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The Philosophical Implications of Set Theory in Infinity

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Abstract

What does the term "Infinity" mean? There are mathematical, physical and metaphysical definitions of the concept of limitlessness. This study will focus on the scripting of the three philosophical foundations of mathematics-formalism, intuitionism and logicism-in set theory.

Examples will also be provided of the concept of infinity for these three schools of thought. However, none of them cannot prove whether there is an infinite set or the existence of infinity. It forms the foundational crisis of mathematics. Further elaboration on these schools of philosophy leads to the ideas of actual, potential and absolute boundlessness. These correspond to three basic definitions of infinity. These will correspond to Roger Penrose's Three World Philosophy and hence implies a quantum mind. By employing those special "rational proof" and set theory in this thesis, one may even build a real thinking computer. Indeed, by using Basic Metaphor Infinity, cognitive mechanisms such as conceptual metaphors and aspects, one can appreciate the transfinite cardinals' beauty fully. This implies the portraiture for endless is anthropomorphic. In other words, because there is a connection between art and mathematics through infinity, one can enjoy the elegance of boundlessness. In the essence, this is what mathematics is: the science of researching the limitless. To conclude, the thinking machine's behind in this paper is very beautiful.

Keywords: Three world philosophy; Quantum mind; Infinity

Introduction

What is the concept of infinity? According to the Encyclopaedia Britannica, one may refer to infinity as the concept about something which is endless, unlimited, or without bounds. In 1657, English mathematician John Wallis introduced the common symbol "∞" for infinity. One can categorize the concept according to three schools of thought:

1. Mathematically: the number of points on a continuous line; or the size of endless sequence in counting numbers: 1,2,3,….

2. Physically: whether the number of stars is infinite; or whether the universe will last forever.

3. Meta-Physically: the discussion of God[1].

It is therefore important to begin investigating the term "Infinity" with these definitions in mind. Moreover, in the 19th century, Georg Cantor proposed what he called "transfinite numbers" from set theory. This in turn caused a great dispute between himself and Leopold Kronecker. Thus, one needs to understand how the philosophical implications for infinity in modern set theory.'

Literature Review

When one tries to resolve the conflict between Cantor and Kronecker, one needs first to understand the arguments behind them. The following sections will describe their differing perceptions in the foundations of mathematical philosophy concerning infinity.

Cantor's formalization of infinity

What is Cantor's mathematical definition of infinity? He developed the idea through set algebra and proposed what can be termed as "Infinity Arithmetic". Indeed, Cantor tried earlier to formalize his ideas and that connected infinity and infinite sets[2]. The origin of Cantor's ideas and their subsequent development will be explained in more depth in the following paragraph.

[1]http://global.britannica.com/topic/infinity-mathematics#toc252429
[2]https://en.wikipedia.org/wiki/Infinity

Origin of Cantor's Theory: Galileo's Paradox: Galileo Galilei (1564-1642) was a Physicist, mathematician and astronomer[3]. He noticed that there are equally as many natural numbers as there are perfect squares of natural numbers [1]. One can draw one-to-one correspondence. For example:

1 is assigned to 1,

2 is assigned to $2^2=4$,

3 is assigned to $3^2=9$…..etc.

It should be noted that f is said to be one-to-one if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, however there may exist $y \in Y$ but one cannot find an $x \in X$ and $f(x)=y$. Therefore, the number of X ≤ the number of Y. However, the set of square natural numbers is properly contained in the set of all-natural numbers [1]. That is the number of Y ≤ the number of X. Therefore, one may conclude the number of X=number of Y. There must be a one-to-one and onto mapping (bijection) between these two sets of numbers. To be more precise, one will have the following (as described by Velickovic):

Definition: For any given two sets X and Y

One may write $X \preccurlyeq Y$ if there exists an injection from X to Y; One may write $X \approx Y$ if there exists a bijection between X and Y[4] (Figure 1).

Theorem (Cantor - Bernstein): Suppose $X \leq Y$ and $Y \leq X$, then $X \approx Y$.

[3]http://www.math.vanderbilt.edu/%7Eschectex/courses/infinity.pdf
[4]https://en.wikipedia.org/wiki/Bijection,_injection_and_surjection

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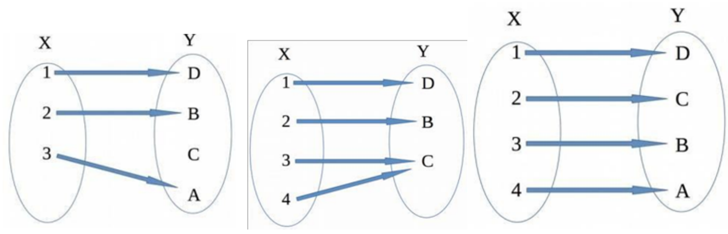

Figure 1: For any given two sets X and Y.

Brief Proof: This means that there is an injection mapping from X to Y. Also, there is an injection mapping from Y to X [2]. Hence by definition, for every element in Y, there is at most one pre-image element in X such that f(x)=y. i.e. number of X ≤ number of Y. Similarly, for every element in X, there is at most one pre-image element in Y s.t. g(y)=x. i.e. the number of elements in Y ≤ the number of elements in X. Thus, the number of X=number of Y. There must be a bijection mapping between X and Y. Details of the proof will be shown as follows[5].

