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The Systematic Formation of High-Order Iterative Methods

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Abstract

Fixed point iteration and the Taylor-Lagrange formula are used to derive, some new, efficient, high-order, up to octic, methods to iteratively locate the root, simple or multiple, of a nonlinear equation. These methods are then systematically modified to account for root multiplicities greater than one. Also derived, are super-quadratic methods that converge contrarily, and super-linear and super-cubic methods that converge alteratingly, enabling us, not only to approach the root, but also to actually bound and bracket it.

Keywords: Fixed point iteration; The generation of high order iterative functions; The Taylor-Lagrange formula; High-order iterative methods; Undetermined coefficients; Contrary and alternating convergence; Root bracketing

Fixed Point Iteration

Consider the paradigmatic fixed point iteration

$$x_1 = F(x_0) \tag{1}$$

to locate fixed point a, F(a)=a of contracting function F(x). We write $x_1-a=F(x_0)$ a and have by power series expansion that

$$x_1 - a = F'(a)(x_0-a) + \frac{1}{2!}F''(a)(x_0-a)^2 + \frac{1}{3!}F'''(a)(x_0-a)^3 + \ldots.. \tag{2}$$

implying that if 0<|F'(x)|<1 near x=a, namely, if F(x) is indeed contracting, then the fixed point iteration converges linearly, and if F'(a)=0, then the fixed point iteration converges quadratically, and so on.

Suppose now that we are seeking root a, f(a)=0, f'(a) ≠ 0, of function f(x). To secure a quadratic iterative method we rewrite f(x)=0 as the equivalent fixed point problem

$$x = F(x),\ F(x) = x + w(x)f(x) \tag{3}$$

for weight function w(x),w(a) ≠ 0, which we seek to fix to our advantage. For a quadratic method we need w(x) to be such that

$$F'(x) = 1 + w'(x)f(x) + w(x)f'(x) = 0 \tag{4}$$

for x near a. Since f (a)=0, we choose to ignore w'(x) f(x) in the above equation, to have w(x)=−1/f'(x), and with it, Newton's method

$$x_1 = x_0 - u_0,\ u_0 = \frac{f_0}{f_0'},\ f_0 = f(x_0) \tag{5}$$

which is actually quadratic

$$x_1 - a = \frac{1}{2}\frac{f''}{f'}(x_0-a)^2 + o((x_0-a)^3) \tag{6}$$

where f'=f'(a) ≠ 0, f"=f"(a) < ∞, and where x_0 is the iterative input and x_1 the iterative output.

From the two zero conditions

$$F'(x) = 1 + f'(x)w(x) + f(x)w'(x) = 0,\ F''(x) = f''(x)w(x) + 2f'(x)w'(x) + f(x)w''(x) = 0 \tag{7}$$

we obtain, after ignoring f(x) w"(x) in the second of equations (7), the system of equations

$$\begin{bmatrix} f' & f \\ f'' & 2f' \end{bmatrix}\begin{bmatrix} w \\ w' \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{8}$$

which we solve for w(x) as

$$w = \frac{\det\begin{bmatrix} -1 & f \\ 0 & 2f' \end{bmatrix}}{\det\begin{bmatrix} f' & f \\ f'' & 2f' \end{bmatrix}} \tag{9}$$

to have Halley's method

$$x_1 = x_0 + \frac{\det\begin{bmatrix} -1 & f_0 \\ 0 & 2f_0' \end{bmatrix}}{\det\begin{bmatrix} f_0' & f_0 \\ f_0'' & 2f_0' \end{bmatrix}} f_0 = x_0 - \frac{2f_0'}{2f_0'^2 - f_0 f_0''} f_0 = x_0 - \frac{1}{1-\frac{1}{2}(f_0''/f_0')u_0}u_0,\ u_0 = \frac{f_0}{f_0'} \tag{10}$$

