Journal of
Applied & Computational Mathematics

Tamamgar, J Appl Computat Math 2014, 3:4
DOI: 10.4172/2168-9679.1000171

Review Article Open Access

The Numerical Solution of Linear FredholmIntegro-Differential Equations via Parametric Iteration Method

Tamamgar M*

Department of Mathematics, Islamic Azad University, Gonabad Branch, Gonabad, Iran

Abstract

In this paper a parametric iteration method (PIM) is purposed for solving Linear FredholmIntegro-differential equations (LFIDEs). The solution process is illustrated by some examples. Comparisons are made between PIM and exact solution and CAS wavelet method. The results show the simplicity and efficiency of PIM. Also, the convergence of this method is studied in this work.

Keywords: Parametric iteration method; Linearintegro-differential equations; Numerical solution

Introduction

It is well known that many events in scientific fields deal with integro-differential equations. The Linear integro-differential equations play a major role in many physical processes such as Nano hydrodynamic [1], drop wise condensation [2], biologic [3] and others. The various numerical methods exist for solving LFIDEs for example variation iteration method [4], Adomian decomposition method [5], Chebyshev Polynomials [6], Bernstein's approximation [7]. PIM was applied successfully for solving boundary value problems [8]. We consider linear integro-differential equations as the following:

$$U^{(m)}(x) = g(x) + \lambda \int_a^b k(x,t)u(t)dt, \qquad a \le x \le b \tag{1}$$

And the initial value for both of this equation is as the following:

$$U^{(i)}(x_0)=y_i, \quad (i=0,1,\ldots,m-1). \tag{2}$$

In this work, the numerical solution of (1) is possible by PIM when g,k,u are continuous functions. Parametric iteration method provides solution for LFIDEs as a sequence of iterations. In this study, some examples are given and we solve them using parametric iteration method and compare the obtained results with exact solution. In all these cases, the present technique worked excellently, as it will be shown in this study.

The Basic Idea of the Parametric Iteration Method

In this section, we describe PIM for solving Linear Fredholmintegro-differential equations. Then the l convergence of this method is discussed.

Parametric Iteration Method

The PIM provides the solution for Equation (1) as a sequence of approximations. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes. We assume L_1 and L_2 are the linear operators on $C^m[a,b]$. To explain the basic idea of PIM, we consider Equation (1) as below:

$$L_1(u)+L_2(u)=f(x), \tag{3}$$

where L_1 with the property $L_1(g)\equiv 0$ when $g\equiv 0$ denotes the so-called auxiliary linear operator with respect to u, L_2 is a Linear continuous operator with respect to u and f(x) is the known continuous function and $u\in C^m[a,b]$. The basic essence of this method is to construct a family of iterative processes for Equation (1) as :

$$L_1[u_{k+1}(x)-u_k(x)]=hH(x)A[u_k(x)], \tag{4}$$

With the initial conditions

$$U^{(i)}(x0)=y_i, \quad (i=0,1,\ldots,m-1)$$

Where,

$$A[u_k(x)] = L_1[(u_k(x)] + L_2[u_k(x)] - f(x) = u_k^{(m)} - \lambda\int_a^b k(x,t)u_k dt - f(x), \quad k=0,1\ldots, \tag{5}$$

And $u_0(x)$ is the initial guess which can be chosen arbitrarily but the suitable selection is positively affect for the rate of convergence [9], or it can also be solved from its corresponding linear homogeneous equation $L_1[u_0(x)]=0$ or linear non homogeneous equation $L_1[u_0(x)]=f(x)$ The parameter $h\neq 0$ and function $H(x)\neq 0$ denote the so-called auxiliary parameter and auxiliary function. The selection of h, H(x) was described in [9]. Also, we are free to choose the auxiliary linear operator L_1, the auxiliary parameter h, the auxiliary function H(x) and the initial approximation $u_0(x)$. Therefore if the successive approximation $su_k(x)$, $k \ge 0$ where are obtained by PIM in terms of the auxiliary parameter h then exact solution may be given by $u(x) = \lim_{k\to\infty} u_k(x)$.

