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The Proof of the Infinity of Twin Primes

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Abstract

This paper shows that there are infinitely many twin primes through a focused analysis of twin primes whose last digits are 1 and 3. As a method, the original Infinite Game and Floor Line Arrangement are added to the refined one of Sieve of Eratosthenes. In the first section, we describe the extraction of twin prime numbers by the Sieve of Eratosthenes from the combination of natural numbers whose last digits are 1 and 3 and the numbers differ by 2. If such a twin prime number is finite, from some point all numbers are marked.

In the second section, we use the Infinite Game, but what we do here is the same as the Sieve of Eratosthenes above, except for one point. The only exception is that we can choose where to mark regardless of the prime number. In the third section, we show that it is impossible to mark all numbers even when we artificially select a place to mark in the Infinite Game, by using the method of the Floor Line Arrangement. In the fourth section, we conclude that as a result, it is revealed that there are infinitely many twin prime numbers.

Keywords: Proof infinity • Twin primes • Infinite game • Floor line arrangement

Introduction

This paper uses new attempts to clarify that there are infinitely many twin primes. The new attempts are mainly the following three points.

First, we limited the twin primes handled here to a combination of numbers with the last digits 1 and 3. When trying to prove that there are infinitely many, it seems contradictory to tighten the conditions. But that is the key to this proof. The second is the use of a game named "Infinite Game", along with the Sieve of Eratosthenes [1-8] from ancient days though a refined version. It's just a game, but surprisingly this leads to this proof. Third, use the method named "Floor Line Arrangement". This serves to assist, the proof through the "Infinite Game".

Share the joy of solving the difficult problem that has plagued us over the past 2000 years.

Experimental

Application of the Sieve of Eratosthenes

The Sieve of Eratosthenes [1-8] is known as an ancient algorithm for finding all prime numbers up to an arbitrary number. Here, we apply this algorithm to extract twin prime numbers from combinations of natural numbers whose last digits are 1 and 3 and whose numerical values are different by 2.

Sieve multiples of 3 in advance. Mark and remove each set of natural numbers, including multiples of 3.The remaining sets are (11, 13), (41, 43), (71, 73), (101, 103) and more. Then, the smaller number in the natural numbers set is used as the candidate number. They are numbers such as 11, 41, 71, 101 and more, that can be expressed as 30x + 11 (x is 0 or a natural number). At this point, sieving against multiples of 2, 3 and 5 is complete.

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Here we start the work similar to the Sieve of Eratosthenes [1-8] from a multiple of 7, but there are four differences from the original one.

The first is to continue the sieve infinitely. When the combination of twin primes with the last digit being 1 and 3 is finite, then from some point all numbers are marked.

The second is to use many kinds of multiples of prime numbers in parallel as sieves. Third, when sieving with multiples of p, it marks not only when the candidate number is a multiple of p. But also put a mark when the partner of candidate number, that is, a number 2 larger than the candidate number is a multiple of p. For example, when sieving with a multiple of 7, it marks not only when the candidate number is a multiple of 7, like 161, 371, 581 and more, but also when the partner of it is a multiple of 7, such as 131, 341, 551 and more.

Fourth, when sieving with multiples of p, start with p^2. There is also the expression "refinement" for this [2,3]. A good reason for this, it is described that all smaller multiples of p are marked at that point [1-3]. However, in fact, this is a crucial point in this proof. Therefore, we state this in other words as follows.

"The sieve by the multiples of p is useless below p^2." Table 1 shows the situation where the candidate numbers are sieved according to the above rules. Here, the candidate numbers are limited to less than 1000.

The Infinite Game

Significance

Here, let's consider the following game named "Infinite Game" to show that all the numbers from a certain point are never marked by the Sieve of Eratosthenes of the previous section. What we do in this game is the same as the Sieve of Eratosthenes above, except for one point. The only exception is that we can choose where to mark regardless of the prime number. That way we aim to mark all numbers. If it is impossible to mark all the numbers by artificially selecting where to mark, how can you mark them with the Sieve of Eratosthenes above-mentioned?

Rule

The basic rules for advancing this game are as follows.