which is, indeed, cubic

$$x_1 - a = \frac{1}{12}\frac{3f''^2 - 2f'f'''}{f'^2}(x_0-a)^3 + o((x_0-a)^4) \tag{11}$$

provided that f(a)=0, but f'(a) ≠ 0

Requesting that F(a)=a, F'(a)=0, F"(a)=0, F'"(a)=0, we similarly obtain the method

$$x_1 = x_0 + \frac{\det\begin{bmatrix} -1 & f_0 & 0 \\ 0 & 2f_0' & 1f_0 \\ 0 & 3f_0'' & 3f_0' \end{bmatrix}}{\det\begin{bmatrix} f_0' & f_0 & 0 \\ f_0'' & 2f_0' & f_0 \\ f_0''' & 3f_0'' & 3f_0' \end{bmatrix}} f_0 = x_0 - \frac{6f_0'^2 - 3f_0 f_0''}{6f_0'^3 - 6f_0 f_0' f_0'' + f_0^2 f_0'''} f_0 \tag{12}$$

which is quartic

$$x_1 - a = \frac{1}{24}\frac{3f''^3 - 4f'f''f''' + f'^2 f''''}{f'^3}(x_0-a)^4 + o((x_0-a)^5) \tag{13}$$

provided that f'=f'(a) ≠ 0.

Higher order single-point methods are readily obtained by

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requesting higher order derivatives of F(x)=x+w(x) f(x) to to zero at x=a, f (a)=0 [1].

A Recursive Determination of the Higher Order Iterative Function

There are various ways to recursively generate a new higher order iterative function F(x) of eq. (1) from a known lower order one. Traub [2] has suggested such a rational recursive formula. If, for example, $F(x)=F_2(x)$ is such that

$$F_2(a)=a,\ \ F_2'(a)=0, \tag{14}$$

then

$$F_3(x)=\frac{nF_2(x)-xF_2'(x)}{n-F_2'(x)}, n=2 \tag{15}$$

is such that

$$F_3(a)=a, F_3'(a)=0, F_3''(a)=0 \tag{16}$$

with which the iterative method $x_1=F_3(x_0)$ to locate fixed point a, F(a)=a becomes cubic

$$x_1-a=-\frac{1}{12}F'''(a)(x_0-a)^3+O((x_0-a)^4). \tag{17}$$

Instead of rational formula (15) we suggest the product formula

$$F_3(x)=F_2+\frac{1}{n}F_2'(F_2-x),\ \ n=2 \tag{18}$$

with which we still have third order convergence

$$x_1-a=\frac{1}{12}(3F''(a)^2-F'''(a))(\mathrm{x}_0-\mathrm{a})^3+\mathrm{O}((\mathrm{x}_0-\mathrm{a})^4). \tag{19}$$

For example, for Newton's method $F_2(x)=x-f(x)/f'(x)$. Using formula (18) we obtain by it the method

$$x_1=F_3(x_0), F_3(x_0)=x_0-u_0-\frac{f_0''}{2f_0'}u_0^2,\ \ u_0=\frac{f_0}{f_0'} \tag{20}$$

which is, indeed, cubic

$$x_1-a=\frac{1}{6}\frac{3f''^2-f'\,f'''}{f'^2}(x_0-a)^3+O((x_0-a)^4) \tag{21}$$

provided that f'=f'(a) ≠0.

Iterative method (20) is also obtained from Halley's method of eq. (10) using the approximation

$$(1-\frac{1}{2}\frac{f_0''}{f_0'}\mathrm{u}_0)^{-1}=1+\frac{1}{2}\frac{f_0''}{f_0'}u_0. \tag{22}$$