According to [9], let $V=\{u : u\in C_m[a, b]\}$ be the solution space and $\{e_j(x) : e_j(x) \in V, j=0,1,\ldots\}$ denote the set of base functions. Hence we can represent the solution in the series $u(x) = \sum_{j=0}^{\infty} \alpha_j e_j(x)$, where α_j is a coefficient belonging to real numbers. As long as the set of base functions is determined, the auxiliary linear operator L_1, the initial approximation $u_0(x)$ and the auxiliary function H(x) must be chosen in such a way that all solutions of the corresponding PIM equations (4) exist and it can be expressed by this set of base functions. Now, in order to avoid expensive computational works for solving (1) via PIM, it is straightforward to use the following set of base functions

***Corresponding author:** Tamamgar M, Department of Mathematics, Islamic Azad University, Gonabad Branch, Gonabad, Iran, Tel: +13103159823; E-mail: tamamgar_m@yahoo.com

Received May 10, 2014; **Accepted** June 27, 2014; **Published** July 03, 2014

Citation: Tamamgar M (2014) The Numerical Solution of Linear FredholmIntegro-Differential Equations via Parametric Iteration Method. J Appl Computat Math 3: 171 doi:10.4172/2168-9679.1000171

Citation: Tamamgar M (2014) The Numerical Solution of Linear FredholmIntegro-Differential Equations via Parametric Iteration Method. J Appl Computat Math 3: 171 doi:10.4172/2168-9679.1000171

$\{(x-1)^j|\ j=0,1,\ldots\}$ (6)

That $u(x)=\sum_{j=0}^{\infty}\alpha_j(x-a)^j$ (7)

Where $\alpha_j\in$ Rare unknown coefficients to be determined and a is a constant belong to real numbers. Now, we set the auxiliary operator L as the following:

$L[u(x)]=u^{(m)}(x)$ (8)

The initial guess is to form combination of m-term of (7) i.e.

$$u_0(x)=\alpha_0+\alpha_1(x-a)+\alpha_2(x-a)^2+\cdots+\alpha_j(x-a)^j. \quad (9)$$

According to (9), the initial conditions (3), and with due attention to $L_1[u_0(x)]=f(x)$, the coefficients $\alpha_0, \alpha_1, \alpha_2,\ldots, \alpha_m$ will be determined. Also, we set H(x)=1, (the selecting of H(x) is arbitrary, but the suitable selection depends on the base functions for solution [9], and we use the PIM processes to compute the approximation solutions of (1).

The Valid Region of h

Assume that we gain a family of solution series in terms of the auxiliary parameter h by means of PIM. We consider this solution as a function in terms of h, x then we derive (once or more) this function respect to x in x=β that β∈ [a, b] i.e. we let U=G(x, h) be the solution of (1) or (2) then we set

$$\Omega=\frac{\partial^i G}{\partial x^i}\Big|_{x=\beta},\quad \beta\in[a,b],\quad (i=1,2,..)$$

Therefore Ω will be in terms of h, now we plot Ωcurve, according to these h curves, it is easy todiscover the valid region of h, which corresponds to the line segments nearly parallel to the horizontal axis. This region is called valid region of h which we note with R_h. We ensure the solution series converge for any h∈ R_h.

Analysis of convergence of the parametric iteration formula

In this section we study the local convergence of approximate solution provided by PIM for solving (1). Initially, We let $u^{(i)}(0)=0$, (i=0,1,...,m-1) and set $L[u(x)]=u^{(m)}(x)$ therefore, we have from (5) the following parametric iteration formula:

$$u_{k+1}(x)=(1+h)u_k(x)-\frac{h}{(m-1)!}\int_0^x(x-t)^{m-1}\left[g(t)+\lambda\int_a^b k(t,s)u_k(s)\,\mathrm{ds}\right]dt. \quad (10)$$

The iterative formula (10) expressed by sequence makes a recurrence sequence $\{u_k(x)\}$. Obviously, the limit of the sequence will be the solution of Equation (1) if the sequence is convergent. In order to prove the sequence $\{u_k(x)\}$ is convergent, we construct a series

$U_0(x) + [u_1(x)-u_0(x)] + \ldots + [u_k(x)-u_{k-1}(x)]+\ldots.$ (11)

Noticing that

$S_{k+1}=u_0(x) + [u_1(x) - u_0(x)] + \ldots + [u_k(x) - u_{k-1}(x)] = u_k(x).$ (12)

The sequence {uk(x)} will be convergent if the series is convergent.