Proposition: X is infinite iff $X \approx X/\{x\}$, for any $x \in X$.

Definition: X is countable if $X \approx \mathbb{N}$.

The author notes that a set X is countable if one can find a bijection between X and ℕ or some subset of ℕ. Thus, one may conclude that for any finite set X is countable (when a bijection exists between X and subset {0, 1, 2,…..m-1} where m=size of X).

Cantor's transfinite number: As previously mentioned, the number of elements in a set is important in determining the countable properties of it. Thus, one has the following circumscriptions:

Definition: The number of elements in a set X is defined as cardinality and denoted as |X|. Where the cardinal is called the transfinite number.

Definition [3]: $|\mathbb{N}|$ is denoted by $\aleph_0$.

Theorem (Cantor): The set of rational numbers ℚ is countable.

Proof: It is obvious that $\mathbb{N} \subset \mathbb{Q}$. Then $|\mathbb{N}| \leq |\mathbb{Q}|$. Therefore, our goal is to prove $\mathbb{Q} \subset \mathbb{N}$ or $|\mathbb{Q}| \leq |\mathbb{N}|$. One may erect a $f: \mathbb{Q} \to \mathbb{N}$ with the later diagram of a spiral transmits the concept of this function [4] (Figure 2):

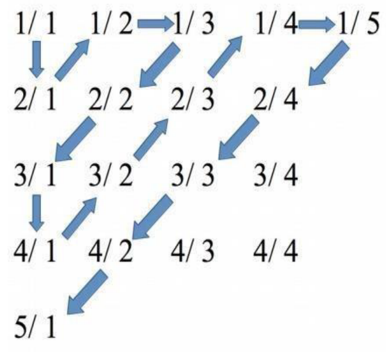

Figure 2: Diagram of a spiral transmits.

One may further construct the following ordered pairs: (1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 1)….. Then write each pair as a fraction: 1/1, 2/1, 1/2, 1/3, 2/2, 3/1….. before deleting those repeated pair and mapping them to 1, 2, 3, 4, 5…..

$$\begin{array}{lllll} 1/1, & 2/1, & 1/2, & 1/3, & 3/1\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5\ldots \end{array}$$

Obviously, the above mapping is an injection between ℚ and ℕ and hence $|\mathbb{Q}| \leq |\mathbb{N}|$. One will have $\mathbb{Q} \preccurlyeq \mathbb{N}$ Therefore, with theorem 2.1.1.2, one will get $\mathbb{Q} \approx \mathbb{N}$ and 2.1.1.4, ℚ is countable. It seems straightforward but the violation found was so amazing such that Cantor said "I see this result, but just cannot believe it [5]!".

Thus, the members of the set fraction's collection is dense, but the number of members and the difference in the distance of natural numbers are equal. Studying this, Cantor decided to give a label to the collection of a countable number, which is called them a power of $\aleph_0$. (ℵ is pronounced as "aleph", the first Hebrew letter). A collection of power $\aleph_0$ and natural numbers have the same number, so are countable [5].

In addition, from the above proof, one may ask (as did Cantor): "Are all infinite sets countable?" To answer this question, this study suggests using the following theorem below:

Theorem: The set of real numbers ℝ is uncountable.

Proof: On the contrary, suppose there is a bijection $f: \mathbb{N} \to \mathbb{R}$ [0, 1]. One can enumerate the infinite list in general as follows [4,5]:

$$r_1 = 0.a_1a_2a_3\ldots\ldots$$
$$r_2 = 0.b_1b_2b_3\ldots\ldots$$
$$r_3 = 0.c_1c_2c_3\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{where } a_i, b_i, c_i \text{ is selected from } \mathbb{N}$$

Then one may choose those diagonals such that where it forms a new real number $r=0.a_1b_2c_3$….. Now consider the real number "s" obtained by modifying every digit of r, say by replacing each digit x_i with x_i+2 mod 10. This study claims that "s" will not appear in the above list of infinite real numbers. Conversely, suppose that "s" existed, and it was the nth number in the list. Then r is differed from s in the n^{th} digit x_n: the n^{th} digit of s is the n^{th} digit of r plus 2 mod 10. That is $s \neq r_n$ for any r_n in [0, 1]. Hence "s" is a real number but obviously not in the range of f. This will contradict the fact that f is a bijection. ℝ is therefore uncountable. Cantor used the letter "C" to represent the power ℵ for set rational numbers [5].

[5]http://math.gmu.edu/~tlim/bernstein.pdf

In such a way, Cantor developed hierarchical orders for infinity. For example, the power ℵ named C in such a set is higher than the hierarchy of set with power $\aleph_\circ$. However, there will not be an infinite set with hierarchy of power lower than $\aleph_\circ$; even if one tries to form a set of $\mathbb{N}^2$ and deleted all the non- $\mathbb{N}^2$, the result remains the same.

The arithmetic of transfinite cardinal number: Consider the subsets of set S={a, b, c}, one can have {a}, {b}, {c}, {a, b}, {a, c}, {b, c} together with empty set { } and the original set {a, b, c}. In total there are eight subsets from the set which contains three elements. One calls the collection of these subsets power set P(S). There are 2^3 such subsets. The result can be extended to any finite set consisting of n elements and has 2^n elements for the power set P(N). Similarly, Cantor expanded the consequence to the infinite sets. When one considers the power set P(inf) of any infinite set, the numbers of elements of such a set P(inf) is greater than that of original set. This way of thinking was revolutionary and unbelievable in Cantor's time.