Further, if $F_3(x)$ is such that

$$F_3(a)=a, F_3'(a)=0, F_3''(a)=0 \tag{23}$$

then

$$F_{n+1}(x)=F_n+\frac{1}{n}F_n'(F_n-x),\ \ n=3 \tag{24}$$

is such that

$$F_4(a)=a,\ \ F_4'(a)=0, F_4''(a)=0, F_4'''(a)=0 \tag{25}$$

and the iterative method $x_1=F_4(x_0)$ to locate fixed point a is quartic

$$x_1-a=\frac{1}{72}F^{(4)}(x_0-a)^4+O((x_0-a)^5). \tag{26}$$

It is well known that the modified Newton's method

$$x_1=F_2(x_0), F_2(x)=x-m\frac{f(x)}{f'(x)} \tag{27}$$

converges quadratically to a root of any multiplicity m ≥ 1. From equation (24) we derive the third order method

$$x_1=F_3(x_0), F_3(x)=x-\frac{1}{2}m(3-m)u-m^2\frac{f''(x)}{2f'(x)}u^2, u=\frac{f(x)}{f'(x)} \tag{28}$$

Indeed, assuming that

$$f(x)=(x-a)^m g(x),\ \ g(a)\neq 0 \tag{29}$$

we obtain for the method in eq. (28)

$$x_1-a=\frac{(3+m)B^2-mAC}{2m^2A^2}(x_0-a)^3+O((x_0-a)^4) \tag{30}$$

where A=g(a), B=g'(a), C=g"(a), and m is the multiplicity index of repeating root a [3].

From eq. (29) we have

$$\left(\frac{f(x)}{f'(x)}\right)'=\frac{1}{m}-\frac{2}{m^2}\frac{g'(a)}{g(a)}(x-a)+O((x-a)^2) \tag{31}$$

by which we may, knowing an x close to a, estimate m.

A One-Sided Third-Order Two-Step, or Chord, Method

Having computed $x_1=x_0-f_0/f_0$ we return to correct it as the mid-point method

$$x_2=x_0-\frac{f(\mathrm{x}_0)}{f'\left(\frac{1}{2}x_0+\frac{1}{2}x_1\right)}=x_0-\frac{f(\mathrm{x}_0)}{f'(\mathrm{x}_0-\frac{1}{2}u_0)},\ \ u_0=\frac{f_0}{f_0'} \tag{32}$$

which is now cubic, or third order

$$x_2-a=\frac{1}{24}\frac{6f''^2-f'\,f'''}{f'^2}(\mathrm{x}_0-\mathrm{a})^3+\mathrm{O}((\mathrm{x}_0-\mathrm{a})^4). \tag{33}$$

See also Traub [2] page 164 eq. (8-12).

The modified method

$$x_2=x_0-\frac{4f_0}{f_0'+3f'(x_0-\frac{2}{3}u_0)} \tag{34}$$

is cubic

$$x_2-a=\frac{1}{4}(\frac{f''}{f'})^2(\mathrm{x}_0-\mathrm{a})^3+\mathrm{O}((\mathrm{x}_0-\mathrm{a})^3) \tag{35}$$

and one sided. At least asymptotically, if $x_0-a>0$, then also $x_2-a>0$, and if $x_0-a<0$, then also $x_2-a<0$

Using the approximation

$$f'(x-\frac{1}{2}u)=\frac{f(x)-f(x-u)}{u} \tag{36}$$

equation (32) becomes the two-step, or chord, method

$$x_2=x_0-\frac{1}{1-r}u_0\ \ or\ \ x_2=x_0-(1+r)u_0\ \ or\ \ x_2=x_0-(1+\mathrm{r}+\mathrm{r}^2)\mathrm{u}_0,\ \ \mathrm{r}=\frac{f_1}{f_0}, \tag{37}$$

where $u_0=f_0/f_0$, $x_1=x_0-u_0$, $f_1=f(x_1)$. All three methods of eq. (37) are cubic

$$x_2-a=\frac{1}{4}(\frac{f''}{f'})^2(\mathrm{x}_0-\mathrm{a})^3,\ \ \mathrm{x}_2-\mathrm{a}=\frac{1}{2}(\frac{f''}{f'})^2(\mathrm{x}_0-\mathrm{a})^3, x_2-a=\frac{1}{4}(\frac{f''}{f'})^2(\mathrm{x}_0-\mathrm{a})^3+O((x_0-a)^3). \tag{38}$$

See Traub [2-4].