Theorem: If g(x) ∈ C[a,b] and $|\lambda|\le\frac{1}{M}$ then the series of (11) is convergent, i.e., the sequence $\{u_k(x)\}$ is convergent for x∈ [a,b] .

Proof: According to (10), note that

$$|u_1(x)-u_0(x)|=\left|h\left[u_0(x)-\frac{1}{(m-1)!}\int_0^x(x-t)^{m-1}\left[g(t)+\lambda\int_a^b k(t,s)u_0(s)ds\right]dt\right]\right|$$
$$\le|h|\left[L_1+\frac{1}{(m-1)!}(L_2L_3b+\frac{L_2L_1}{2}(b-a))\right]=|h|r \quad (13)$$

Where

$$r:=L_1+\frac{1}{(m-1)!}(L_2L_3b+\frac{L_2L_1}{2N}(b-a)),\ M=\max_{a\le t\le x\le b}|k(x,t)|,\ L_1=\max_{a\le t\le x\le b}|u_0(t)|,\ L_2=\max_{a\le t\le x\le b}|(x-t)^{m-1}|,$$
$$L_3=\max_{a\le t\le x\le b}|g(t)|, \quad (14)$$

Now, we have from (10)

$$|u_2-u_1|=\left\|\left[(1+h)(u_1-u_0)-\frac{h}{(m-1)!}\int_0^x(x-t)^{m-1}\left[\lambda\int_0^t k(t,s)(u_1-u_0)ds\right]dt\right]\right\|$$
$$\le\left||1+h||u_1-u_0|+\frac{|h|}{(m-1)!}\int_0^x(\mathrm{x-t})^{(\mathrm{m-1})}\left[\lambda N\int_0^t k(t,s)|u_1-u_0|ds\right]dt\right| \quad (15)$$
$$\le|h|r\left[|1+h|=\frac{L_2|h|}{2(m-1)!}b^2\right]=|h|r\sum_{n=0}^{1}\binom{1}{n}|1+h|^{1-n}\left(\frac{L_2|h|}{2(m-1)!}b^2\right)^n$$

$$|u_3-u_2|=\left\|\left[(1+h)(u_2-u_1)-\frac{h}{(m-1)!}\int_0^x(x-t)^{m-1}\left[\lambda\int_0^t k(t,s)(u_2-u_1)ds\right]dt\right]\right\|$$
$$\le\left||1+h||u_2-u_1|+\frac{|h|}{(m-1)!}\int_0^x(\mathrm{x-t})^{(\mathrm{m-1})}\left[\lambda N\int_0^t k(t,s)|u_2-u_1|ds\right]dt\right| \quad (16)$$
$$\le|h|r\left[|1+h|^2+\frac{2L_2|h||1+\mathrm{h}}{2(m-1)!}b^2+\frac{L_2^2|h|^2}{(2(m-1!)^2}b^4\right]=|h|r\sum_{n=0}^{2}\binom{2}{n}|1+h|^{1-n}\left(\frac{L_2|h|}{2(m-1)!}b^2\right)^n$$

$$|u_{k+1}-u_k|\le(|h|r)\sum_{n=0}^{k}\binom{k}{n}|1+h|^{k-n}\left(\frac{L_2|h|b^2}{2(m-1)!}\right)^n \quad (17)$$

In view of (18), the convergence of the series (11) can be concluded for the solution domain x>0 and |1+h<1 with the help of some mathematical software. Therefore the series of (12) is absolute convergence, i.e., the sequence $\{u_k(x)\}$ is convergent for x∈ [a,b].

Illustrative Examples

Now, we use PIM to solve three examples of the kind of (1) and compare the obtained results with exact solution and CAS wavelet method [10], to show the efficiency of PIM.

