The Numerical Solution of Linear Fredholm Integro-Differential Equations via Parametric Iteration Method

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Abstract

In this paper a parametric iteration method (PIM) is proposed for solving Linear Fredholm Integro-differential equations (LFIDEs). The solution process is illustrated by some examples. Comparisons are made between PIM and exact solution and CAS wavelet method. The results show the simplicity and efficiency of PIM. Also, the convergence of this method is studied in this work.

Keywords: Parametric iteration method; Linear integro-differential equations; Numerical solution

Introduction

It is well known that many events in scientific fields deal with integro-differential equations. The linear integro-differential equations play an major role in many physical processes such as nano hydrodynamic [1], drop wise condensation [2], biological [3] and others. The various numerical methods exist for solving LFIDEs for example variation iteration method [4], Adomian decomposition method [5], Chebyshev Polynomials [6], Bernstein’s approximation [7]. PIM was applied successfully for solving boundary value problems [8]. We consider linear integro-differential equations as the following:

\[ L_1(u) + L_2(u) = f(x), \quad a \leq x \leq b \]  
(1)

And the initial value for both of this equation is as the following:

\[ U^0(x) = y_0, \quad (i=0,1,...,m-1). \]  
(2)

In this work, the numerical solution of (1) is possible by PIM when \( g, k, u \) are continuous functions. Parametric iteration method provides solution for LFIDEs as a sequence of iterations. In this study, some examples are given and we solve them using parametric iteration method and compare the obtained results with exact solution. In all examples, the present technique worked excellently, as it will be shown in this study.

The Basic Idea of the Parametric Iteration Method

In this section, we describe PIM for solving Linear Fredholm integro-differential equations. Then the convergence of this method is discussed.

Parametric Iteration Method

The PIM provides the solution for Equation (1) as a sequence of approximations. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes. We assume \( L_1 \) and \( L_2 \) are the linear operators on \( C^m[a,b] \). To explain the basic idea of PIM, we consider Equation (1) as below:

\[ L_1(u) + L_2(u) = f(x), \]  
(3)

where \( L_1 \) with the property \( L_1(g) \equiv 0 \) when \( g \equiv 0 \) denotes the so-called auxiliary linear operator with respect to \( u \), \( L_2 \) is a Linear continuous operator with respect to \( u \) and \( f(x) \) is the known continuous function and \( u \in C^m[a,b] \). The basic essence of this method is to construct a family of iterative processes for Equation (1) as:

\[ L_1[u_{k+1}(x)] - u_k(x)] = h H(x) A[u_k(x)], \]  
(4)

With the initial conditions

\[ U^0(x) = y_0, \quad (i=0,1,...,m-1) \]

Where,

\[ A[u_k(x)] = L_2[u_k(x)] + L_1[u_k(x)] - f(x) = u_k^{(m)} - \int_a^b k(x,t)u_k(t)dt - f(x), \quad k \geq 0 \]  

And \( u_k(x) \) is the initial guess which can be chosen arbitrarily but the suitable selection is positively affect for the rate of convergence [9], or it can also be solved from its corresponding linear homogeneous equation \( L_1[u_k(x)] = 0 \) or linear non-homogeneous equation \( L_1[u_k(x)] = f(x) \). The parameter \( h \neq 0 \) and function \( H(x) \neq 0 \) denote the so-called auxiliary parameter and auxiliary function. The selection of \( h, H(x) \) was described in [9]. Also, we are free to choose the auxiliary linear operator \( L_1 \), the auxiliary parameter \( h \), the auxiliary function \( H(x) \) and the initial approximation \( u_0(x) \). Therefore if the successive approximation \( u_k(x) \), \( k \geq 0 \) where are obtained by PIM in terms of the auxiliary parameter \( h \) then exact solution may be given by

\[ u(x) = \lim_{k \to \infty} u_k(x). \]

According to [9], let \( V = (u : u \in C^m[a,b]) \) be the solution space and \( \{c_j(x) : c_j(x) \in V, j = 0,1,...\} \) denote the set of base functions. Hence we can represent the solution in the series

\[ u(x) = \sum_{j=0}^{\infty} \alpha_j c_j(x), \]

where \( \alpha_j \) is a coefficient belonging to real numbers. As long as the set of base functions is determined, the auxiliary linear operator \( L_1 \), the initial approximation \( u_0(x) \) and the auxiliary function \( H(x) \) must be chosen in such a way that all solutions of the corresponding PIM equations (4) exist and it can be expressed by this set of base functions. Now, in order to avoid expensive computational works for solving (1) via PIM, it is straightforward to use the following set of base functions

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Received May 10, 2014; Accepted June 27, 2014; Published July 03, 2014


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\[(x-1)! = j=0,1,...\]
\[
That \ u(x) = \sum_{j=0}^{\infty} a_j (x-a)^j.
\]