1. Arrange natural numbers n such that n = 30x + 11 (where x is 0 or a natural number) in order, starting from a favorite number.
2. A plate with a length of p, and p dots (p is a prime number of 7 or more) can be used infinitely. Here, let's call such plates PL (p). For example, a

plate with a length of 7, and 7 dots is PL (7).

3. From the dots on each plate, you can freely select two places to mark. However, it is necessary to unify all mark positions on the same length plate.
4. Arrange the plates parallel to the numbers, and arrange the plates of the same length at the end without a gap. Any number of plates may be arranged, and any number of plates may be added later.
5. A plate of a certain length cannot be placed on the side of a number that is smaller than the square of its length. You can put the plate in any place you like, if that is the side of the number equal or greater than the square of its length.
6. Regardless of the mark of which plate, a number with a mark on the side is marked.
7. The goal of the game is to mark numbers infinitely and continuously.
8. If you come to a number that cannot be marked, it is a "dead end" and a "defeat".

Examples of Playing

Here are two examples of how to proceed with this game.

(Example 1) When you starting from 71

In order to mark 71 and 101, it is necessary to place a mark of a plate, on the side of the numbers. The only plate that can be used at this place is PL (7), so we take the only option.Next, in order to mark 131 and 161, the only plate that can be used in this scene is PL (11), so we take the only option again. Furthermore, in order to mark 191 and 221, the only option is to use PL (13), so here too, we take such an option (Figure 1).

But here, we get the "dead end". The reason is that 251 cannot be

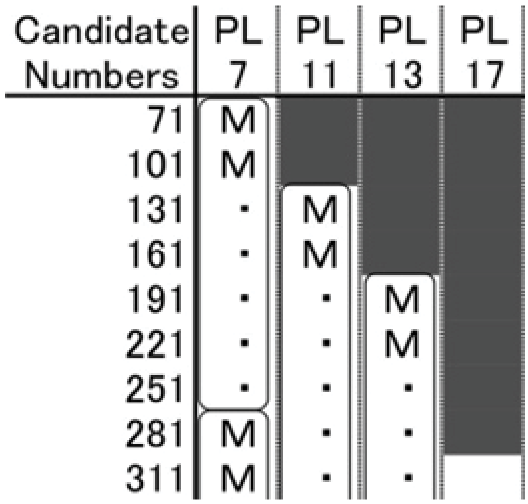

Figure 1. When you starting from 71.

Table 1. Sieving against the candidate numbers, Sbm: Sieving by the multiples of, M: Marked.

X	Sbm 7	Sbm 11	Sbm 13	Sbm 17	Sbm 19	Sbm 23	Sbm 29	Sbm 31	Status
11									Remained
41									Remained
71									Remained
101									Remained
131	M								
161	M								
191									Remained
221			M						
251		M							
281									Remained
311									Remained
341	M	M							
371	M								
401			M						
431									Remained
461									Remained
491				M					
521									Remained
551	M				M				
581	M	M							
611			M						
641									Remained
671		M							
701					M				
731				M					
761	M								
791	M		M						
821									Remained
851						M			
881									Remained
911		M							
941						M			
971	M								

marked depending on the plates used so far, and we cannot use PL (17) because 251 is less than 289, the square of the length of the plate. Therefore, we face the "defeat" here.

(Example 2) When you starting from 1001

When you start from 1001, you have many choices. The way of playing shown in Figure 2 is merely an example, and is not mean the best strategy. In this example, there is a "dead end" at 2591, but it may be postponed depending on the idea. However, we cannot avoid the "defeat".

Floor Line Arrangement

Definition of "Floor Line Arrangement" and its significance

Here we consider the Floor Line Arrangement to prove that players in the Infinite Game definitely lose. It is, the plates used in the Infinite Game are separated from the rules and arranged one by one as shown in Figure 3, connecting the ends to each other.

Where we start the Floor Line Arrangement with the same number as the player of the Infinite Game, it is proven that the plates of player of the Infinite Game always collide the ceiling at last, when the following two points are always true:

- A plate closest to the ceiling of the player of the Infinite Game is never farther from the ceiling than the plate on the Floor Line Arrangement placed at the same distance from the starting point.