Cantor concluded: If one starts from any finite or infinite set S, one can create a new set P(S) which has more elements than set S [5]. By repeating this process, one can make more new power set. Thus, one can have infinitely hierarchical infinite sets, each new power set introduces a larger new set with larger ℵ. Cantor used $2^{\aleph_\circ}$, $4^{\aleph_\circ}$ where in the hierarchy, those sets with same cardinal number are with one to one correspondence, while those sets with different numbers cannot be one to one.

The cardinal number $\aleph_\circ$

$$2^{\aleph_\circ}$$

$$(2^2)^{\aleph_\circ}$$

Cantor named transfinite cardinal numbers. In the case of a finite set, the hierarchical relationship is:

n < 2n < (22)n <…... When one expands these transfinite cardinal numbers, one will have

$\aleph_\circ >$

$2\aleph_\circ >$

$(22)\aleph_\circ >$…..

Hence, by extending the above comparison, Cantor established so called arithmetic transfinite cardinal numbers which have some strange rules:

$1 + \aleph = {}_\circ\aleph_\circ$, ℵ 0 + ℵ 0 =

ℵ 0 * ℵ 0 * ℵ 0

= ℵ 0

To conclude, Cantor tried to develop those philosophies concerning infinite sets of different sizes. Nevertheless, his formalism caused a great deal of controversy with Kronecker's Finitism – an extreme case of intuitionism. This will be discussed more in depth in the next section.

Kronecker's finitism and intuitive objections to transfinite number: Kronecker's Finitism is a form of mathematical philosophy that accepts the presence of only finite mathematical objects. It is therefore best understood when one compares it to the conventional philosophy of mathematics where infinite mathematical objects (e.g., infinite sets) are considered as reasonable. The major concept of finitistic mathematics is asserting the non-existence of infinite objects, e.g. infinite sets. One will consider all-natural numbers as existing. However, the set of all-natural numbers is thought to be non-existing as a mathematical object. Therefore, any quantification over infinite domains is not relevant. One of the most famous mathematical theories in finitism is Skolem's primitive recursive arithmetic[6].

Indeed, one can prove the non-existence of infinite sets through the following counter example[7]:

Consider a set S={66454517, 3, 507}, one can rewrite it as:

66454517

03

507

1. Construct a number from a diagonal where one will take the "units" digit from the first number,

i.e. "7" in this case.

2. Next one gets the "tens" digit from the following number by adding a zero before "3".

3. Continue the recursive process; one will have "507" which is just in the set S.

4. The last step is to change each digit in the "found" number to any other digit. For example, 5 changes to 9, 0 changes to 1 and 7 could change to 1 also.

5. The result is "911".

However, the number is not in the set. This counter example tells us that one cannot have a complete set of ALL-natural numbers as one will always find some numbers not in the set.

Cantor's proof assumes that "infinitely many" is a valid idea and is employed in disproving the above counter argument. According to Cantor, one can always create a number that is not in the set S but has "infinitely many" digits. Hence the created number is not a natural one and consequently must not be in the set S.

Even if one accepts that "infinitely many" is valid, the counter example is still true. This is because some digits created by the recursive process have a finite number of digit places away from the lowest-value digit. At the same time, other digits must have an infinite number of places away from the lowest-value digit. Obviously, there is an inconsistency which should not occur since the recursive process is uniform. Therefore, finitism demonstrates that Cantor's formalization of an infinite set is invalid and only finite sets exist.

These serious criticisms were one of the reasons why Cantor became gloomy. His former teacher, famous mathematician Leopold Kronecker, tried to attack transfinite numbers even more. In fact, Kronecker was very conservative and was not only resisting the concept of the infinite, but also commenting on those mathematical theories based on natural numbers. The reason for these attacks was not purely academic but also due to jealousy. He saw his student's reputation becoming greater than his own. Sadly, Cantor died in a mental hospital in 1918.

To summarize, through Kronecker's finitism (the non-existence of infinite sets) and an intuitive recursive process, one can illustrate that there are no Cantor transfinite numbers and hence all philosophies mentioned in section 2.1 fail. After reviewing the relation between

[6]https://en.wikipedia.org/wiki/Finitism

[7]http://www.extremefinitism.com/blog/infinity-and-infinite-sets-the-root-of-the-problem/

infinite sets and formalism and intuitionism, this study will proceed to discuss Dedekind and logicism in the next section.

Dedekind's Infinite set and Logicism

According to Dedekind8:

"A set X is infinite if only if it is equivalent to a proper subset of itself[8]."

Given sets S and T, they are said to be equivalent if only if there exists a bijection *f*:

$S \rightarrow T$

between elements of S and those of T[9]. Proof:

Case I: "if" part,

Let X be an infinite set. Since every infinite set has countably infinite subset[10], thus one can possibly construct one from X.

Suppose $S=\{a_1, a_2, a_3, \ldots\}$ is a countably infinite subset of X.