Where \(a \in \mathbb{R}\) are unknown coefficients to be determined and \(a\) is a constant belong to real numbers. Now, we set the auxiliary operator \(L\) as the following:
\[
L[u(x)] = u^{(m)}(x)
\]

This region is called valid region of \(h\) which we note with \(R_h\). We ensure \(Lh \leq 1\) i.e. we let \(U = G(x, h)\) be the solution of \((1)\) with \(x = \beta\) that \(\beta \in [a, b]\) for \(x > 0\) and \(|1 + h < 1\) with the help of ODE. This region is called valid region of \(h\) which we note with \(R_h\). We ensure the solution series converge for any \(h \in R_h\).

\[
\Omega = \frac{\partial G}{\partial h} \bigg|_{h=\beta}, \ \beta \in [a, b], \quad (i = 1, 2, \ldots)
\]

Therefore \(\Omega\) will be in terms of \(h\), now we plot \(\Omega\) curve, according to these \(h\) curves, it is easy to find the valid region of \(h\), which corresponds to the line segments nearly parallel to the horizontal axis. This region is called valid region of \(h\) which we note with \(R_h\). We ensure the solution series converge for any \(h \in R_h\).

### Analysis of convergence of the parametric iteration formula

In this section we study the local convergence of approximate solution provided by PIM for solving \((1)\). Initially, We let \(u(0) = 0\), \((i = 0, 1, \ldots, m-1)\) and set \(L[u(x)] = u^{(m)}(x)\) therefore, we have from (5) the following parametric iteration formula:
\[
u_{i+1}(x) = (1 + h\nu_i(x)) - \frac{h}{(m-1)!} \int_{a}^{b} (x-t)^{m-1} g(t) + \lambda \int_{a}^{b} k(t, s) \nu_i(s) ds \ dz \ dt.
\]

The iterative formula (10) expressed by sequence makes a recurrence sequence \(\{\nu_i(x)\}\). Obviously, the limit of the sequence will be the solution of Equation (1) if the sequence is convergent. In order to prove the sequence \(\{\nu_i(x)\}\) is convergent, we construct a series
\[
\nu_0(x) + \nu_1(x) - \nu_0(x) + \ldots + [\nu_i(x) - \nu_{i-1}(x)] + \ldots
\]

Noticing that
\[
S_{i+1} = \nu_{i+1}(x) + [\nu_i(x) - \nu_{i-1}(x)] + \ldots + [\nu_i(x) - \nu_{i-1}(x)] = \nu_i(x)
\]

The sequence \(\{\nu_i(x)\}\) will be convergent if the series is convergent.

**Theorem**: \(g(x) \in C[a,b]\) and \(|\lambda| \leq \frac{L}{r}\) then the series of (11) is convergent, i.e., the sequence \(\{\nu_i(x)\}\) is convergent for \(\forall x \in [a,b]\).

**Proof**: According to (10), note that
\[
|\nu_i(x) - \nu_{i-1}(x)| = \left| \nu_i(x) - \nu_{i-1}(x) \right| \leq \left| \int_{a}^{b} (x-t)^m g(t) + \lambda \int_{a}^{b} k(t, s) \nu_i(s) ds \ dz \ dt \right|
\]

\[
\leq h \left[ L + \frac{1}{(m-1)!} \left( (Lh) + \frac{L^2}{2} (h-a) \right) \right] |h| r
\]

Where
\[
r = L + \frac{1}{(m-1)!} \left( (Lh) + \frac{L^2}{2} (h-a) \right), M = \max \{ |g(x)|, L \}, L_i = \max \{ |\nu_i'|, \nu_i'' \}, L_r = \max \{ (1-r)^{m-1} \}
\]

Now, we have from (10)
\[
|\nu_i - \nu_0| = \left| (1 + h\nu_i(x)) - \frac{h}{(m-1)!} \int_{a}^{b} (x-t)^{m-1} g(t) + \lambda \int_{a}^{b} k(t, s) \nu_i(s) ds \ dz \ dt \right|
\]

\[
\leq L + \frac{1}{(m-1)!} \left( (Lh) + \frac{L^2}{2} (h-a) \right) \left( h \right)^{m-1} \left( (1 + h)^{m-1} \right) \lambda \int_{a}^{b} \int_{a}^{b} k(t, s) \nu_i(s) ds \ dz \ dt
\]

\[
\leq h \left[ L + \frac{1}{(m-1)!} \left( (Lh) + \frac{L^2}{2} (h-a) \right) \right] |h| r
\]

In view of (18), the convergence of the series (11) can be concluded for the solution domain \(x > 0\) and \(|1 + h < 1\) with the help of some mathematical software. Therefore the series of (12) is absolute convergence, i.e., the sequence \(\{\nu_i(x)\}\) is convergent for \(\forall x \in [a,b]\).

