

Now, the assertion follows by induction on j.

Lemma 2.2. Let L be a finite dimensional nilpotent Lie algebra of class d ≥ 2. Let 0→R→F→L→0 be a free presentation of L, then $\dfrac{\gamma_{c+1}(\gamma_d(F)+R,F)}{\gamma_{c+1}(R,F)}$ is a homomorphic image of $\gamma_d(L)\otimes\underbrace{\frac{L}{Z_{d-1}(L)}\otimes\cdots\otimes\frac{L}{Z_{d-1}(L)}}_{c-times}$.

Proof. Put $Z_k(L)=T_k/R$ for $0\le k\le d$. Now consider the following chain

$$S=T_d\supseteq\ldots\supseteq T_k\supseteq T_{k-1}\supseteq\ldots\supseteq T_1\supseteq T_0=R.$$

Since $[T_k,F]\subseteq T_{k-1}$, then by Lemma 2.1,

$$[T_{d-1},[\gamma_{d-2}(F),F]]\subseteq T_{d-1-(d-2+1)}=T_0=R.$$

Therefore,

$$[\gamma_d(F)+R,\underbrace{T_{d-1},\ldots,T_{d-1}}_{c-times}]\subseteq[\gamma_d(F),\underbrace{T_{d-1},\ldots,T_{d-1}}_{c-times}]+[R,\underbrace{T_{d-1},\ldots,T_{d-1}}_{c-times}]$$

$$\subseteq[\gamma_d(F),\underbrace{T_{d-1},\ldots,T_{d-1}}_{c-times}]+\gamma_{c+1}(R,F)$$

$$\subseteq[[[T_{d-1},F],\gamma_{d-1}(F)],\underbrace{T_{d-1},\ldots,T_{d-1}}_{(c-1)-times}]$$

$$+[[[T_{d-1},[\gamma_{d-2}(F),F]],F],\underbrace{T_{d-1},\ldots,T_{d-1}}_{(c-1)-times}]+\gamma_{c+1}(R,F)$$

$$\subseteq[[T_{d-2},\gamma_{d-1}(F)],\underbrace{T_{d-1},\ldots,T_{d-1}}_{(c-1)-times}]+\gamma_{c+1}(R,F)$$

$$\vdots$$

$$\subseteq[[T_0,\gamma_1(F)],\underbrace{T_{d-1},\ldots,T_{d-1}}_{(c-1)-times}]+\gamma_{c+1}(R,F)$$

$$\subseteq\gamma_{c+1}(R,F).$$

The latter inclusion gives the following epimorphism

$$\frac{\gamma_d(F)+R}{R}\times\underbrace{\frac{F}{T_{d-1}}\times\cdots\frac{F}{T_{d-1}}}_{c-times}\to\frac{\gamma_{c+1}(\gamma_d(F)+R,F)}{\gamma_{c+1}(R,F)}$$

$$(x+R,f_1+T_{d-1},\ldots,f_c+T_{d-1})\mapsto[x,f_1,\ldots,f_c]+\gamma_{c+1}(R,F).$$

Corollary 2.3. Under the assumptions and notation of the above Lemma, we have

$$\dim M^{(\,)}(L)+\dim(\gamma_d(L)+\gamma_{c+1}(L))\le \dim M^{(c)}(L/\gamma_d(L))+\dim\gamma_d(L)\left[\dim\left(\frac{L}{Z_{d-1}(L)}\right)\right]^c.$$

Proof. In Corollary 1.2(i), taking $N=\gamma_d(L)=\dfrac{\gamma_d(F)+R}{R}$. Now by Lemma 2.2, we have

$$\dim M^{(c)}(L)+\dim(\gamma_d(L)+\gamma_{c+1}(L))=\dim M^{(c)}(L/\gamma_d(L))+\dim\left(\frac{\gamma_{c+1}(\gamma_d(F)+R,F)}{\gamma_{c+1}(R,F)}\right)$$

$$\le\dim\Big(\gamma_d(L)\otimes\underbrace{\frac{L}{Z_{d-1}(L)}\otimes\cdots\otimes\frac{L}{Z_{d-1}(L)}}_{c-times}\Big)$$

$$+\dim M^{(c)}(L/\gamma_d(L))$$

$$=\dim M^{(c)}(L/\gamma_d(L))+\dim\gamma_d(L)\left[\dim\left(\frac{L}{Z_{d-1}(L)}\right)\right]^c.$$

In following, we give another an inequality for the dimension of the c-nilpotent multiplier of finite dimensional nilpotent Lie algebras.

Theorem 2.4. Let L be a finite dimensional nilpotent Lie algebra of class ≥ 1, then

$$\dim M^{(c)}(L)\le\dim M^{(c)}(L/L^2)+\dim L^2[\dim(L/Z(L))-\dim(L/Z(L))^2]^c-\dim\big(\gamma_{c+1}(L)\big).$$