Convergence of this method is also one sided.

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Construction of High-Order Iterations by Undetermined Coefficients

Halley's method, or for that matter any other higher order method, can be constructed by writing δx, $x_1=x_0+\delta x$, as a power series of $u_0=f_0/f'_0$, or even of merely $f_0=f(x_0)$. For example, we write the quadratic

$$x_1 = x_0 + Pf_0 + Qf_0^2 \tag{39}$$

and then sequentially fix the undetermined coefficients P and Q for highest attainable order of convergence.

Thus, at first we have from eq. (39) that

$$x_1 - a = (1 + Pf')(x_0 - a) + O((x_0 - a)^2) \tag{40}$$

and we set $P=-1/f'_0$. With this P we have next that

$$x_1 - a = (\frac{f''}{2f'} + f'^2 Q)(x_0 - a)^2 + O((x_0 - a)^3) \tag{41}$$

and we set

$$P = -\frac{1}{f'_0},\ Q = -\frac{f''_0}{2f_0'^3} \tag{42}$$

with which the polynomial method of eq. (20) is recovered.

Doing the same to the rational method

$$x_1 = x_0 + \frac{P + Qf_0}{R + Sf_0} f_0 \tag{43}$$

we determine by power series expansion that cubic convergence is assured for P=−1/f00,

Q=0, R=1, $S=-f''_0/(2f'^2_0)$, with which the classical Halley's method of eq. (10) is recovered.

Quartic and Quintic Multistep Methods

The rational two-step method (a generalization of the method in eq. (37)) of Ostrowski [5] appendix G,

$$x_2 = x_0 - \frac{1-r}{1-2r} u_0 = x_0 - (1 + r + 2r^2 + 4r^3 + ...)u_0,\ f_1 = f(x_0 - u_0),\ r = \frac{f_1}{f_0} \tag{44}$$

is quartic

$$x_2 - a = \frac{1}{24} \frac{f''(3f''^2 - 2f'\ f''')}{f'^3} (x_0 - a)^4 + O((x_0 - a)^5). \tag{45}$$

Traub [2,3,6-8]

See also page 184 eq. (8-78).

The polynomial in r method

$$x_2 = x_0 - (1 + r + 2r^2)u_0,\ u_0 = \frac{f_0}{f'_0}, \tag{46}$$

is also quartic

$$x_2 - a = \frac{1}{24} \frac{f''}{f'^3} (15f''^2 - 2f'\ f''')(x_0 - a)^4 + O((x_0 - a)^5). \tag{47}$$

The multistep method

$$x_2 = x_0 - \frac{1}{1-r} u_0,\ x_3 = x_2 - \frac{1}{1-2r} \frac{f_2}{f_0}, r = \frac{f_1}{f_0} \tag{48}$$

is quintic

$$x_3 - a = \frac{1}{24} \frac{f''^2(3f''^2 - f'\ f''')}{f'^4} (x_0 - a)^5 + O((x_0 - a)^6). \tag{49}$$

Sextic and Octic Multistep Methods

The multistep method

$$x_2 = x_0 - (1 + r + 2r^2)u_0,\ x_3 = x_2 - \frac{1-r}{1+3r} \frac{f_2}{f'_0},\ r = \frac{f_1}{f_0},\ u_0 = \frac{f_0}{f'_0} \tag{50}$$

is sextic

$$x_3 - a = \frac{1}{144} \frac{f'\ f'''(-15f''^2 + 2f'\ f''')}{f'^4} (x_0 - a)^6 + O((x_0 - a)^7). \tag{51}$$