One can create a partition of S into:

$S1=\{a_1, a_3, a_5, \ldots\}$ and $S_2=\{a_2, a_4, a_6, \ldots\}$

In addition, assume that there exists a bijection established between S and S_1 s.t.

$a_n \leftrightarrow a_{n-1}$

One can further extend the bijection between:

$S\cup(X\backslash S)=X$ and $S_1\cup(X\backslash S)=X\backslash S_2$

One can demonstrate that a bijection can be created between T and one of its proper subset $T \backslash S_2$. Hence one can conclude if X is infinite, then it is equivalent to one of its proper subsets.

Case II: "only if" part,

On the contrary, suppose X is equivalent to one of its proper subsets, say X0,

i.e. $X0\subset X$ and there exists a bijection *f*: $X \rightarrow X0$. Nevertheless, there is no bijection between finite set and its proper subset[11]. Therefore, X must be infinite.

Indeed Dummett, offered a critique of Dedekind's Infinite set theory:

"Dedekind's philosophy of mathematics was that mathematical objects are 'free creations of the human mind'. The idea, widely shared by his contemporaries, was that abstract objects are created by the operations of our minds. This would seem to lead to a solipsistic conception of mathematics; but it is implicit in this conception that each subject is entitled to feel assured that what he creates by means of his own mental operations will coincide, at least in its properties, with what others have created by means of analogous operations [6]. For Frege, such an assurance would be without foundation: for him, the contents of our minds are wholly subjective; since there is no means of comparing them, I cannot know whether my idea is the same as yours."

After reviewing the three categories of philosophy, one finds that none of them can fully explain the concept of infinity in mathematics. Further work should be done between mathematicians and philosophers to create a model for it. However, one should still appreciate the beauty in the method of the various proofs about infinity in these schools. In addition to these proofs, there are also daily examples of teaching infinity through mathematical philosophy; these will be discussed in detail in the next section.

Examples of teaching infinity using mathematical philosophy

During every day learning at university, the present author has encountered several different concepts of infinity in mathematics. These will now be discussed one by one through the application of mathematical philosophy.

Teaching calculus using the concept of limit in formalism

One of the most famous examples in teaching the idea of infinity is calculus. It allows one to teach formally the concept of a limit. To begin, one should understand the well-known Paradoxes of motion:

"Achilles allows the tortoise a head start of 100 meters, for example. If we suppose that each racer starts running at some constant speed (one very fast and one very slow), then after some finite time, Achilles will have run 100 meters, bringing him to the tortoise's starting point. During this time, the tortoise has run a much shorter distance, say, 10 meters. It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther; and then more time still to reach this third point, while the tortoise moves ahead. Thus, whenever Achilles reaches somewhere the tortoise has been, he still has farther to go. Therefore, because there are an infinite number of points Achilles must reach where the tortoise has already been, he can never overtake the tortoise[12]."

Long before Cantor, Zeno's paradoxes (Zeno of Elea, 490 – 425 BC) shows how poor people's understanding was of the concept of infinity:

"There is no motion, because to get anywhere you'd first have to get halfway, and before that you'd have to get a quarter of the way, etc [4]. (Figure 3)."

The figure above shows one of Zeno's Paradoxes. It is used to tell us how worst the knowing of infinity earlier on Cantor is.

Nowadays, one can explain Zeno's Paradox as:

$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots=1$ which is an infinite series with the sum equal to the limit of finite partial sums.

Indeed, one will never get 1. The infinite series should be rewritten as:

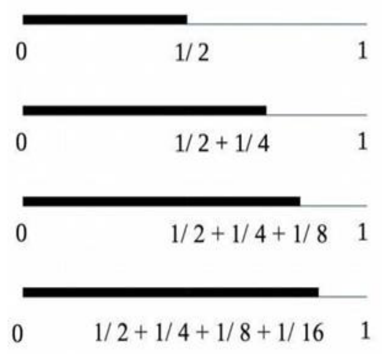

Figure 3: one of Zeno's Paradoxes.

[8] https://proofwiki.org/wiki/Infinite_Set_Equivalent_to_Proper_Subset
[9] https://proofwiki.org/wiki/Definition:Set_Equivalence
[10] https://proofwiki.org/wiki/Infinite_Set_has_Countably_Infinite_Subset
[11] https://proofwiki.org/wiki/No_Bijection_between_Finite_Set_and_Proper_Subset

[12] https://en.wikipedia.org/wiki/Zeno%27s_paradoxes

…..+ 1/32 + 1/16 + 1/8 + 1/4 + 1/2=1

This means that one will have infinitely many steps before one gets ½ ways or gets 1/4 way. If each step takes 1 second, one will never get anywhere.

Nevertheless, when one take an in-depth look at the infinite series, if there is a sufficient number of items with common ratio equals to "1/2", then the sum of the series can be approached arbitrarily closer to 1. Suppose the runner has a constant speed "s", then the time needed between two points is directly proportional to the distance travelled. Therefore, the total period required for the whole journey is the sum of the series multiplied by the constant "s". Hence only limited time is used for the travelling and this solves the controversy. The ancient Greek did not believe the infinite series to have a convergence to a limit.