### Illustrative Examples

Now, we use PIM to solve three examples of the kind of (1) and compare the obtained results with exact solution and CAS wavelet method [10], to show the efficiency of PIM.

**Example 1.** We consider the Linear Fredholm integro-differential of the second kind with exact solution \(u(x) = xe^{-x}\) as the following:
\[
u'(x) = xe^{-x} - xe^{-x} + \int_{a}^{b} u(t) dt, \ 0 \leq x \leq 1
\]
\(u(0) = 0\)

According to PIM proceeding, we define:
\[
L_1[u(x)] = u(x)
\]

And
\[
g(x) = xe^{-x} + e^{-x}
\]

Then we have from (8) \(u_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\) using initial condition and \(L_1[u(x)] = g(x)\) gives us that \(a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{2}\).

Now, If we set \(H(x) = 1\) then we obtain from (10),
\[
u_{i+1}(x) = (1 + h\nu_i(x)) - \frac{h}{(m-1)!} \int_{a}^{b} (x-t)^{m-1} g(t) + \lambda \int_{a}^{b} k(t, s) \nu_i(s) ds \ dz \ dt
\]

The obtained result for the 12th iteration was shown in Table 1 and compare with obtained results of CAS wavelet method [10]; also the valid region of \(h\) is depicted in Figure 1.

**Example 2:** For second example we consider the Linear Fredholm integro-differential of the second kind with exact solution \(u(x) = \cos(2\pi x)\) as the following:
\[
u'(x) = u(x) - \cos(2\pi x) - 2\sin(2\pi x) - \frac{1}{2} \sin(4\pi x) + \int_{0}^{b} \sin(4\pi x) + 2\pi u(t) dt
\]
\(u(0) = 1\)

For applying PIM, we get:
\[
L_1[u(x)] = u(x)
\]

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And
\[
g(x) = -\cos(2\pi x) - 2\pi \sin(2\pi x) - \frac{1}{2} \sin(4\pi x)
\] (26)

We have from (8), \(u_0(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3\). Using initial condition
and \(L_1[u_0(x)] = g(x)\) gives us that \(\alpha_0 = 1, \alpha_1 = -1, \alpha_2 = -\frac{16\pi^2 - 8\pi^2}{8\pi}, \alpha_3 = \frac{2}{3}\).

Now, if we set \(H(x) = 1\) then we obtain from (10),
\[
u_k(x) = (1 + h\nu_{k-1}(x)) + \frac{\pi - 1 - 4\sin(2\pi x) + 4\cos(2\pi x)\pi \cos(4\pi x)}{8\pi} \int_0^x \sin(4\pi t + 2\pi x)\nu_{k-1}(x) \, dt \, dx, \quad k = 1, \ldots
\] (27)

The obtained result for the 4th iteration was shown in Table 2 and 3 and compare with obtained results of CAS wavelet method [10]; also the valid region of \(h\) i.e. \(R_h\) was presented in Figure 2.

**Example 3:** we consider the Linear Fredholm-integro-differential of the second kind solution \(u(x) = \sin x\) as the following:
\[
u''(x) = -\sin x + x - \int_0^x t\nu(t) \, dt, \quad 0 \leq x \leq 1,
\] (28)
\[
u(0) = 0, \nu'(0) = 1
\] (29)

According to PIM procedure we set;

\[L_1[u(x)] = u'(x)
\] (30)
And
\[g(x) = x - \sin(x)
\] (31)
Then we have from (8) \(u_0(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3\), (25), (26) gives us that
\[\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 0, \alpha_3 = -\frac{1}{6}
\]
Now similar to example1, the iteration scheme is as the following:

\[ u_{n+1}(x) = (1 + h)u_n(x) - h \left[ \frac{1}{5} x^4 + \sin x \right] \int_0^1 \left[ \frac{1}{5} x^2 u_n(s) \right] ds, \quad k = 0, 1, \ldots \]  

(32)

The results of example 2 for 10th are available in Table 2. \( R_h \) for Equation (23) is presented in Figure 2 and 3.

**Results and Figure**

In this section we present the results of examples 1, 2 in two tables and plot the h-curve to determine \( R_h \). All the computations have been done with Maple 13.

**Conclusion**

In this article, parametric iteration method was applied to solve the Linear Fredholm-integro-differential equations. In order to illustrate the method we solve three examples. PIM results compared to CAS wavelet method shows that the former is easier in practice and more accurate for LFIDEs. Also we show the high accuracy of PIM in example 3 with comparing the obtained results with exact solution. Further the convergence of PIM for solving LFIDEs in the valid region of \( h (R_h) \) was presented. Additionally, if we increase the number of iterations by PIM scheme it seems the results will have more accuracy in solutions.

**References**