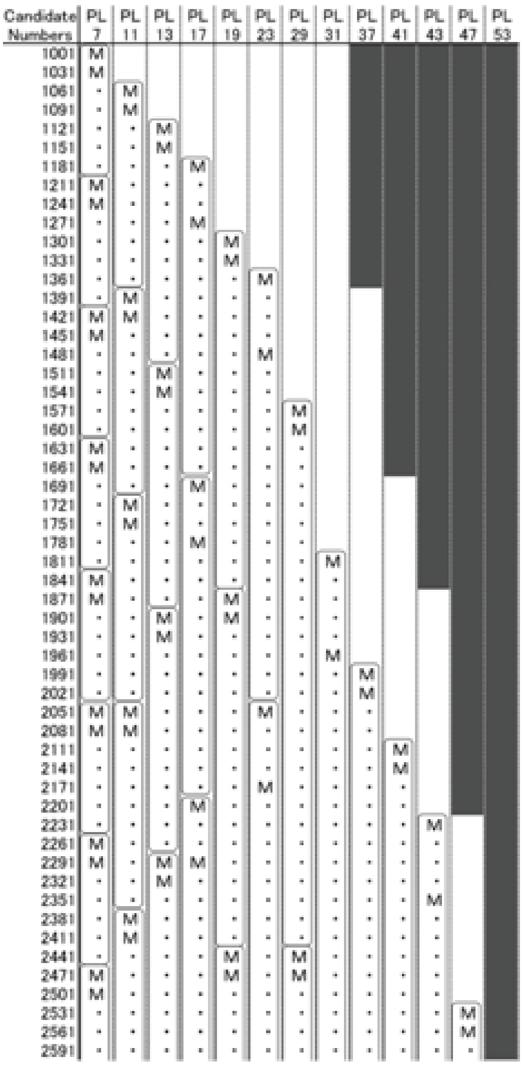

Figure 2. When you starting from 1001.

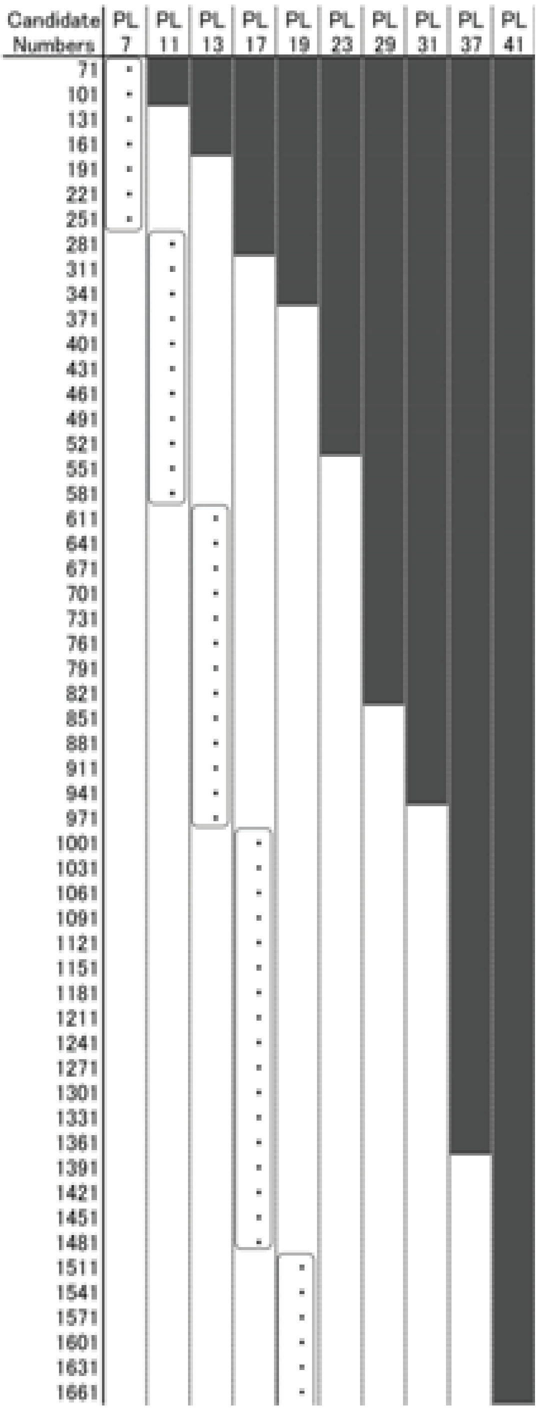

Figure 3. Floor Line Arrangement from 71.