Proof. We use induction on the class of L. If L is of class 1, then $L^2=0$ and the result holds. Assume the result for nilpotent Lie algebras of class to be less than d and let L have class m= d-1. Note that $\gamma_m(L)\subseteq Z(L)$, $L^2\subseteq Z_{m-1}(L)$, $(L/\gamma_m(L))^2=L^2/\gamma_m(L)$ and $Z(L)/\gamma_m(L)\subseteq Z(L/\gamma_m(L))$. For convenience, let $A=(L/\gamma_m(L))/Z(L/\gamma_m(L))$ and $B=L/Z(L)=(L/\gamma_m(L))/(Z(L)/\gamma_m(L))$. Since A is a homomorphic image of B, it follows that $\dim A/A^2\le\dim B/B^2$. By induction,

$$\dim M^{(c)}(L/\gamma_m(L))\le\dim M^{(c)}\left((L/\gamma_m(L))/(L/\gamma_m(L))^2\right)$$

$$+\dim(L/\gamma_m(L))^2\left[\dim\left(A/A^2\right)\right]^c-\dim(\gamma_{c+1}(L/\gamma_m(L)))$$

$$\le\dim M^{(c)}(L/L^2)+\dim(L^2/\gamma_m(L))\left[\dim(B/B^2)\right]^c-\dim(\gamma_{c+1}(L/\gamma_m(L))).$$

By Corollary 2.3,

$$\dim M^{(c)}(L)\le\dim M^{(c)}(L/\gamma_m(L))+\dim(\gamma_m(L))\left[\dim(L/Z_{m-1}(L))\right]^c$$

$$-\dim(\gamma_m(L)\cap\gamma_{c+1}(L)).$$

Also, $\dim(L/Z_{m-1}(L))\le\dim\left(L/(L^2+Z(L))\right)=\dim\left(B/B^2\right)$.

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Therefore,

$$\dim M^{(c)}(L) \le \dim M^{(c)}(L/L^2) + \dim(L^2/\gamma_m(L))\left[\dim(B/B^2)\right]^c - \dim(\gamma_{c+1}(L/\gamma_m(L)))$$

$$+\dim(\gamma_m(L))\left[\dim(B/B^2)\right]^c - \dim(\gamma_m(L)\cap\gamma_{c+1}(L))$$

$$\le \dim M^{(c)}(L/L^2) + \dim L^2\left[\dim(B/B^2)\right]^c - \dim(\gamma_{c+1}(L)).$$

Since $\dim(B/B^2) \le \dim(L/L^2)$, we obtain:

Corollary 2.5. Under the assumptions and notation of the above Theorem, we have

$$\dim M^{(c)}(L) \le \dim M^{(c)}(L/L^2) + \dim L^2\left[\dim(L/L^2)\right]^c - \dim\gamma_{c+1}(L).$$

Now, we compare our results to upper bound given [7], when c = 1.

Theorem 2.6. Let L be a finite dimensional nilpotent Lie algebra of class m and d=d(m). Then

$$\dim M(L) \le \sum_{k=1}^{m} l_d(k+1).$$

Example 2.7. Let F be a free Lie algebra on 2 generators and $L = F/F^3$. Then L is a Lie algebra of 2 generators and class 2. Thus $\dim L/L^2 = l_2(1) = 2, \dim L^2/L^3 = l_2(2) = 1$ and dim L=3. By Theorem 2.6,

$$\dim M(L) \le \sum_{j=1}^{2} l_2(j+1) = l_2(2) + l_2(3) = 1 + \frac{1}{3}\left(\mu(1)2^3 + \mu(3)2\right) = 1 + \frac{1}{3}(6) = 3.$$

Note that L is a finite dimensional nilpotent Lie algebra of maximal class (1 + 1) and $Z(L)=L^2$. By Corollary 1.4 and Lemma 1.3,

$$\dim M(L) \le \dim M(L/Z(L)) + 1 = \dim M(L/L^2) + 1 = l_2(2) + 1 = 1 + 1 = 2.$$

Also, by Corollary 2.5,

$$\dim M(L) \le \dim M(L/L^2) + \dim L^2[\dim(L/L^2)] - \dim L^2 = l_2(2) + 1(2) - 1 = 2.$$

Example 2.8. Let F be a free Lie algebra on 2 generators and $L = F/F^4$. Then L is a Lie algebra of 2 generators and class 3. Thus $\dim L = 5$, $\dim L^2 = 3$ and $\dim L/L^2 = l_2(1) = 2$. By Theorem 2.6

$$\dim M(L) \le \sum_{j=1}^{3} l_2(j+1) = l_2(2) + l_2(3) + l_2(4)$$

$$= 1 + 2 + \frac{1}{4}\left(\mu(1)2^4 + \mu(2)2^2 + \mu(4)2\right)$$

$$= 1 + \frac{1}{4}(16-4) = 6.$$

Also, by Corollary 2.5,

$$\dim M(L) \le \dim M(L/L^2) + \dim L^2[\dim(L/L^2)] - \dim L^2 = l_2(2) + 3(2) - 3 = 4.$$

In this two examples, we see that our results give two better upper bounds for dim(L) than the previously known result.

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References

1. Baer R (1945) Representations of groups as quotient groups, I, II, III. Trans Amer Math Soc 58: 295-419.
2. Burns J, Ellis G (1998) Inequalities for Baer invariants of finite groups. Canad Math Bull 41: 385-391.
3. Leedham-Green CR, McKay S (1976) Baer-invariant, isologism, varietal laws and homology. Acta Math 137: 99-150
4. Batten P, Stitzinger E (1996) On covers of Lie algebras. Comm Alg 24: 4301-4317.
5. Batten P, Moneyhun K, Stitzinger E (1996) On characterizing nilpotent Lie algebras by their multipliers. Comm Alg 24: 4319-4330.
6. Salemkar AR, Edalatzadeh B, Araskhan M (2009) Some inequalities for the dimension of the c-nilpotent multiplier of Lie Algebras. J Algebra 322: 1575-1585.
7. Bahturin Y (1987) Identical relations in Lie algebras. VNU Science Press BV.