Really for the sequence 1, 1/2, 1/4, …., 1/n, the limit is zero. As n increases, the value in the sequence becomes smaller and tends to zero but will never equal zero. It means if one has sufficiently large n, the sequence a_n can approach arbitrarily close to zero. For example, if one needs a_n to be smaller than one in a thousand, then one requires n to be larger than one thousand. Similarly, one can make an even as close as one in ten billions.

However, those mathematicians did not like such kind of lengthy explanation. They preferred a simpler description:

$a_n \to L$ when $n \to \infty$ or $\lim_{n \to \infty} a_n = L$

Thus, one can represent the sequence by $1/n \to 0$ when $n \to \infty$ or $\lim 1/n=0$. $n\to\infty$

The term $\lim a_n$ is a kind of symbol under mathematical rules meaning "limiting value of…." $n\to\infty$

When n tends to infinity. It obviously connects the ideas of endlessness and formalism. To be more precise, an abstract definition of limit may be as follows:

Suppose there exists a finite value "a", then for any given positive integer ε, there is always a positive integer N(ε) such that n > N, $|xn - a| < \varepsilon$, then one says the sequence $\{x_n\}$ is with limit "a"; $\lim X_n=a$; $x_n \to a$, $n\to\infty$

After viewing the example for origin, linking and teaching concepts between limit (a kind of infinity as well as calculus) and formalism, one may proceed to the inferior part [7].

Using geometry to learn infinity mathematics intuitively

Consider the following function: $y=1/x$

Where the diagram is likely described as below:

(Figure 4) The above figure on the left shows a hyperbola (positively branch). Suppose it is rotating around the x-axis, then it becomes a hyperbolic rotator as shown on the right. If one integrates the surface area from x=1 to infinity of the rotator, the area is infinitely large. To be more precise, one can show that when calculate the corresponding surface area from $x=1$ to some value greater than 1 such as t and letting "t" tend to infinity, there will be no bound to the area. However, the corresponding rotator's volume will tend to a certain limited value. In other words, there is a limit in volume for the infinite 3D figure. If one wants to paint on the surface of the 3D figure, the job cannot be finished as one needs an infinite amount of paint. On the other hand, if one paints the surface inside of the rotator, then it is finite. This paradox cannot find a simple explanation and tells us that when anything is related to the concept of infinity, one's intuition can produce errors.

Furthermore, relationships exist between these paradoxes and so-called "Morbid Functions".

For example, one considers $y=\sin(1/x)$ which has some special properties:

1. When x approaches to zero, the graph of the function's oscillation becomes larger, thus it can never be depicted completely. If one compares it with the hyperbola's graph, there is also a "discontinuity" when x=0. However, the difference is that this function's graph will not tend to infinity but only the oscillation frequency becomes infinite.

2. If one considers the related function $y=x\sin(1/x)$, the singular point x=0 will disappear and hence the function becomes continuous (Figure 5).

Consider the function $y=f(x)$ where

$$f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q} \\ -1 & \text{when } x \notin \mathbb{Q} \end{cases}$$

Then for the upper line y=1, there are an uncountably number of holes since $\mathbb{R}$ is uncountable. On the other hand, for y=-1, there are a countable number of holes as $\mathbb{Q}$ is countable. This special function discontinues everywhere and leads to controversies. As a result, the concept of "continuous" is redefined.

To conclude, from Jones [8], "We define 'geometrical intuition' as a skill to create and manipulate geometrical figures in the mind, to see geometrical properties, to relate images to concepts and theorems in geometry, and decide where to start when solving problems in geometry. We suggest that tasks that require students to imagine and manipulate geometrical figures can link geometrical intuition more directly with geometrical theory and involve active use of imagination skills."

Applying contradictory logicism in studying infinity algebra

When one is concerned with contradiction in infinity, one might refer to an analogy proposed by philosopher William Lane Craig who was involved in the calculations of certain kinds of infinite sets:

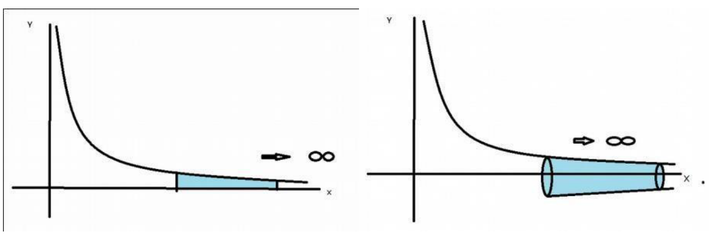

Figure 4: Left shows a hyperbola and hyperbolic rotator shown on the right.

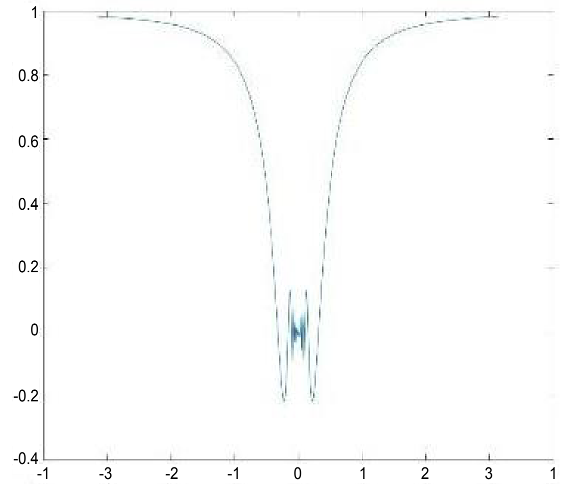

Figure 5: MathLab scripting of plotting function "xsin(1/x)" x=[-pi:0.01:pi]; y=(x).* (sin(1./x)); plot(x,y).