- No matter which number the Floor Line Arrangement starts from, the plates on there cannot avoid collision with the ceiling.

Here, we prove these two things.

Proof about that the plate closest to the ceiling of the player of the Infinite Game is never farther from the ceiling than the plate on the Floor Line Arrangement placed at the same distance from the starting point

When a player tries to delay the defeat, it is rational to use the strategy shown in the examples of playing, such that, using from the shortest plate in sequence and marking the unmarked numbers immediately. Here, we formulate the progression of the plates when such strategy is adopted. First, define the position of PL (p_n), as the distance from the starting point to the tail of the PL (p_n), that the player first uses or to the edge of the plate closest to the starting point. And the position of PL (p_n), when adopting a rational strategy is expressed as RSV (p_n). RSV is an abbreviation for Rational Strategy Value. As shown in Figure 2, the values of RSV (7), RSV (11), RSV (13), RSV (17) and RSV (19), are 0, 2, 4, 6 and 10, respectively.

Next, assuming a theoretically ideal game progression, the position of PL (P_n), in that case is expressed as TIV (p_n). TIV is an abbreviation for Theoretical Ideal Value. Then, the value is calculated by the following formula.

$$TIV(p_n) = 2 + \sum_{\substack{p \text{ prime} \\ \delta(13) \leq \delta(p) \leq \delta(p_n)}}$$

where,

$$\delta(p_n) = 2 \prod_{\substack{p \text{ prime} \\ 7 \leq p \leq p_{n-2}}} \frac{p}{p-2}$$

The meaning of this formula is to find out how many dots it takes to insert two marks for each plate and then add the numbers in order. Here we take a closer look at the first few.

First, the value of TIV (7) is 0. Because the game starts with PL (7), that is the starting point. Then the value of TIV (11) is 2. There is no marked number in PL (7) since there is no plate used before it, so there is no choice but to mark the first two dots. So far, the value of TIV (p_n) is the same as RSV (p_n).

TIV (p_n), a mathematical formula that predicts the position of a plate despite its very irregular progression, has its own effects from PL (13). The value of RSV (13) is 4, but the value of TIV (13) is estimated as follows. First, consider how many dots are needed to insert two marks in PL (11). It is equivalent to how many dots are needed before two unmarked numbers appear in PL (7). The ratio of unmarked numbers in PL (7) is five-seventh, so the required number of dots is 2.8. As a result, the value of TIV (13) is 4.8. The meaning of this formula is to repeat this operation from then on.

As shown in Table 2, the value of TIV (p_n) fluctuates in the range larger than the value of RSV (p_n), at the beginning. However, it slowly converges to a range of values that are moderately larger than the value of RSV (p_n). The reason for such a convergence is that the strategy underlying the "Example of Playing" is by no means the best strategy in every situation, even if it is "rational".

For example, as shown in Figure 4, when we start the "Infinite Game" from 161, the strategy we adopted faces the "dead end" at 521, but another strategy defers it to 731. Therefore, the value of RSV (p_n). Is somewhat smaller than the value of TIV (p_n), which is the theoretical upper bound.In other words, with the exception of the early stages, there are temporarily better strategies for every situation.

On the other hand, when the position where PL (p_n) is used on the Floor Line Arrangement is expressed as FLA (p_n), the following formula is established.

$$FLA(p_n) = \sum_{\substack{p \text{ prime} \\ 7 \leq p \leq p_{n-1}}}$$

Here we present the following inequality.

$$TIV(p_n) < 2 + \frac{2\left(\sum_{\substack{p \text{ prime} \\ 7 \leq p \leq p_{n-2}}}\right)}{5} < FLA(p_n)$$

It is clear that this inequality holds.