The method

$$x_3 = x_2 - \frac{f(x_2)}{g'}, g' = (1 - 2r + 3r^2 - s(1 + 2r^2))f'0, r = \frac{f_1}{f_0}, s = \frac{f_2}{f_1} \tag{52}$$

is octic

$$x_3 - a = \frac{1}{1152} \frac{f''^2(-15f''^2 + 2f'\ f''')(27f''^3 + 2f'\ f''\ f''' - f'^2 f'''')}{f'^7} (x_0 - a)^8 + O((x_0 - a)^9). \tag{53}$$

[6,9]

Contrarily converging super-quadratic methods

We write

$$x_2 = x_0 - (1 + \Pr)u_0, u_0 = \frac{f_0}{f_0}, r = \frac{f_1}{f_0}, f_1 = f(x_1), x_1 = x_0 - u_0 \tag{54}$$

for undetermined coefficient P, and have

$$x_2 - a = \frac{1}{2} \frac{f''}{f'} (1 - P)(x_0 - a)^2 + O((x_0 - a)^3). \tag{55}$$

We request that

$$\frac{f''}{f'} (1 - P) = 2k(\frac{f''}{f'})^2 \tag{56}$$

for parameter k, by which the iterative method in eq. (54) turns into

$$x_2 = x_0 - (1 + r)u_0 + 4kr^2 \tag{57}$$

for any constant k, and

$$x_2 - a = k(\frac{f''}{f'})^2 (x_0 - a)^2 + O((x_0 - a)^3). \tag{58}$$

This super-quadratic method converges from above if k>0, and from below if k<0

The interest in the method

$$x_2 = x_0 - \frac{1}{1-r} u_0, x_1 = x_0 - u_0, f_1 = f(x_1), r = \frac{f_1}{f_0}, u_0 = \frac{f_0}{f'_0} \tag{59}$$

is that it ultimately converges oppositely to Newton's method,

$$x_2 - a = -\frac{1}{2} \frac{f''}{f'} (x_0 - a)^2 + O((x_0 - a)^3) \tag{60}$$

as is seen by comparing eq. (60) with eq. (6).

The average of Newton's method and the method of eq. (59) is cubic,

$$\frac{1}{2}(x_1 + x_2) - a = \frac{1}{6} \frac{f'''}{f'} (x_0 - a)^3 + O((x_0 - a)^4). \tag{61}$$

Alternaingly Converging Super-Linear and Super-Cubic Methods

We start by modifying Newton's method

$$x_1 = x_0 - (1 + k) \frac{f_0}{f'_0}, k \ge 0 \tag{62}$$

to have

$$x_1 - a = -k(x_0 - a) + O((x_0 - a)^2) \tag{63}$$

indicating that, invariably, the method converges, at least asymptotically,

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alternatingly. For k > 0, if $x_0 - a > 0$, then $x_1 - a < 0$, and vice versa. For a higher-order alternating method we rewrite the originally quartic method of eq. (46) as

$$x_2 = x_0 - (1 + r + Qr^2)u_0, u_0 = \frac{f_0}{f'_0}, r = \frac{f_1}{f_0}, f_1 = f(x_0 - u_0) \quad (64)$$

for the undetermined coefficient Q, and have that

$$x_2 - a = -k(\frac{f''}{f'})^2 (x_0 - a)^3 + O((x_0 - a)^4), k = \frac{1}{4}(Q-2). \quad (65)$$

This super cubic method converges alternatingly if parameter k > 0.

Correction for Multiple Roots by Undetermined Coefficients

We rewrite Newton's method as

$$x_1 = x - Pu_0 \quad u_0 = \frac{}{}, \quad (66)$$

for undetermined coefficient P, and have that for a root of multiplicity m ≥ 1

$$x_1 - a = (1 - \frac{P}{m})(x_0 - a) + \frac{P}{m^2}\frac{B}{A}(x_0 - a)^2 + O((x_0 - a)^3) \quad (67)$$

where A=g(a), B=g'(a) for g(x) in eq. (29). Quadratic convergence is restored, as is well known, with P=m.