"Imagine that one has an infinite number of marbles in his possession and he wants to give other some of them. In addition, suppose one (say A) wants to give other one (say B)an infinite number of marbles [9]."

Case I: One (A) could do that by giving all marbles to the other one (B). Hence, A will have zero marbles left for him. i.e. $\aleph 0 - \aleph 0 = 0$

Case II: Alternatively, A gives B all the odd numbered marbles. Then A will still have an infinite number left for himself and B will also have an infinite set too. i.e. $\aleph 0 - \aleph 0 = \aleph 0$

Obviously, the above two cases show contradictory results (0 and ℵ 0). However, subtraction and dividing sets of equal amounts should not produce them. These collisions cast doubt on the idea that infinite can be treated as a coherent notion. Nevertheless, other philosophers such as Morrision and Gumiski believe that there are discrepancies between subtracting infinite sets in transfinite mathematics (which will normally result in absurdities). The "removal" of one infinite set from another does not happen in the present real world. In addition, the present author notes that the paradox leads to the logical contradiction of an infinite set being both "Divisible" and yet "not divisible" which results in both a mathematical and a logical contradiction [9].

There cannot be infinite set of marbles or an infinite set of anything. Thus, this may lead to the concept of indefiniteness which can resolve the problem.

After discussing the views and examples from the three categories of philosophy of mathematical infinity, one turns to the philosophical implications.

Discussions

Types of infinity and their philosophical implications

For each kind of mathematical philosophy, there are matters of corresponding philosophical significance such as the countability of numbers, the start of the universe and the big bang as well as the existence of God.

Cantor's theory implies actual limitless: Countability of numbers

What is actual limitlessness? One usually refers to the concept as an ongoing process which is repeated over and over but it is conceived as being "completed" or as having a final resultant state [10]. For instance, one can contemplate the sequence of regular polygons with an increasing number of sides where the distance from the centre to any of the vertices remains constant. Indeed, one always begins with a triangle, then a square, a pentagon, a hexagon and so on endlessly until it completes a circle. This is because after each iteration, there is an increase in one to the number of sides, the side's length becomes smaller but the distance "r" between the centre of the polygon and the vertices remains the same. As one continues the process, the area and the perimeter of the polygon gets closer and closer to the value in πr2 and 2πr respectively. The circle has all the prototypical properties that circles must have but conceptually it is a polygon. "The main theme of Cantor's Grundlagen is that there are multiple actual infinities, because there is a realm of an actual, but increasable infinite known as the transfinite [11]." This is the reason why it is proposed here that there is a connection between actual limitlessness and Cantor's Theory of transfinite numbers. Through these cardinals, one can tell the countability of "Natural", "Rational" and even "Real" numbers as mentioned earlier.

Kronecker's disagreement suggests potential boundless: Big bang theory

On the contrary, potential boundless means "a non-terminating process (such as "add 1 to the previous number") produces an unending "infinite" sequence of results, but each individual result is finite and is achieved in a finite number of steps[13] ." It is proposed here that this definition is related to Kronecker's finitism as well as Skolem's primitive recursive arithmetic. In fact, according to Craig[14], what the past is made up of a series of events (E1, E2…En). Usually, one refers to "the past" as the set of all these events. Hence, the past cannot be an actual infinity. If one begins with a single event and adds one another gradually, one will get never get an actual infinite number of events. What one ends up with is a set whose number of constituents becomes ever larger and tends towards infinity but never actually reaches it[15]. This is defined as a "potential infinity". Therefore, it appears to the present author that there must be a beginning of time. What are entailed by "potential boundlessness" are the Big Bang theory and thus a "Start of the Universe".

Recently, some astronomers have combined observations and mathematical models in order to develop a workable theory of how the Universe came to be. According to the theory, our universe previously existed as a "singularity" 13.7 billion years ago. Singularities are thought to be "black holes" with intense gravitational pressure where finite matter is crushed into infinite density. Our Universe is believed to have started as an infinitely hot, infinitesimally small, infinitely dense, something-a singularity. Beyond initial shape, the singularity seemingly inflated (the so called "Big Bang"), cooled and expanded, and began changing from being extremely small and extremely hot, to the current temperature and size of the universe. The process is ongoing and human beings are a part of it-incredible creatures that are living on a unique planet, circling with a beautiful star. The planet is also clustered with some hundred billion other stars in a galaxy soaring through the cosmos. All this happening inside an expanding universe that began as an infinitesimal singularity which appeared out of nowhere and for reasons unknown. This is one suggestion what the Big Bang theory is.