Therefore, the plate closest to the ceiling placed by the player of the Infinite Game is never farther from the ceiling than the plate on the Floor Line Arrangement placed at the same distance from the starting point.

Table 2. Comparison of the value of RSV (p_n) and TIV (p_n).

p_n	RSV(p_n)	TIV(p_n)	TIV(p_n)/RSV(p_n)
7	0	0	
11	2	2	1
13	4	4.8	1.2
17	6	8.22	1.37
19	10	12.27	1.23
23	12	16.85	1.4
29	19	21.97	1.16
31	27	27.58	1.02
37	33	33.61	1.02
41	37	40.05	1.08
43	41	46.86	1.14
47	51	54.02	1.06
53	53	61.53	1.16
59	55	69.37	1.26
61	61	77.52	1.27
67	66	85.96	1.3
71	73	94.68	1.3
73	76	103.67	1.36
79	97	112.92	1.16
83	103	122.43	1.19
101	130	152.42	1.17
199	327	396.68	1.21
307	555	649.82	1.17
401	769	916.79	1.19
499	1027	1206.47	1.17
601	1297	1495.82	1.15
701	1574	1821	1.16
797	1782	2096.63	1.18
907	2160	2448.55	1.13
997	2421	2743.99	1.13
1103	2782	3142.14	1.13
1201	3000	3430.78	1.14

Proof that the plates on the Floor Line Arrangement cannot avoid collision with the ceiling

Next, we prove that the plates on the Floor Line Arrangement collide the ceiling somewhere, regardless of its starting number.

When starting the Floor Line Arrangement from 71 as shown in Figure 3, PL (5478371) collides the ceiling. That is, the height of the ceiling when using this plate is 5478367 as follows.

$$\left[71 + 30\left(\sum_{\substack{p \text{ prime} \\ 7 \leq p \leq 5478337}}\right)\right]^{\frac{1}{2}} \sim 5478367 < 5478371$$

Furthermore, apart from the rules of the Infinite Game, we place the plates of the Floor Line Arrangement from 0. One scale remains at 30.Then, even if the plates are arranged from 0, the same plate as those arranged from 71 collides the ceiling. The height of the ceiling at this position is used as the basic unit and is represented by the symbol c. And, the value of c is approximately 5.5E +6.

In understanding this proof, it is important to recognize that this basic unit c is the following three approximate numbers. They are the following three elements of the position where the plate of the Floor Line Arrangement that starts from 0 collides with the ceiling:

- Height of the ceiling.
- Length of the plate that collided.
- The square root of the numerical value that is 30 times the total length of the plates used up to that point

