Time transformation and reversibility of Nambu–Poisson systems

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Abstract

A time transformation technique for Nambu–Poisson systems is developed, and its structural properties are examined. The approach is based on extension of the phase space $\mathscr{P}$ into $\bar{\mathscr{P}} = \mathscr{P} \times \mathbb{R}$, where the additional variable controls the time-stretching rate. It is shown that time transformation of a system on $\mathscr{P}$ can be realised as an extended system on $\bar{\mathscr{P}}$, with an extended Nambu–Poisson structure. In addition, reversible systems are studied in conjunction with the Nambu–Poisson structure. The application in mind is adaptive numerical integration by splitting of Nambu–Poisson Hamiltonians. As an example, a novel integration method for the rigid body problem is presented and analysed.

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1 Introduction

In 1973 Nambu [23] suggested a generalisation of Hamiltonian mechanics, taking the Liouville condition on volume preservation in phase space as a governing principle. Nambu postulated that the governing equations for a dynamical system on $\mathbb{R}^n$ should have the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j_1,\dots,j_{n-1}} \epsilon_{ij_1\dots j_{n-1}} \frac{\partial H_1}{\partial x_{j_1}} \frac{\partial H_2}{\partial x_{j_2}} \cdots \frac{\partial H_{n-1}}{\partial x_{j_{n-1}}} \tag{1.1}$$

where ϵ is the Levi–Civita tensor over n indices, and $H_1, \dots, H_{n-1}$ are smooth real valued functions on $\mathbb{R}^n$ called Hamiltonian functions. Notice that the vector field in equation (1.1) is source free (its divergence is zero), which implies that the corresponding phase flow is volume preserving.

Later Takhtajan [27] formalised Nambu's framework by introducing the concept of Nambu–Poisson brackets on general phase space manifolds. Based on Takhtajan's work the geometry of Nambu–Poisson structures has been explored in several papers [6, 4, 5, 21, 11, 22, 28, 29].

In this paper we study time transformation of Nambu–Poisson systems. Such transformations are important in the construction and analysis of adaptive structure preserving numerical time stepping methods [26, 10, 3, 7, 24, 18, 2, 20, 19]. The idea is to obtain time step adaptivity by equidistant discretisation in the transformed variable, which corresponds to non-equidistant discretisation in the original time variable. Although numerical integration is a main motivation, the focus in the paper is not on numerical issues, but rather on structural properties.

The current section continues with a brief review of Nambu–Poisson mechanics, and of a time transformation method by Hairer and Söderlind [9]. The main results are in Section 2, where time transformation for Nambu–Poisson systems is developed. In Section 3, the Nambu–Poisson structure is studied in conjunction with reversibility. As an application, we show in Section 4 how to construct fully explicit, adaptive numerical integration methods based on splitting of the Nambu–Poisson Hamiltonians. In particular, a novel method for the free rigid body. Conclusions are given in Section 5.

We adopt the following notation. $\mathscr{P}$ denotes a phase space manifold of dimension n, with local coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$. The algebra of smooth real valued functions on $\mathscr{P}$ is denoted $\mathcal{F}(\mathscr{P})$. Further, $\mathfrak{X}(\mathscr{P})$ denotes the linear space of vector fields on $\mathscr{P}$. The Lie derivative along $X \in \mathfrak{X}(\mathscr{P})$ is denoted $\mathcal{L}_X$. If $X, Y \in \mathfrak{X}(\mathscr{P})$ then the commutator $[X, Y] = \mathcal{L}_X Y$ supplies $\mathfrak{X}(\mathscr{P})$ with an infinite dimensional Lie algebra structure. Its corresponding Lie group is the set $\mathrm{Diff}(\mathscr{P})$ of diffeomorphisms on $\mathscr{P}$, with composition as group operation. (See McLachlan and Quispel [16] and Schmid [25] for issues concerning infinite dimensional Lie groups.) If $\Phi \in \mathrm{Diff}(\mathscr{P})$ then Φ^* denotes the pull-back map and Φ_* the push-forward map imposed by Φ.

1.1 Nambu–Poisson mechanics

In Hamiltonian mechanics, the phase space manifold $\mathscr{P}$ is equipped with a Poisson structure, defined by a bracket operation $\{\cdot_1, \cdot_2\} : \mathcal{F}(\mathscr{P}) \times \mathcal{F}(\mathscr{P}) \to \mathcal{F}(\mathscr{P})$ that is skew-symmetric, fulfils the Leibniz rule and the Jacobi identity. Nambu–Poisson mechanics is a generalisation.

Definition 1.1. A Nambu–Poisson manifold of order k consists of a smooth manifold $\mathscr{P}$ together with a multilinear map

$$\{\cdot_1, \ldots, \cdot_k\} : \underbrace{\mathcal{F}(\mathscr{P}) \times \ldots \times \mathcal{F}(\mathscr{P})}_{k \text{ times}} \to \mathcal{F}(\mathscr{P})$$

that fulfils:

- total skew-symmetry

$$\{H_1, \ldots, H_k\} = \mathrm{sgn}(\sigma)\{H_{\sigma_1}, \ldots, H_{\sigma_k}\} \tag{1.2a}$$

- Leibniz rule

$$\{GH_1, \ldots, H_k\} = G\{H_1, \ldots, H_m\} + H_1\{G, H_2, \ldots, H_k\} \tag{1.2b}$$

- fundamental identity

$$\begin{aligned}\{H_1, \ldots, H_{k-1}, \{G_1, \ldots, G_k\}\} = \{&\{H_1, \ldots, H_{k-1}, G_1\}, G_2, \ldots, G_k\} \\ &+ \{G_1, \{H_1, \ldots, H_{k-1}, G_2\}, G_3, \ldots, G_k\} + \ldots \\ &+ \{G_1, \ldots, G_{k-1}, \{H_1, \ldots, H_{k-1}, G_k\}\}\end{aligned} \tag{1.2c}$$

Remark 1.1. The case $k = 2$ coincides with ordinary Poisson manifolds.

The first two conditions, total skew-symmetry (1.2a) and Leibniz rule (1.2b), are straightforward: they imply that the bracket is of the form

$$\{H_1, \ldots, H_k\} = \eta(\,\mathrm{d}H_1, \ldots, \,\mathrm{d}H_k)$$

for some totally skew-symmetric contravariant k–tensor η [27]. The third condition, the fundamental identity (1.2c), is more intricate. The range of possible Poisson–Nambu brackets is heavily restricted by this condition [27].