[13]https://en.wikipedia.org/wiki/Actual_infinity
[14]https://en.wikipedia.org/wiki/Actual_infinity
[15]http://ieet.org/index.php/IEET/more/danaher20141015

Dedekind's concept describes absolute endlessness: The existence of god

What is the definition of absolute endlessness? Some philosophers believe that the class V of all sets, the mindscape, the cosmos, and God are all examples of what "absolute" is[16]. Indeed, one may define the term "absolute" in the sense of "non-relative, non-subjective". The absolute itself exists together with the highest form of completeness. In actual, there is a relationship between the limitlessness of God and mathematical infinity. As St. Gregory said, "No matter how far our mind may have progressed in the contemplation of God, it does not attain to what He is, but to what is beneath Him [12]." In such a case, one is now in the rudiments of the infinite dialectic process since one is trying to establish an image of the whole mindscape. The dialectic process happens in the following way:

1. First, one collects a group of thoughts into a single thought T.
2. When one is in a conscious state of mind T, a new thought is then constructed that one has not yet accounted for previously.
3. One's mindscape is improved for thought including the elements of T plus T itself.

In mathematically symbolic term, one may consider the *n*th thought T_n, one can define inductively: $To=\emptyset$ and $Tn + 1=Tn \cup \{Tn\}$, for any sets *A* and *B*. *A*U*B* means the set of all the sets that are members of *A* or of *B*. Nevertheless, one may have another inductive definition: $T_n=\{Tm$: $m< n\}$, which means "*Tn* is the set of all *Tm* such that *m* is less than *n*."*17* However, T plus "T" is not always different from the thought T. In the case of a mind M, it is already fully self-aware, and therefore M plus "M" is no different from M. i.e. M U {M}=M. Rucker discusses in detail "absolute" in terms of the rational and mystical. Indeed, during 1887 one of Cantor's friends, Richard Dedekind, issued a proof and claimed that the mindscape is infinite. Dedekind's word for mindscape was *Gedankenwelt,* which liberally means "thought-world". By continuing the repetitive process, Dedekind proved the infinitude of the mindscape:

{*s, s* is a possible thought, *s* is a possible thought is a possible thought….}

Therefore, this shows that the class of all sets, the mindscape and the class of all true propositions are all infinite. In similar terms, one can tell whether God is endless. There are comments about God's existence in Rucker's work, however, this author believes in the existence of God and His "Absolute".

Significance of This Research

Indeed, the most important of this paper is: this author is the first mankind tries to use mathematical set theory in terms of symbols to connect body, mind and spirit world together for the building of our first thinking computer (the proof is ac a special form of mathematical rationalization). It is because the consequence of such an action implies Penrose's three world philosophy and thus a quantum mind ("There is the inverse square law, which is a mathematical consequence of the Universe existing in 3 spatial dimensions that allows for extraordinarily stable orbits over astronomical time frames. Then there is quantum mechanics, which appears to underpin all of physical reality and can only be revealed in the language of mathematics.") It is no doubt that the development of a thinking machine will create an ethical controversy. In classical humanities, one may have a famous statement "I think therefore I am." It suspects the existence of us. But the suspect itself can also be suspected by somebody else. Thus, the process is similar in this author's section "Absolute Endlessness". Finally, this may lead to the Skepticism. The aim of it is to work against those who is suspecting our present knowledge. Then if you believe in Platonic mathematics and set theory, it is easy to convert everything (the special "rational" proof in this paper) into electronic and computing parts. Hence, a real man-made thinking computer can be achieved (Figure 6).

Conclusions

Infinity acts as a link between art and mathematics

After reviewing, providing examples and discussing infinity, the question remains "What is the role of limitlessness? This study finds that it acts as a connection between art and mathematics. For example, one can appreciate the beauty of transfinite cardinals and hence imply the portrait of infinity has human face:

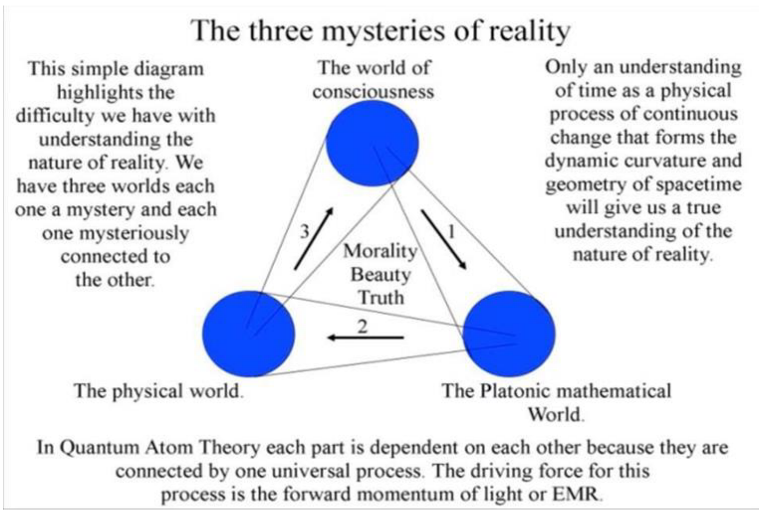

Figure 6: Roger Penrose's three world philosophy implies human beings' quantum mind.

[16]http://science.nasa.gov/astrophysics/focus-areas/what-powered-the-big-bang/
16http://www.big-bangtheory.com

1. Aspectual systems consist of continuative and iterative processes, those ideal and non-ideal structures with a starting status and completion status and so on.
2. Conceptual Metaphors such as in the case of Cantor's Metaphor Same Number as Is Pairability.
3. Conceptual Blending such as the use of multiple implicit Basic Mappings of infinity in Cantor's proofs.