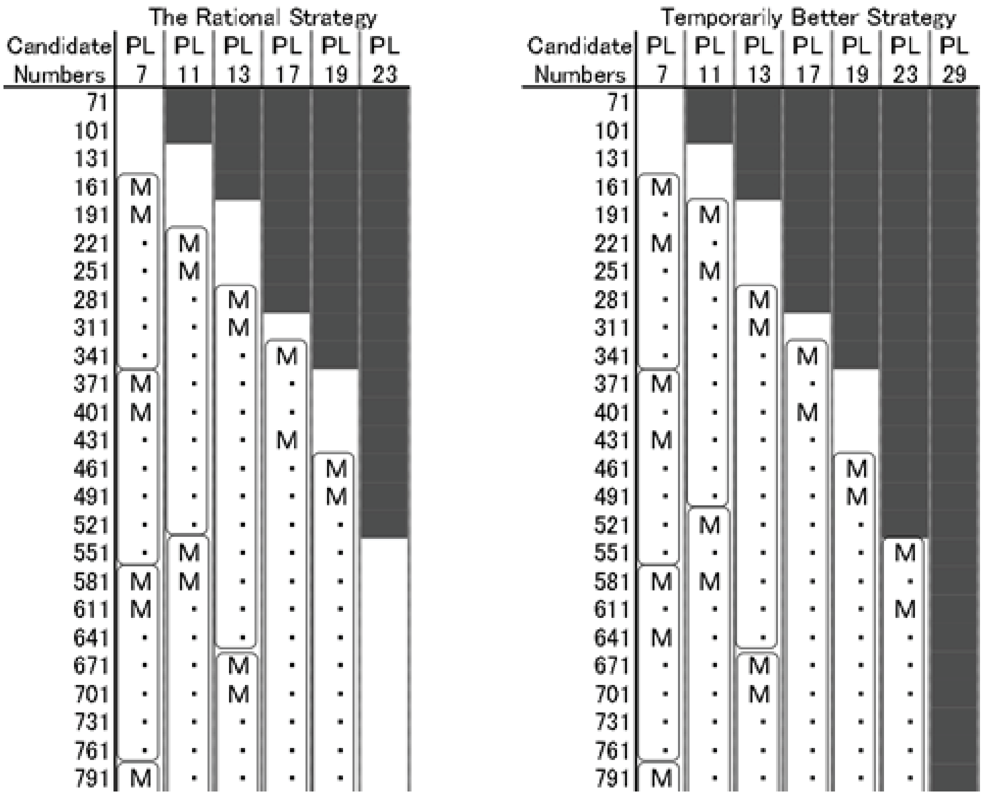

Figure 4. The Rational Strategy and the Temporarily Better Strategy.

In addition, to make it easier to handle huge number of plates, the total length of the plates used is represented by a mathematical formula that includes a logarithmic integral as follows. This is the key to this proof.

$$FLA(p_n) = \sum_{\substack{p \text{ prime} \\ 7 \le p \le p_{n-1}}} \sim \frac{1}{2} p_{n-1} \int_7^{p_{n-1}} \frac{dt}{\ln t}$$

The meaning of this formula is that the number of prime numbers from 7 to p_{n-1} approaches the next value by the prime number theorem [8-10].

$$\int_7^{p_{n-1}} \frac{dt}{\ln t}$$

And the average value of them is roughly ½.p_{n-1}.Therefore, the product of the number of prime numbers and their average value approximates the total length of the plate. That's a pretty rough estimate, but it is not any problem for this discussion.

Now, find the value of c, which is the basic unit, using the formula that uses this logarithmic integral. The height of the ceiling is the square root of the value obtained by multiplying the total length of the arranged plates by 30 times, so it is as follows.

$$5478371 \sim \left(15 * 5478337 \int_7^{5478337} \frac{dt}{\ln t}\right)^{\frac{1}{2}} = 5583634$$

Here, when b is larger than a, and each is a sufficiently large number, the following formula generally holds.

$$\frac{\int_7^b \frac{dt}{\ln t}}{\int_7^a \frac{dt}{\ln t}} < \frac{b}{a}$$

The fact that this inequality holds is equivalent to the following inequality holds.

$$\frac{15b \int_7^b \frac{dt}{\ln t}}{15a \int_7^a \frac{dt}{\ln t}} < \frac{b^2}{a^2}$$

Therefore, no matter what number the Floor Line Arrangement starts with, it always collides with the ceiling somewhere.

Here, we repeat the same content, showing a concrete example. As mentioned above, c is a symbol that has three contents, and is approximately 5.5E + 6. The coordinate on the horizontal axis where the ceiling height is c is c^2, which is approximately 3E+13. The position where the ceiling height is 10 times, that is, the coordinate on the horizontal axis where the ceiling height is 10c or 5.5E +7, is $100c^2$, which is approximately 3E + 15.

On the other hand, when we arrange up to the plate with a length of 10c or 5.5E +7 on the Floor Line Arrangement, the total length on horizontal axis coordinate of the plates is 2.7E + 15. It is as the following formula.

$$2.7E + 15 \sim 15(5.5E + 7) \int_7^{5.5E+7} \frac{dt}{\ln t}$$

This means that even when we start arranging the plates on the Floor Line Arrangement from 3E + 14, which is one-tenth of the whole, it collides with the ceiling of 5.5E + 7 at 3E + 15. Similarly, in order to hit the plates to the ceiling of 5.5E + 13 at 3E + 27, we can start the Floor Line Arrangement from 1.5E + 27, half of the whole.In this way, no matter what number the Floor Line Arrangement starts with, the plates always collide with the ceiling somewhere.

Thus, the ceiling continues too low to bring victory to the player of the game.