A Nambu–Poisson system on a Nambu–Poisson manifold of order k is determined by $k-1$ Hamiltonian function $H_1, \ldots, H_{k-1} \in \mathcal{F}(\mathscr{P})$. The governing equations are

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{H_1, \ldots, H_{k-1}, F\} \qquad \forall\, F \in \mathcal{F}(\mathscr{P}) \tag{1.3a}$$

which may also be written

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = X_{H_1,\ldots,H_{k-1}}(\boldsymbol{x}) \tag{1.3b}$$

where $X_{H_1,\ldots,H_{k-1}} \in \mathfrak{X}(\mathscr{P})$ is defined by $\mathcal{L}_{X_{H_1,\ldots,H_{k-1}}} F = \{H_1, \ldots, H_{k-1}, F\}$. The corresponding flow map is denoted $\varphi^t_{H_1,\ldots,H_{k-1}}$. Notice that due to skew symmetry of the bracket, all the Hamiltonians $H_1, \ldots, H_{k-1}$ are first integrals, which follows from equation (1.3a).

Due to the fundamental identity (1.2c), Nambu–Poisson systems fulfil certain properties which have direct counterparts in Hamiltonian mechanics (the case $k = 2$).

Theorem 1.1 (Takhtajan [27]). *The set of first integrals of system* (1.3) *is closed under the Nambu–Poisson bracket. That is, if $G_1, \ldots, G_k$ are first integrals, then $\{G_1, \ldots, G_k\}$ is again a first integral.*

Theorem 1.2 (Takhtajan [27]). *The flow of system* (1.3) *preserves the Nambu–Poisson structure. That is,*

$$\{G_1, \ldots, G_k\} \circ \varphi^t_{H_1,\ldots,H_{k-1}} = \{G_1 \circ \varphi^t_{H_1,\ldots,H_{k-1}}, \ldots, G_n \circ \varphi^t_{H_1,\ldots,H_{k-1}}\} \qquad \forall\, G_1, \ldots, G_k \in \mathcal{F}(\mathscr{P})$$

or equivalently

$$\mathcal{L}_{X_{H_1,\ldots,H_{k-1}}} \eta = 0 \tag{1.4}$$

Remark 1.2. The set of vector fields that fulfils equation (1.4) is denoted $\mathfrak{X}_\eta(\mathscr{P})$. Clearly $\mathfrak{X}_\eta(\mathscr{P})$ is closed under linear combinations, so it is a sub-space of $\mathfrak{X}(\mathscr{P})$. Further, since $\mathcal{L}_{[X,Y]}\eta = \mathcal{L}_X(\mathcal{L}_Y \eta) - \mathcal{L}_Y(\mathcal{L}_X \eta)$ it is also closed under the commutator. Thus, $\mathfrak{X}_\eta(\mathscr{P})$ is a Lie sub-algebra of $\mathfrak{X}(\mathscr{P})$. Correspondingly, $\mathrm{Diff}_\eta(\mathscr{P})$ denotes the Lie sub-group of $\mathrm{Diff}(\mathscr{P})$ that preserves the Nambu–Poisson structure. An element $\Phi \in \mathrm{Diff}_\eta(\mathscr{P})$ is called an η–map.

Remark 1.3. It is important to point out that in general not every $X \in \mathfrak{X}_\eta(\mathscr{P})$ corresponds to a Nambu–Poisson system, i.e., a system of the form of equation (1.3). The reason is that the set of vector fields of the form of equation (1.3) is not closed under linear combinations.

There are also fundamental differences between Hamiltonian and Nambu–Poisson mechanics, i.e., between $k = 2$ and $k \geq 3$. In particular there is the following result, conjectured by Chatterjee and Takhtajan [4] and later proved by several authors.

Theorem 1.3 ([6, 1, 22, 11, 13]). *A totally skew-symmetric contravariant tensor of order $k \geq 3$ is a Nambu–Poisson tensor if and only if it is locally decomposable about any regular point. That is, about any point $\boldsymbol{x} \in \mathscr{P}$ such that $\eta(\boldsymbol{x}) \neq 0$ there exist local coordinates $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)$ such that*

$$\eta = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_k}$$

Thus, every Nambu–Poisson tensor with $k \geq 3$ is in essence a determinant on a sub-manifold of dimension k. It is not so for Poisson tensors.

1.2 Time transformation of dynamical systems

In this section we review the time transformation technique developed in Hairer and Söderlind [9]. Consider a dynamical system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = X(\boldsymbol{x}), \qquad X \in \mathfrak{X}(\mathscr{P}) \tag{1.5}$$

Its flow map is denoted φ_X^t. Introduce an extended phase space $\bar{\mathscr{P}} = \mathscr{P} \times \mathbb{R}$, with local coordinates $\bar{\boldsymbol{x}} = (\boldsymbol{x}, \xi)$. The projection $\bar{\mathscr{P}} \ni \bar{\boldsymbol{x}} \mapsto \boldsymbol{x} \in \mathscr{P}$ is denoted Π, and $\bar{\mathscr{P}} \ni \bar{\boldsymbol{x}} \mapsto \xi \in \mathbb{R}$ is denoted π. Let $Q \in \{F \in \mathcal{F}(\mathscr{P}); F > 0\}$ and consider the extension of system (1.5) into

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = X(\boldsymbol{x})/\xi \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}\tau} = (\mathcal{L}_X Q)(\boldsymbol{x})/Q(\boldsymbol{x}) \end{cases} \qquad \text{or shorter} \qquad \frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}\tau} = \bar{X}(\bar{\boldsymbol{x}}) \tag{1.6}$$

The flows of the two systems are related by a reparametrisation $t \leftrightarrow \tau$.

Theorem 1.4 (Hairer and Söderlind [9])**.** *The flow of the extended system* (1.6) *restricted to* $\mathscr{P}$ *is a time transformation of the flow of system* (1.5)*. That is,*

$$\Pi\varphi_{\bar{X}}^{\tau}(\bar{\boldsymbol{x}}) = \varphi_X^{\sigma(\tau,\bar{\boldsymbol{x}})}(\boldsymbol{x}), \quad \forall\, \bar{\boldsymbol{x}} \in \bar{\mathscr{P}}, \tau \in \mathbb{R} \qquad \textit{where} \qquad \sigma(\tau,\bar{\boldsymbol{x}}) \equiv \int_0^{\tau} \frac{\mathrm{d}s}{\pi\varphi_{\bar{X}}^{s}(\bar{\boldsymbol{x}})}$$