With P=m(1–k), k<0 the modified Newton's method of eq. (66) becomes superliner and ultimately of alternating convergence [10].

Next, we rewrite the method in eq. (37) as

$$x_2 = x_0 - \frac{P}{Q - r}u_0, r = \frac{f_1}{f_0}, f_1 = f(x_1), x_1 = x_0 - u_0 \quad (68)$$

and seek to fix correction coefficients P and Q so that convergence remains cubic even in the event that root a is of multiplicity m > 1. By power series expansion we determine that

$$P = Q = (\frac{m-1}{m})^{m-1}, m > 1, P = Q = 1 \; if \; m = 1 \quad (69)$$

upholds cubic convergence

$$x_2 - a = \frac{mB^2 - 2(m-1)AC}{2m^2A^2}(x_0 - a)^3 + O((x_0 - a)^4) \quad (70)$$

where A=g(a), B=g'(a),C=g"(a) for g(x) in eq. (29)

The method

$$x_2 = -(P + Qr)u_0, P = m(2-m), Q = \frac{m^{m+1}}{(m-1)^{m-1}}, m > 1 \quad (71)$$

is also cubic

$$x_2 - a = \frac{(m+2)B^2 - 2(m-1)AC}{2m^2A^2}(x_0 - a)^3 + O((x_0 - a)^4) \quad (72)$$

where A=g(a), B=g'(a), C=g"(a) for g(x) in eq. (29) [11-13].

Correction of Halley's Method for Multiple Roots

We rewrite Halley's method of eq. (10) for the undetermined coefficient P and Q as

$$x_1 = x_0 - \frac{Pf'_0}{Qf'^2_0 - f_0 f''_0} f_0 \quad (73)$$

and determine by power series expansion that for

$$P = 2, Q = 1 + \frac{1}{m} \quad (74)$$

convergence remains cubic for a root of any multiplicity m ≥ 1

$$x_1 - a = \frac{(m+1)B^2 - 2mAC}{2m^2A^2}(x_0 - a)^3 + O((x_0 - a)^4) \quad (75)$$

where A=g(a), B=g'(a),C=g"(a) for g(x) in eq. (29) [11,12].

Use of the Taylor-Lagrange formula

We write the second order version of the Taylor-Lagrange formula

$$f(x) = f(x_0 + \delta x) = f(x_0) + \delta x f'(x_0) + \frac{1}{2}\delta x^2 f''(\xi), x_0 < \xi < x_0 + \xi x \quad (76)$$

and take $f(x_1 = x_0 + \delta x) = 0$, $\xi = x_0$ to obtain the iterative method

$$x_1 = x_0 + \delta x, 0 = f(x_0) + \delta x f'(x_0) + \frac{1}{2}\delta x^2 f''(x_0). \quad (77)$$

We approximate the solution of the increment equation

$$f_0 + \delta x f'_0 + \frac{1}{2}\delta x^2 f''_0 = 0 \quad (78)$$

or, for that matter, any such higher order algebraic equation, by the power series

$$\delta x = (P + Qf_0 + Rf_0^2 + Sf_0^3 + Tf_0^4 + ...)f_0 \quad (79)$$

and have upon substitution and collection that

$$(1 + Pf'_0) + (Qf'_0 + \frac{1}{2}P^2 f''_0)f_0 + (Rf'_0 + PQf''_0)f_0^2 + (Sf'_0 + \frac{1}{2}Q^2 f''_0 + PRf''_0)f_0^3 + ... = 0 \quad (80)$$

from which we have that

$$P = -\frac{1}{f'_0}, Q = -\frac{1}{2}P^2 s_0, R = -PQs_0, S = -(\frac{1}{2}Q^2 + PR)s_0, T = -(QR + PS)s_0 \quad (81)$$