Conclusion

In this paper, we first showed how to extract twin prime numbers from combinations of two natural numbers whose last digits are 1 and 3 and whose numerical values differ by 2, by applying the Sieve of Eratosthenes. And we confirmed that, where we continue this sieve infinitely, when the combination of twin primes with the last digit being 1 and 3 is finite, then from some point all numbers are marked.

Next, we saw a game called "Infinite Game". What we did here was the same as the Sieve of Eratosthenes above, with one exception. The only exception was that we could choose the mark point artificially, regardless of whether it was a twin prime number. It was a game that aimed to mark all numbers in that way from a certain number. And we confirmed that if we couldn't mark all the numbers even if we chose the mark points artificially, it would be impossible to mark all the numbers with the Sieve of Eratosthenes. Then we saw that marking all numbers in this game is almost impossible.

Finally, it proved impossible to mark all numbers in "Infinite Game" by intervening of the technique of "Floor Line Arrangement". In the world shown here, the plates arranged in "Floor Line Arrangement" collide with the ceiling somewhere no matter which number we start, and space is always lost. In them, the plates of player of the "Infinite Game" that started from the same number as the "Floor Line Arrangement" cannot dive under the floor, and as a result, it is forced to collide with the ceiling.

The fact that we cannot mark all numbers in the "infinite game" means, as we have already confirmed, that we cannot mark all candidates with the Sieve of Eratosthenes, that is, there are infinitely many twin primes whose last digits are 1 and 3. Through the same method, it can be shown that there are infinitely many twin primes whose last digits are 7 and 9, and 9 and 1. In this way, there are infinitely many twin prime numbers.

Treasures from this proof

As one precedent, the useful treasures left to other researchers by this proof are the influence of simplification, the significance of reconfirming the conditions, the utility of a new kind of method for reduction to absurdity, and the value of constructing a proper perspective.

An example of the influence of simplification is that the analysis target is limited to twin prime numbers whose last digits are 1 and 3. Regarding the limitation, we mentioned that "it seems contradictory to tighten the conditions" in the introduction, but actually, simplification made it easier to see the number structure.

The significance of reconfirming the conditions is the importance of re-understanding the given situation more accurately. Even if it is the world in the infinite number of numbers, there are various restrictions. The ceiling that appears in this paper is one of them. This constraint has been known since ancient times, but it seems that it has not been perceived as a real feeling to the fact that the ceiling continues too low, even if we have known that the graphs of $y=\sqrt{x}$ and $y = 1/10000 * \sqrt{x}$ are similar figures. When conducting research, it is necessary to reconfirm the conditions carefully and eliminate not only obvious mistakes and oversights but also sensory deviations.

The utility of a new kind of method for reduction to absurdity is as follows. This paper can also be interpreted as being logically developed by a certain absurdity method. That is, when we think that "We assumed that “it is possible to mark all candidates for twin prime numbers with the Sieve of Eratosthenes” and led to the contradiction", then we can interpret that way. However, in the conventional absurd law, the judgment standard has been either "true" or "false", and although it could be repeated, it has been basically a world of dichotomy. But, in this paper, "Infinite Game" implies "maybe false" to the hypothesis at first. Then, the hypothesis is finally judged as "false" through the "Floor Line Arrangement".

Here, what we should take notice of is the role of the "Infinite Game". Even if the "Infinite Game" is not enough to judge whether the hypothesis is true or false by itself, it is impossible to reach the conclusion without it, that is, only with the Sieve of Eratosthenes and the "Floor Line Arrangement". Thus, what we showed here was an absurdity not just based on "true" or "false", or, in other words, an absurdity not based solely on the dichotomy.

We can say that a new kind of absurd law made it possible to develop more detailed logic.

Constructing a proper perspective is, in this paper, a presentation of "Infinite Game" and "Floor Line Arrangement". In this paper, the new kind of absurdity method described above was made possible by the construction of proper perspectives. It would be nice if this paper is remembered as one of the example that proactively creating tools and constructing new mechanisms can be a breakthrough to a difficult problem.

We hope that this paper will bring many researchers with various hints and inspirations.

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