Indeed, Cantor's theories and proofs often worked against the mainstream mathematicians' thought, and hence why he surprised many people [13]. His idea also caused discomfort among certain mathematics professionals. Meschkowski so eloquently wrote, "Cantor's theorem is thus a beautiful example of a mathematical paradox, of a true statement which seems to be false to the uninformed" [14].

Nevertheless, what a mathematician focuses on with infinity is that "Mathematics takes us into the region of absolute necessity to which not only the actual world, but every possible world, must conform [15]."

From a philosopher's point of view: "Mathematics is an ideal world and an eternal edifice of truth. In the contemplation of its serene beauty man can find refuge from the world full of evil and suffering" [16].

From the astronomer James Jeans (1877-1946), "God is a mathematician" and most mathematicians say "God made the numbers. All the rest is made by human [17]."

In a nutshell, it is suggested here that mathematics is the science of studying infinity. When one is talking about its daily applications, these may include calculating the gravitational force for infinite mass, determining whether infinity exists in our physical universe, infinite regression arguments, computing such as arithmetic overflow and division by zero etc[17]. Below is a poem that can best depict boundlessness:

"L'infinito" (Giacomo Leopardi)[18]

This lonely hill was always dear to me, and

this hedgerow, which cuts off the view of

so much of the last horizon. But sitting here and

gazing, I can see beyond, in my mind's eye,

unending spaces,

and superhuman silences, and depthless calm,

till what I feel

is almost fear. And when I hear the wind stir in

these branches, I begin comparing that endless

stillness with this noise: and the eternal comes

to mind, and the dead seasons, and the present

living one, and how it sounds.

So my mind sinks in this immensity: and foundering is sweet in such a sea. (Translated by Jonathan Galassi)

[17]https://en.wikipedia.org/wiki/Infinity#Physics
[18]https://en.wikipedia.org/wiki/L%27infinito

Limitations

There are limitations to this thesis. They are:

Cantor's theory may not be true but it can be modified to contain no bugs. One of Cantor's Theory defects is Russell's paradox, as follows:

"If S were the set of all sets then $P(S)$ would at the same time be bigger than S and a subset of S."

To avoid such a paradox, one may extend the set theory into the 'New Foundation' (NF) set theory. Instead of assuming a set, we consider $\{s \in S: s \notin f(s)\}$ as a local 'type theory' and is a set in NF. One can then easily show by proof of contradiction that:

$|P_1(S)| < |P(S)|$

And eliminate Russell's Paradox.

Since my special 'rationalization' is based on the assumptions of both set theory and logic, it may suffice to fail if one challenges the validation of the premises. This is the question of Platonism and anti-Platonism. Another main result of Cantor's Theory is the famous 'Continuum Hypothesis' problem: Does any set exist which has cardinality between natural number and real number? One may even extend the above problem as follows: Does any set exist that has a size between $|S|$ and $|P(S)|$ for some infinite S?-Generalized Continuum Hypothesis problem. This author's answer is: there are various mathematical structures stay between natural and real numbers. One may subdivide them as small as possible. Their cardinality converges to either card (N) or card (R). The problem of continuum hypothesis is just our analog world which is continuous (or our real and human world). But the digital world is only "0"-card (N) and "1"-card (R). At the same time, one can make it as detail as one may want since there are structures between N and R.

References

1. Velickovic B (2010) The Mathematical Theory of Infinity. Sino-European Winter School in Logic, Language and Computation.
2. Rao S, Tse D (2009) Discrete Mathematics and Probability Theory. Pp1-5.
3. Bulatov A (2012) Bijection and Cardinality. Discrete Mathematics Pp1-17.
4. Schechter E (2002) Georg Cantor: The man who tamed infinity. Vanderbilt University Pp 1-33.
5. Maor BE (1986) To Infinity and Beyond: A Cultural History of Infinite. Princeton University Press.
6. Dummett MAE (1991) Frege: Philosophy of Mathematics. Harvard University Press Pp1-352.
7. Fudan University (1978) Mathematical Analysis.
8. Jones K, Fujita T, Yamamoto S (2004) Geometrical Intuition and the Learning and Teaching of Geometry. ICME-10 Pp1-7.
9. Sewell KK (2010) The case against Infinity. Pp1-53.
10. Nűńez RE (2005) Creating mathematical infinities: Metaphor, blending, and the beauty of transfinite cardinals. J Pragmat 37: 1717–1741.
11. Newstead A (2009) Cantor on Infinity in Nature, Number, and the Divine Mind. American Catholic Philosophical Quarterly 83: 533-553.
12. Rucker R (2013) Infinity and the Mind: the Science and Philosophy of the Infinite. Princeton Science Library. Pp368.
13. Carey PH (2005) Beyond Infinity: Georg Cantor and Leopold Kronecker's Dispute over Transfinite Numbers. Boston College.
14. Dauben JW (1979) Georg Cantor: His Mathematics and Philosophy of the Infinite. Harvard University Press Pp424.
15. Egner R, Denonn L (2009) The Basic Writings of Bertrand Russell. London and New York.

16. Copleston FC (1966) A history of philosophy/8. Bentham to Russell. London: Search Press.
17. Peat ED (2002) From Certainly to Uncertainly: The Story of Science and Ideas in the twentieth Century. Washington DC Joseph Henry Press.