Further, $Q(\boldsymbol{x})/\xi$ *is a first integral of system* (1.6)*.*

Proof. From equation (1.6) it follows directly that $\Pi_*\bar{X}$ is parallel with X. Thus, $\Pi\varphi_{\bar{X}}^{\tau}$ and φ_X^t define the same phase diagrams. It remains to find the relation between t and τ. Since $\mathrm{d}x/\mathrm{d}t = (\mathrm{d}t/\mathrm{d}\tau)(\mathrm{d}x/\mathrm{d}\tau)$ it follows from equation (1.6) that $\mathrm{d}t/\mathrm{d}\tau = 1/\xi$. Integration of this relation gives $\sigma(\tau, \bar{\boldsymbol{x}})$. Further, straightforward calculations and utilisation of the governing equations (1.6) show that $\mathrm{d}(Q(\boldsymbol{x})/\xi)/\mathrm{d}\tau = 0$. □

Remark 1.4. It is clear that the time transformation is determined by Q. Since Q is strictly positive, the map $\sigma(\,\cdot\,, \bar{\boldsymbol{x}}) : \mathbb{R} \to \mathbb{R}$ is bijective, i.e., the reparametrisation $t \leftrightarrow \tau$ is bijective.

In Hairer and Söderlind [9], the motivation for the extended time transformation (1.6) is to construct explicit adaptive numerical integrators for *reversible systems*. The key is that under reversibility of Q, the extended time transformation (1.6) preserves reversibility. First, recall the basic definitions of reversible systems.

Definition 1.2. Let $R \in \mathrm{Diff}(\mathscr{P})$.

- A vector field $X \in \mathfrak{X}(\mathscr{P})$ is called reversible with respect to R if $R_* \circ X = -X \circ R$, or equivalently $\mathrm{d}(R(\boldsymbol{x}))/\mathrm{d}t = -(X \circ R)(\boldsymbol{x})$.
- A diffeomorphism $\Phi \in \mathrm{Diff}(\mathscr{P})$ is called reversible with respect to R if $R \circ \Phi = \Phi^{-1} \circ R$.

It is a well known result that the flow of a system is reversible if and only if its corresponding vector field is reversible [12, 8]. Now, concerning time transformation of reversible systems, it is straightforward to check the following result.

Theorem 1.5 (Hairer and Söderlind [9])**.** *If* $X \in \mathfrak{X}(\mathscr{P})$ *is reversible with respect to* R *and* $Q \in \mathcal{F}(\mathscr{P})$ *fulfils* $Q = Q \circ R$*, then the vector field* $\bar{X} \in \mathfrak{X}(\bar{\mathscr{P}})$ *in equation* (1.6) *is reversible with respect to* $\bar{R} : \bar{\boldsymbol{x}} \mapsto (R(\boldsymbol{x}), \xi)$*.*

2 Nambu–Poisson extensions and time transformations

In this section we develop a time transformation technique for Nambu–Poisson systems. Let $\mathscr{P}$ be a Nambu–Poisson manifold of order k and η its Nambu–Poisson tensor. Consider again the extended phase space $\bar{\mathscr{P}} = \mathscr{P} \times \mathbb{R}$. Our first goal is to introduce a Nambu–Poisson structure on $\bar{\mathscr{P}}$. The most natural extension of the Nambu–Poisson tensor η is given by

$$\bar{\eta} = \eta \wedge \frac{\partial}{\partial \xi} \tag{2.1}$$

It is not obvious that the bracket corresponding to $\bar{\eta}$ will fulfil the fundamental identity (1.2c). For example, in the canonical Poisson case, i.e., $k = 2$, it is not so if $n \geq 3$.

Lemma 2.1. *If $k \geq 3$ or $k = n = 2$, then $\bar{\eta}$ given by equation (2.1) defines a Nambu–Poisson structure of order $k+1$ on $\bar{\mathscr{P}}$.*

Proof. If $k \geq 3$ then it follows from Theorem 1.3 that η is decomposable about its regular points, and when $k = n = 2$ it is obviously so. Thus, $\eta \wedge \frac{\partial}{\partial \xi}$ is also decomposable about its regular points, so the assertion follows from Theorem 1.3. □

The bracket associated with $\bar{\eta}$ is denoted $\bar{\{}\cdot, \ldots, \cdot\bar{\}}$.

Let $H_1, \ldots, H_{k-1} \in \mathcal{F}(\mathscr{P})$ be the Hamiltonians for a Nambu–Poisson system on $\mathscr{P}$, i.e., of the form of system (1.3). Further, let $G \in \mathcal{F}(\bar{\mathscr{P}})$ and consider the system on $\bar{\mathscr{P}}$ given by

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = \bar{\{}H_1, \ldots, H_{k-1}, G, F\bar{\}}, \quad \forall\, F \in \mathcal{F}(\bar{\mathscr{P}}) \tag{2.2}$$

Remark 2.1. A functions $H \in \mathcal{F}(\mathscr{P})$ is considered to belong to $\mathcal{F}(\bar{\mathscr{P}})$ by the natural extension $\bar{\boldsymbol{x}} \mapsto H(\boldsymbol{x})$. Likewise, $\bar{H} \in \mathcal{F}(\bar{\mathscr{P}})$ is considered to be a function in $\mathcal{F}(\mathscr{P})$ depending on the parameter ξ. Thus, $\bar{\{}\cdot, \ldots, \cdot\bar{\}}$ is defined also for elements in $\mathcal{F}(\mathscr{P})$ and vice versa.

We continue with the main result in the paper. It states that time transformation of a Nambu–Poisson system can be realised as an extended Nambu–Poisson system.

Theorem 2.1. *Let $G \in \mathcal{F}(\bar{\mathscr{P}})$ and assume the conditions in Lemma 2.1 are valid. Then:*

1. *The extended system (2.2) is a Nambu–Poisson system.*
2. *Its flow restricted to $\mathscr{P}$ is a time transformation, determined by the additional first integral G, of the flow of system (1.3). That is,*

$$\Pi\varphi^{\tau}_{H_1,\ldots,H_{k-1},G}(\bar{\boldsymbol{x}}) = \varphi^{\sigma(\tau,\bar{\boldsymbol{x}})}_{H_1,\ldots,H_{k-1}}(\boldsymbol{x}), \quad \forall\, \bar{\boldsymbol{x}} \in \bar{\mathscr{P}}, \tau \in \mathbb{R}$$

where

$$\sigma(\tau, \bar{\boldsymbol{x}}) \equiv \int_0^{\tau} \frac{\partial G}{\partial \xi}\big(\varphi^{s}_{H_1,\ldots,H_{k-1},G}(\bar{\boldsymbol{x}})\big)\,\mathrm{d}s$$