where $s_0 = f''_0 / f_0$

The methods

$$x_1 = x_0 + Pf_0 + Qf_0^2 \; and \; x_2 = x_0 + Pf_0 + Qf_0^2 + Rf_0^3 \quad (82)$$

or

$$x_1 = x_0 - (1 + \frac{1}{2}u_0)u_0, x_2 = x_0 - (1 + \frac{1}{2}u_0 + \frac{1}{2}u_0^2)u_0, u = \frac{f}{f'}, s = \frac{f''}{f'}, u = us \quad (83)$$

are both cubic

$$x_1 - a = \frac{1}{6}\frac{3f''^2 - f' f'''}{f'^2}(x_0 - a)^3 + O(x_0 - a)^4, x_2 - a = -\frac{1}{3}\frac{f'''}{f'}(x - a_0)^3 + O((x_0 - a)^4), \quad (84)$$

provided that f'(a) ≠0.

The method

$$x_1 = x_0 - (P + Qu_0)u_0, u_0 = \frac{f_0}{f'_0}, u_0 = \frac{f_0 f''_0}{f'^2_0}, P = \frac{m(3-m)}{2}, Q = \frac{1}{2}m^2 \quad (85)$$

converges cubically as well to a root of any multiplicity m ≥ 1

$$x_1 - a = \frac{(3+m)B^2 - 2mAC}{2A^2m^2}(x_0 - a)^3 + O((x_0 - a)^4) \quad (86)$$

where A=g(a), B=g'(a),C=g"(a) for g(x) in eq. (29).

Still Higher Order Methods

Starting with

$$f(x) = f(x_0 + \delta x) = f_0 + \delta x f'_0 + \frac{1}{2}\delta x^2 f''_0 + \frac{1}{6}\delta x^3 f'''(\xi), x_0 < \xi < x_0 + \delta x \quad (87)$$

we obtain the iterative method

$$x_1 = x_0 + \delta x, \; f_0 + \delta x f'_0 + \frac{1}{2}\delta x^2 f''_0 + \frac{1}{6}\delta x^3 f'''_0 = 0 \quad (88)$$

where

$$\delta x = (P + Qf_0 + Rf_0^2 + Sf_0^3 + ...)f_0 \quad (89)$$

with

$$P=-\frac{1}{f'}, Q=\frac{1}{2}P^3f'', R=P^2(Qf''+\frac{1}{6}P^2f'''), S=P(\frac{1}{2}Q^2f''+PRf''+\frac{1}{2}P^2Qf'''). \quad (90)$$

The methods

$$x_1=x_0+(P+Qf_0+Rf_0^2)f_0 \ and \ x_2=x_0+(P+Qf_0+Rf_0^2+Sf_0^3)f_0 \quad (91)$$

are both quartic

$$x_2-a=\frac{1}{24}\frac{f''''}{f'}(x_0-a)^4+O((x_0-a)^5) \quad (92)$$

provided that f'(a) ≠ 0.

Unknown Root Multiplicity

The two single-step methods

$$x_1=x_0-m\frac{f_0}{f_0'}, x_2=x_0-\frac{f_0'}{f_0'^2-f_0f_0''}f_0 \quad (93)$$

converge contrarily to root a of any multiplicity m

$$x_1-a=\frac{1}{m}\frac{B}{A}(x_0-a)^2+O((x_0-a)^3), x_2-a=-\frac{1}{m}\frac{B}{A}(x_0-a)^2+O((x_0-a)^3) \quad (94)$$

where A=g(a), B=g'(a) for g(x) in eq. (29). Their average is a cubic method

$$x_3-a=\frac{B^2(m-1)-2ACm}{2A^2m^2}(x_0-a)^3+O((x_0-a)^4), x_3=\frac{1}{2}(x_1+x_2) \quad (95)$$

where A=g(a), B=g'(a),C=g"(a) for g(x) in eq. (29).

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