Proof. The first assertion follows directly from Lemma 2.1, since $\bar{\eta}$ is a Nambu–Poisson tensor. Since H_i for $i = 1, \ldots, k-1$ are independent of ξ, it follows from the definition (2.1) of $\bar{\eta}$ that

$$\bar{\{}H_1, \ldots, H_{k-1}, G, F\bar{\}} = \frac{\partial G}{\partial \xi}\{H_1, \ldots, H_{k-1}, F\} - \frac{\partial F}{\partial \xi}\{H_1, \ldots, H_{k-1}, G\}$$

Thus, for $F = x_1, \ldots, x_n$, the governing equations (2.2) are parallel with those of system (1.3a), i.e., $\Pi\varphi^{\tau}_{H_1,\ldots,H_{k-1},G}$ and $\varphi^{t}_{H_1,\ldots,H_{k-1}}$ defined the same phase diagram. The relation between τ and t is given by $\mathrm{d}t/\mathrm{d}\tau = \partial G/\partial\xi$, which, after integration, gives the desired form of $\sigma(\tau, \bar{\boldsymbol{x}})$. □

It is straightforward to check the following corollary, which shows that the technique used by Hairer and Söderlind [9], reviewed in Section 1.2, is a special case.

Corollary 2.1. *The case* $G(\bar{\boldsymbol{x}}) = \log(\xi/Q(\boldsymbol{x}))$ *coincides with the transformation* (1.6) *applied to system* (1.3)*.*

3 Reversible Nambu–Poisson systems

Recall that the time transformation by Hairer and Söderlind [9] is developed with reversible systems in mind. In the previous section we developed a similar approach, but based on the Nambu–Poisson framework. One may ask under what conditions a Nambu–Poisson system is reversible, and in what sense the time transformation technique developed above preserves reversibility. These questions are studied in this section.

As a first step, we have some results on necessary and sufficient conditions for a Nambu–Poisson system to be reversible.

Proposition 3.1. *Let* $R \in \mathrm{Diff}(\mathscr{P})$*. Then* $X_{H_1,\ldots,H_{k-1}}$ *is reversible with respect to* R *if and only if*

$$\{H_1, \ldots, H_{k-1}, F \circ R\} = -\{H_1, \ldots, H_{k-1}, F\} \circ R, \quad \forall\, F \in \mathcal{F}(\mathscr{P}) \tag{3.1}$$

Proof. Since R is a diffeomorphism it holds that $\mathcal{F}(\mathscr{P}) \circ R = \mathcal{F}(\mathscr{P})$, so the governing equations (1.3a) are equivalent to

$$\frac{\mathrm{d}(F \circ R)}{\mathrm{d}t} = \{H_1, \ldots, H_{k-1}, F \circ R\}, \quad \forall\, F \in \mathcal{F}(\mathscr{P})$$

This is equivalent to

$$\frac{\mathrm{d}(F \circ R)}{\mathrm{d}t} = -\{H_1, \ldots, H_{k-1}, F\} \circ R, \quad \forall\, F \in \mathcal{F}(\mathscr{P})$$

if and only if condition (3.1) holds. The last set of equations is exactly the condition on $X_{H_1,\ldots,H_{k-1}}$ for reversibility with respect to R. □

If R is a Nambu–Poisson map the assertion may be stated in the following way instead.

Corollary 3.1. *Let* R *be a Nambu–Poisson map, i.e.,* $R \in \mathrm{Diff}_\eta(\mathscr{P})$*. Then* $X_{H_1,\ldots,H_{k-1}}$ *is reversible with respect to* R *if and only if*

$$\{H_1, \ldots, H_{k-1}, F\} = -\{H_1 \circ R, \ldots, H_{k-1} \circ R, F\}, \quad \forall\, F \in \mathcal{F}(\mathscr{P}) \tag{3.2}$$

Proof. With F set to $F \circ R$, it is clear that the condition (3.2) is equivalent to the condition (3.1) if $R \in \mathrm{Diff}_\eta(\mathscr{P})$. □

As a generalisation of Theorem 1.5, we now show in what way reversibility of a Nambu–Poisson system is preserved by the time transformed extended system (2.2).

Theorem 3.1. *Let the system* (1.3) *be reversible with respect to R. Then the extended time transformed Nambu–Poisson system* (2.2) *is reversible with respect to $\bar{R} : \bar{\boldsymbol{x}} \mapsto (R(\boldsymbol{x}), \xi)$ if $G \circ \bar{R} = G$.*

Proof. Since $\partial H_i/\partial\xi = 0$ we have

$$\bar{\{}H_1, \ldots, H_{k-1}, G, F \circ \bar{R}\bar{\}} = \frac{\partial G}{\partial \xi}\{H_1, \ldots, H_{k-1}, F \circ \bar{R}\} - \frac{\partial (F \circ \bar{R})}{\partial \xi}\{H_1, \ldots, H_{k-1}, G\}$$

Since $\bar{R}$ maps ξ to ξ it holds that $\partial(F \circ \bar{R})/\partial\xi = \partial F/\partial\xi \circ \bar{R}$. Further, $G = G \circ \bar{R}$ yields $\partial G/\partial\xi = \partial G/\partial\xi \circ \bar{R}$ and

$$\{H_1, \ldots, H_{k-1}, G\} = \{H_1, \ldots, H_{k-1}, G \circ \bar{R}\}$$

Altogether we now have

$$\begin{aligned}
\bar{\{}H_1, \ldots, H_{k-1}, G, F \circ \bar{R}\bar{\}} &= \frac{\partial G}{\partial \xi} \circ \bar{R}\{H_1, \ldots, H_{k-1}, F \circ \bar{R}\} - \frac{\partial F}{\partial \xi} \circ \bar{R}\{H_1, \ldots, H_{k-1}, G \circ \bar{R}\} \\
&= -\frac{\partial G}{\partial \xi} \circ \bar{R}\{H_1, \ldots, H_{k-1}, F\} \circ \bar{R} + \frac{\partial F}{\partial \xi} \circ \bar{R}\{H_1, \ldots, H_{k-1}, G\} \circ \bar{R} \\
&= -\bar{\{}H_1, \ldots, H_{k-1}, G, F\bar{\}} \circ \bar{R}
\end{aligned}$$

where the stipulation that system (1.3) is reversible have been used in conjunction with Proposition 3.1. Application of Proposition 3.1 again completes the assertion. □

4 Application: numerical integration by splitting

The main motivation for extended time transformations is to construct adaptive numerical integration algorithms. By a *numerical integrator* for a dynamical system $X \in \mathfrak{X}(\mathscr{P})$, we mean a family of near identity maps $\Phi_h \in \mathrm{Diff}(\mathscr{P})$, such that Φ_h is an approximation of the exact flow φ_X^h. Numerical solution "paths" are obtained by the discrete dynamical system $\boldsymbol{x}_{k+1} = \Phi_h(\boldsymbol{x}_k)$. The integrator Φ_h is *consistent of order* p if $\Phi_h - \varphi^h = \mathcal{O}(h^p)$, which in particular implies $\Phi_0 = \mathrm{Id}$. It is *explicit* if $\Phi_h(\boldsymbol{x})$ can be computed by a finite algorithm. Notice that Φ_h is not a one parameter group, i.e., $\Phi_h \circ \Phi_s \neq \Phi_{h+s}$.

When constructing numerical integrators, one typically tries to preserve as much as possible of the underlying qualitative structure of the exact flow. In our case, we like Φ_h to preserve the Nambu–Poisson structure, and in presence also reversibility. In addition, time step adaptivity is crucial in order for the integration method to be computationally efficient. Indeed, we would like to vary the step size h during the integration process according to the present local character of the dynamics, without destroying the structural properties of the method. The standard approach, motivating our work, is to utilise a time transformation $t \leftrightarrow \tau$ that preserves the structure of the original system, and then construct a τ–equidistant numerical integrator for transformed system. An equivalent view point is to say that the time transformation should regularise the problem, so that it becomes easier to integrate numerically.

Splitting is a compelling technique for the construction of structure preserving integrators [17]. The basic idea is as follows. Let $\mathfrak{X}_A(\mathscr{P})$ be a Lie sub-algebra of $\mathfrak{X}(\mathscr{P})$, and let $\mathrm{Diff}_A(\mathscr{P})$ be the corresponding Lie sub-group of $\mathrm{Diff}(\mathscr{P})$. Assume that $X \in \mathfrak{X}_A(\mathscr{P})$ can be splitted into explicitly integrable sub-system, each of which is a system in $\mathfrak{X}_A(\mathscr{P})$. That is, $X = X_1 + \ldots + X_k$, where $X_i \in \mathfrak{X}_A(\mathscr{P})$ and $\varphi^t_{X_i}(\boldsymbol{x})$ can be computed explicitly. A numerical integrator for X is obtained by $\Phi_h = \varphi^h_{X_1} \circ \cdots \circ \varphi^h_{X_k}$. It is clear that Φ_h is an approximation of φ^h_X, and that $\Phi_h \in \mathrm{Diff}_A(\mathscr{P})$. Further, by the Baker–Campbell–Hausdorff (BCH) formula, it follows that Φ_h is the exact flow of a modified vector field $\tilde{X}_h \in \mathfrak{X}_A(\mathscr{P})$, i.e., $\Phi_h = \varphi^h_{\tilde{X}_h}$. This information is crucial for the analysis of Φ_h. For example, if $\mathfrak{X}_A(\mathscr{P})$ is the Lie-algebra corresponding to a Poisson structure on $\mathscr{P}$, then Φ_h will exactly conserve a modified Hamiltonian, which is $\mathcal{O}(h^p)$–close to the Hamiltonian for the original problem [8].

Remark 4.1. Due to convergence issues, the BCH formula needs to be truncated, which implies that assertions on Φ_h, coming from $\tilde{X}$, are valid only for exponentially long times, i.e., up to time scales of order $\mathcal{O}(\exp(\mathcal{O}(1/h^p)))$. See Hairer et. al. [8] for details.

Our notion for the construction of integrators is to utilise the results in Section 2–3, and consider splitting of the individual Nambu–Poisson Hamiltonians.

Let η be a Nambu–Poisson tensor. The set of Nambu–Poisson maps which are reversible with respect to R is denoted $\mathrm{Diff}_{\eta,R}(\mathscr{P})$. If $\Phi, \Psi \in \mathrm{Diff}_{\eta,R}(\mathscr{P})$, then in general we have

$$R \circ \Phi \circ \Psi = \Phi^{-1} \circ R \circ \Psi = \Phi^{-1} \circ \Psi^{-1} \circ R \neq (\Phi \circ \Psi)^{-1} \circ R$$

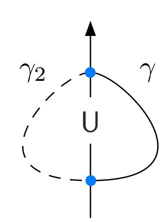
Thus, $\mathrm{Diff}_{\eta,R}(\mathscr{P})$ is not closed under composition, so it is not a sub-group of $\mathrm{Diff}(\mathscr{P})$. However, $\mathrm{Diff}_{\eta,R}(\mathscr{P})$ is closed under the symmetric group operation $(\Phi, \Psi) \mapsto \sqrt{\Phi} \circ \Psi \circ \sqrt{\Phi}$, which we write as $\Phi \circledcirc \Psi$. Further, from the symmetric BCH formula (cf. [15]) it follows that if $X, Y \in \mathfrak{X}_{\eta,R}(\mathscr{P})$, then the vector field Z such that $\varphi^t_Z = \varphi^t_X \circledcirc \varphi^t_Y$ belongs to $\mathfrak{X}_{\eta,R}(\mathscr{P})$.

Remark 4.2. For near identity maps, $\sqrt{\Phi}$ is defined by taking its representation $\Phi = \exp(X)$ and then setting $\sqrt{\Phi} = \exp(X/2)$. In our case, Φ will always be an exact flow φ^t_X, in which case $\sqrt{\varphi^t_X} = \varphi^{t/2}_X$.

We now give a result concerning reversible systems, which is of use for the analysis of periodic numerical paths of reversible splitting methods.

Lemma 4.1. *Let $X \in \mathfrak{X}(\mathscr{P})$ be reversible with respect to $R \in \mathrm{Diff}(\mathscr{P})$. Assume that the set $\mathsf{U} = \{\boldsymbol{x} \in \mathscr{P}; R(\boldsymbol{x}) = \boldsymbol{x}\}$ of fix-points of R is non-empty and that $\gamma : \mathbb{R} \to \mathscr{P}$ is a solution curve of X for which there exists $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ such that $\gamma(t_1), \gamma(t_2) \in \mathsf{U}$. Then γ is periodic.*

Proof. For simplicity assume that $t_1 = 0$ and $t_2 > 0$, which is not a restriction. The curve $\gamma_2(t) = (R \circ \gamma)(-t)$ is also a solution curve due to reversibility. Further, since R restricted to U is the identity map we have the equalities $\gamma(t_1) = \gamma_2(t_1)$ and $\gamma(-t_2) = \gamma_2(t_2)$. Due to uniqueness of solutions the first equality implies $\gamma_2 = \gamma$, which in conjunction with the second equality implies that $\gamma(-t_2) = \gamma(t_2)$. Thus γ returns to the same point twice, so it is periodic.



4.1 Rigid body problem

The Euler equations for the free rigid body is a Nambu–Poisson system on the phase space $\mathbb{R}^3$, equipped with the canonical Nambu–Poisson structure $\eta = \partial/\partial x_1 \wedge \partial/\partial x_2 \wedge \partial/\partial x_3$. Its two

Hamiltonians are total angular momentum $M(\boldsymbol{x}) = \sum x_i^2/2$ and kinetic energy $T(\boldsymbol{x}) = \sum x_i^2/(2I_i)$, where $I_i > 0$ are the principal moments of inertia. Thus, the governing equations are

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{M, T, F\}, \quad \forall\, F \in \mathcal{F}(\mathbb{R}^3) \tag{4.1a}$$

which explicitly reads

$$\begin{aligned} \dot{x}_1 &= a_1 x_2 x_3, \quad a_1 = (I_2 - I_3)/(I_2 I_3) \\ \dot{x}_2 &= a_2 x_3 x_1, \quad a_2 = (I_3 - I_1)/(I_3 I_1) \\ \dot{x}_3 &= a_3 x_1 x_2, \quad a_3 = (I_1 - I_2)/(I_1 I_2) \end{aligned} \tag{4.1b}$$

It is straightforward to check that the system is reversible with respect to the linear diffeomorphism $R_1 : \boldsymbol{x} \mapsto (-x_1, x_2, x_3)$, and in symmetry, also with respect to R_2, R_3 defined analogously. Thus, due to Lemma 4.1, we have the following KAM–like result for the free rigid body.

Theorem 4.1. *Let $\tilde{X}_h \in \mathfrak{X}(\mathbb{R}^3)$ depend smoothly on h such that $\tilde{X}_0 = X_{M,T} \neq 0$. Assume that $\tilde{X}_h$, for each h, is reversible with respect to R_1, R_2 and R_3. Then, for small enough h, the solution paths of $\tilde{X}_h$ are periodic.*

Proof. It is known that if γ is a solution curve of the Euler equations, then it is either an equilibrium, or it is periodic with finite period $t_e > 0$, in which case it crosses either of the planes $\mathsf{U_i} = \{\boldsymbol{x} \in \mathbb{R}^3; R_i(\boldsymbol{x}) = \boldsymbol{x}\}$ every half period [14]. That is, it holds that $\gamma(t_1), \gamma(t_1 + t_e/2) \in \mathsf{U}_k$ for some $k \in \{1, 2, 3\}$ and $t_1 \in [0, t_e/2)$. Further, since $X_{M,T} \equiv 0$ is not allowed, it is known that if γ is an equilibrium, then $\gamma(t) \in \mathsf{U}_k$. Let $\tilde{\gamma}_h$ be a solution curve of $\tilde{X}_h$ and let γ be the solution curve of $X_{M,T}$ such that $\gamma(0) = \tilde{\gamma}_h(0)$. Assume first that γ is not an equilibrium. Then, for any $\delta \in (0, t_e/2)$ it holds that a continuous path between $\gamma(t_1 - \delta)$ and $\gamma(t_1 + \delta)$ must cross the plane U_k. For small enough h it holds that $\tilde{\gamma}_h(t_1 - \delta)$ and $\tilde{\gamma}_h(t_1 + \delta)$ approximates $\gamma(t_1 - \delta)$ and $\gamma(t_1 + \delta)$ well enough to also be separated by U_k. Thus, $\tilde{\gamma}_h(\tilde{t}_1) \in \mathsf{U}_1$ for some $\tilde{t}_1 \in (t_1 - \delta, t_1 + \delta)$. Likewise, $\tilde{\gamma}_h(\tilde{t}_2) \in \mathsf{U}_1$ for some $\tilde{t}_2 \in (t_1 + t_e/2 - \delta, t_1 + t_e/2 + \delta)$. Since $\tilde{X}_h$ is reversible with respect to R_1 it follows from Lemma 4.1 that $\tilde{\gamma}_h$ is periodic. If γ is an equilibrium and $\tilde{\gamma}_h$ is not, then either there exists $s > 0$ such that $\tilde{\gamma}_h(s) \notin \mathsf{U}_k$, in which case the solution curve of $X_{M,T}$ such that $\gamma(s) = \tilde{\gamma}_h(s)$ is periodic, so we are back to the first case, or $\tilde{\gamma}_h(0), \tilde{\gamma}_h(s) \in \mathsf{U}_k$, in which case the assertion follows directly from Lemma 4.1. □

The traditional perception in the literature is to view the rigid body equations (4.1) as a Poisson system, with the non-canonical Poisson tensor $\eta_M = \eta(\,\mathrm{d}M, \cdot_1, \cdot_2)$, induced by the total angular momentum (M is a Casimir, cf. [14], for this Poisson structure). We denote the corresponding bracket by $\{\cdot_1, \cdot_2\}_M$. It is clear that Diff_{η_M} is a sub-group of Diff_η. Consider the Hamiltonian splitting $T = \sum T_i$, where $T_i(\boldsymbol{x}) = x_i^2/(2I_i)$. The sub-system $\dot{F} = \{T_i, F\}_M$ does not affect x_i, i.e., $\dot{x}_i = 0$, and all the quadratic terms contain x_i. Hence, it is in essence a linear system on $\mathbb{R}^2$, and therefore explicitly integrable (since the exponential map is computable for any 2×2–matrix). A well known second order integrator is obtained by the symmetric composition

$$\Phi_h^{\curlyvee T} = \varphi^h_{M,T_1} \circledcirc \varphi^h_{M,T_2} \circledcirc \varphi^h_{M,T_3}$$

This integrator has the following properties:

1. It is reversible with respect to R_1, R_2 and R_3. Thus, its modified vector field $\tilde{X}_h$ is a R_1, R_2, R_3–reversible perturbation of X, so Theorem 4.1 may be used to deduce periodic orbits of the numerical solution.

2. It is a Poisson map, i.e., $\Phi_h^{\curlyvee T} \in \mathrm{Diff}_{\eta_M}(\mathscr{P})$. This implies that its modified vector field $\tilde{X}_h$ is the Hamiltonian vector field of a modified Hamiltonian $\tilde{T}_h = T + \mathcal{O}(h^2)$, so T is nearly conserved. Further, since M is a Casimir of the Poisson structure it is exactly conserved.

Remark 4.3. One may also view the rigid body equations (4.1) as a Poisson system with the Poisson tensor $\eta_T = \eta(\cdot_1, \mathrm{d}T, \cdot_2)$, and then construct an integrator $\Phi_h^{\curlyvee M}$ by splitting of M. This integrator will exactly conserve T, and nearly conserve M.

Following our notion, we now consider Hamiltonian splitting of both M and T. To this end, let $M_i(\boldsymbol{x}) = x_i^2/2$. Since $X_{M_i,T_i} = 0$ it follows that

$$X_{M,T} = X_{M_1+M_2,T_1+T_2} + X_{M_2+M_3,T_2+T_3} + X_{M_3+M_1,T_3+T_1}$$

Each such vector field is integrable by linear extrapolation, for example,

$$\varphi^t_{M_1+M_2,T_1+T_2}(\boldsymbol{x}) = \boldsymbol{x} + tX_{M_1+M_2,T_1+T_2}(\boldsymbol{x})$$

Thus, a second order integrator is obtained by

$$\Phi_h^{\curlyvee M \curlyvee T} = \varphi^h_{M_1+M_2,T_1+T_2} \circledcirc \varphi^h_{M_2+M_3,T_2+T_3} \circledcirc \varphi^h_{M_3+M_1,T_3+T_1}$$

This integrator is computationally cheaper than $\Phi_h^{\curlyvee T}$, since computation of the exponential map, which involves evaluation of sin and cos, is not necessary. Further, it has the following properties:

1. It is reversible with respect to R_1, R_2 and R_3. Thus, its modified vector field $\tilde{X}_h$ is a R_1, R_2, R_3–reversible perturbation of $X_{M,T}$, so Theorem 4.1 may be used to deduce periodic orbits of the numerical solution.
2. It is an η–map, i.e., $\Phi_h^{\curlyvee M \curlyvee T} \in \mathrm{Diff}_\eta$, which implies $\tilde{X} \in \mathfrak{X}_\eta(\mathscr{P})$. However, $\Phi_h^{\curlyvee M \curlyvee T}$ does not correspond to a modified Nambu–Poisson system (see Remark 1.3), so there are no exactly conserved modified Hamiltonians $\tilde{M}$ and $\tilde{T}$. Nevertheless, M and T are still nearly conserved due to the periodicity of the numerical solution.

Consider now time transformation of system (4.1) into an extended Nambu–Poisson system

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = \bar{\{} M, T, G, F \bar{\}}, \quad \forall F \in \mathcal{F}(\mathbb{R}^4) \tag{4.2}$$

We have the following generalisation of Theorem 4.1.

Theorem 4.2. *Let $\tilde{X}_\epsilon \in \mathfrak{X}(\mathbb{R}^4)$ depend smoothly on ϵ such that $\tilde{X}_0 = X_{M,T,G} \neq 0$. Assume that $\tilde{X}_\epsilon$, for each ϵ, is reversible with respect to $\bar{R}_1$, $\bar{R}_2$ and $\bar{R}_3$, and that there exists $\delta > 0$ such that $\partial G/\partial \xi > \delta$. Then, for small enough ϵ, the solution paths of $\tilde{X}_\epsilon$ are periodic.*

Proof. From the definition of $\bar{R}_i$ it follows that $\bar{\mathsf{U}}_i = \{\bar{\boldsymbol{x}} \in \mathbb{R}^4; \bar{R}_i(\bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}}\}$ is a hyper-plane, and that $\mathbb{R}^3 \ni \boldsymbol{x} \in \mathsf{U}_i$ implies $(\boldsymbol{x}, \xi) \in \bar{\mathsf{U}}_i$ for all $\xi \in \mathbb{R}$. Let γ be a solution curve of $X_{M,T,G}$. Since it is a time transformation of a solution curve of $X_{M,T}$ and since $\partial G/\partial \xi > \delta$ it follows that there exists $t_1 < t_2$ and $k \in \{1, 2, 3\}$ such that $\gamma(t_1), \gamma(t_2) \in \bar{\mathsf{U}}_k$. Thus, γ is periodic due to Lemma 4.1. The proof now proceeds exactly as the proof of Theorem 4.1. $\square$

Assume G takes the splitted form $G(\bar{\boldsymbol{x}}) = G_1(\boldsymbol{x}) + G_2(\xi)$. We propose the following adaptive versions of $\Phi_h^{\curlyvee T}$ and $\Phi_h^{\curlyvee M \curlyvee T}$.

$$
\begin{aligned}
\Phi_\epsilon^{\curlyvee T \curlyvee G} &= \varphi^\epsilon_{M,T,G_1} \circledcirc \varphi^\epsilon_{M,T_1,G_2} \circledcirc \varphi^\epsilon_{M,T_2,G_2} \circledcirc \varphi^\epsilon_{M,T_3,G_2} \\
\Phi_\epsilon^{\curlyvee M \curlyvee T \curlyvee G} &= \varphi^\epsilon_{M,T,G_1} \circledcirc \varphi^\epsilon_{M_1+M_2,T_1+T_2,G_2} \circledcirc \varphi^\epsilon_{M_2+M_3,T_2+T_3,G_2} \circledcirc \varphi^\epsilon_{M_3+M_1,T_3+T_1,G_2}
\end{aligned}
$$

Notice that all of the partial flows are explicitly integrable. In particular, $\varphi^\epsilon_{M,T,G_1}(\bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}} + \epsilon X_{M,T,G_1}(\bar{\boldsymbol{x}})$. Further, it holds that

$$
\varphi^\epsilon_{M,T_i,G_2}(\bar{\boldsymbol{x}}) = (\varphi_{M,T_i}^{\epsilon \frac{\partial G_2}{\partial \xi}(\xi)}(\boldsymbol{x}), 0), \quad i = 1,2,3
$$

and correspondingly for $\varphi^\epsilon_{M_i,T_i,G_2}$. These integrators have the following properties:

1. They are reversible with respect to $\bar{R}_1$, $\bar{R}_2$ and $\bar{R}_3$. Thus, their modified vector fields are $\bar{R}_1, \bar{R}_2, \bar{R}_3$–reversible perturbation of $X_{M,T,G}$, so Theorem 4.2 may be used to deduce periodic orbits of the numerical solution. (Assuming $\exists \varepsilon > 0$ such that $\partial G_2/\partial\xi > \varepsilon$.)
2. They are $\bar{\eta}$–maps. However, they do not correspond to a modified Nambu–Poisson system (see Remark 1.3). Nevertheless, M, T and G are still nearly conserved due to the periodicity of the numerical solution. In fact, M is exactly conserved by $\Phi^{\curlyvee T \curlyvee G}$, since each partial flow is an η_M–map.

As an illustration, numerical simulations with $\Phi^{\curlyvee M \curlyvee T}$, $\Phi^{\curlyvee T}$, $\Phi^{\curlyvee M \curlyvee T \curlyvee G}$, and $\Phi^{\curlyvee T \curlyvee G}$ are given. The moments of inertia are $I_1 = 1/2$, $I_2 = 1$, $I_3 = 2$, and initial data are $\boldsymbol{x}_0 = (0, \cos(\theta), \sin(\theta))$, with $\theta = 0.2$, which correspond to rotation "nearly" about the unstable principle axis. For the adaptive integrators the additional Hamiltonian is $G = G_1 + G_2 = -\log(\|X_{M,T}\| + 0.01) + \log(\xi)$, so the steps become smaller as the magnitude of the vector field $X_{M,T}$ increases. The step size $h = 0.15$ is used for the non-adaptive integrators, and for the adaptive integrators ϵ is chosen to yield the same mean time step (i.e. so that the mean of $\epsilon \partial G/\partial \xi$ is h).

A comparison of solutions in the t (non-adaptive) and in the τ (adaptive) variables are given in Figure 1. Notice that the time-stretching makes the solution "smoother". Further, the numerical errors in the Hamiltonians are plotted in Figure 2. Notice that the errors are significantly smaller for the adaptive integrators.

5 Conclusions

A time transformation technique for Nambu–Poisson systems, based on extending the phase space, have been developed (Theorem 2.1). The technique is shown to preserve reversibility under mild conditions on the additional Hamiltonian function (Theorem 3.1). A family of numerical integrators based on splitting of the Nambu–Poisson Hamiltonians is suggested. In particular, a novel approach for numerical integration of the Euler equations for the free rigid body is presented. By backward error analysis, it is shown that periodicity is preserved (Theorem 4.1 and Theorem 4.2).

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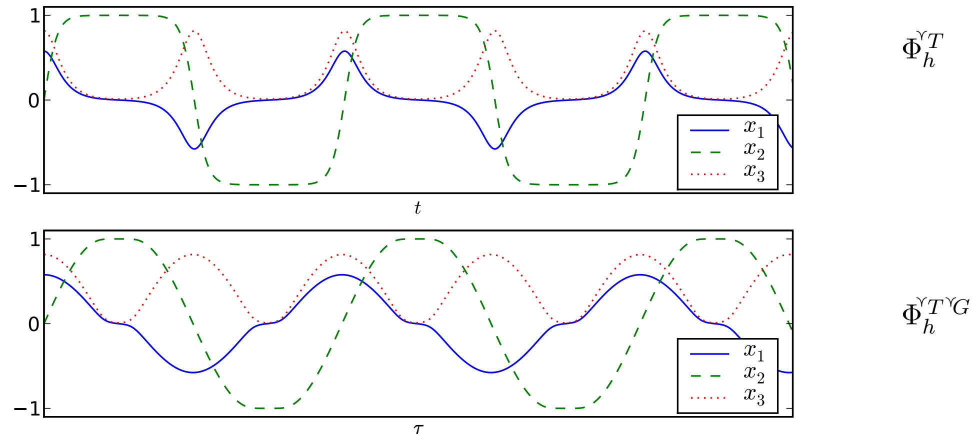

Figure 1. Solution curves for the non-adaptive integrator $\Phi^{\curlyvee T}$, and for the adaptive integrator $\Phi^{\curlyvee T \curlyvee G}$. Notice that the curves in the lower graph, corresponding to $\Phi^{\curlyvee T \curlyvee G}$, are "smoother". This is due to the time-stretching.

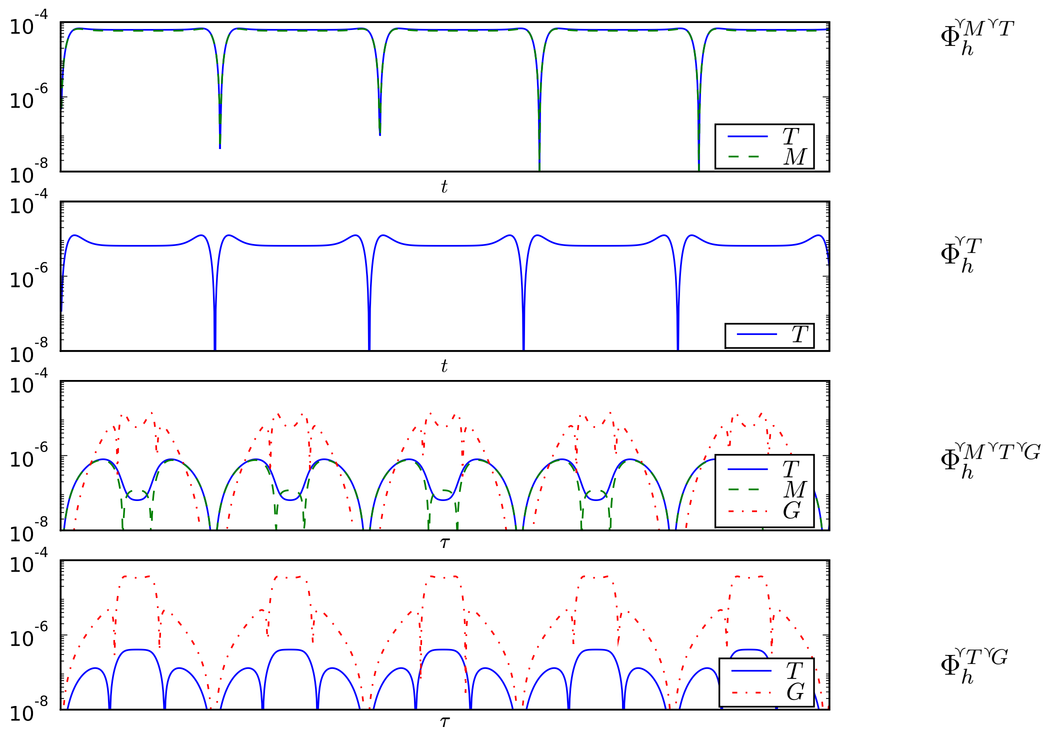

Figure 2. Absolute errors in the Hamiltonians. Notice that the errors in T (and M) are significantly smaller for the adaptive integrators. Thus, increased efficiency due to adaptivity is obtained. (Recall that $\Phi^{\curlyvee T}$ and $\Phi^{\curlyvee T \curlyvee G}$ conserve M up to rounding errors, whence M is not plotted for these.)

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