Example 1. We consider the Linear Fredholmintegro- differential of the second kind with exact solution $u(x)=xe^x$ as the following:

$$u'(x)=xe^x+e^x-x+\int_0^1 xu(t)dt,\ 0\le x\le 1 \quad (18)$$

u(0) = 0 (19)

According to PIM proceeding, we define:

$$L_1\{u(x)]=u'(x) \quad (20)$$

And

$g(x)=xe^x+e^x-x$ (21)

Then we have from (8)$u_0(x)=\alpha_0 + \alpha_1 x + \alpha_2 x_2 + \alpha_3 x_3$. Using initial condition and $L_1[u_0(x)]=g(x)$ gives us that $\alpha_0=0,\alpha_1=1,\alpha_2=\frac{1}{2},\alpha_3=\frac{1}{2}$. Now, If we set H(x)=1 then we obtain from (10),

$$u_{k+1}(x)=(1+h)u_k(x)-h\int_0^x[te^t+e^t-t)+\int_0^1 tu_k(s)ds]dt,\ k=0,1\ldots \quad (22)$$

The obtained result for the 12th iteration was shown in Table 1 and compare with obtained results of CAS wavelet method [10]; also the valid region of h i.e. $R_h$was presented in Figure 1.

Example 2: For second example we consider the Linear Fred holmintegro-differential of the second kind with exact solutionu(x)=cos (2πx) as the following:

$$u'(x)=u(x)-\cos(2\pi x)-2\pi\sin(2\pi x)-\frac{1}{2}\sin(4\pi x)+\int_0^1\sin(4\pi x+2\pi t)u(t)dt \quad (23)$$

u(0)=1 (24)

For applying PIM, we set:

$$L_1[u(x)]=u'(x) \quad (25)$$

x	CAS wavelet	PIM
0.1	1.34917637 × 10^{-3}	1.10517092 × 10^{-1}
0.2	1.15960044 × 10^{-3}	2.44280552 × 10^{-1}
0.3	5.67152531 × 10^{-3}	4.04957642 × 10^{-1}
0.4	5.93105645 × 10^{-3}	5.96729879 × 10^{-1}
0.5	1.32330751 × 10^{-2}	8.24360635 × 10^{-1}
0.6	4.39287720 × 10^{-2}	1.09327129 × 10^{0}
0.7	1.41201624 × 10^{-2}	1.40962689 × 10^{0}
0.8	1.34514117 × 10^{-2}	1.78043274 × 10^{0}
0.9	1.32045209 × 10^{-2}	2.21364280 × 10^{0}

Table 1: The results of example 1 for $u_{10}(x)$ with h=-1.

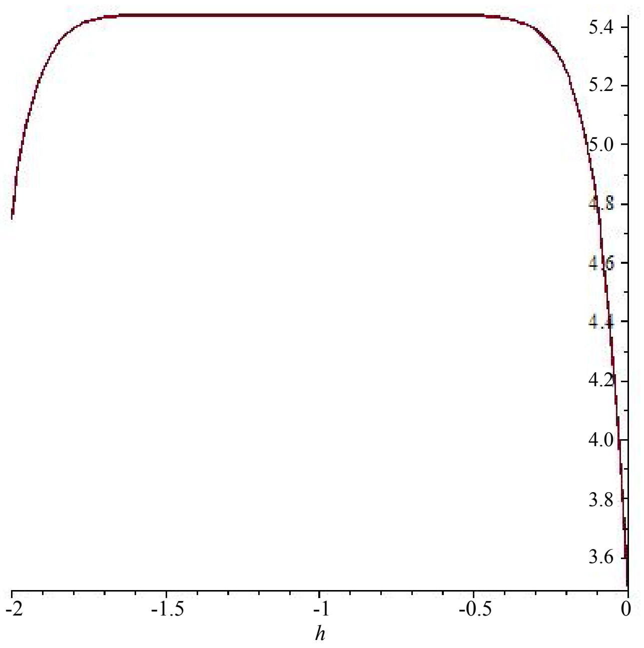

Figure 1: The valid region of h; for example 1 shows -1.5 ≤ h ≤ -0.5.

x	CAS wavelet (k=4, M=1)	PIM (Four iteration)
0.1	2.17942375 × 10^{-4}	8.10481437 × 10^{-1}
0.2	6.38548213 × 10^{-4}	3.07917502 × 10^{-1}
0.3	7.91370487 × 10^{-4}	-3.14332489 × 10^{-1}
0.4	2.15586005 × 10^{-2}	-3.14332489 × 10^{-1}
0.5	4.99358429 × 10^{-3}	-8.16121508 × 10^{-1}
0.6	2.21728810 × 10^{-2}	-1.00658487 × 10^{0}
0.7	1.05645449 × 10^{-4}	-3.24894795 × 10^{-1}
0.8	1.43233681 × 10^{-3}	2.80968929 × 10^{-1}
0.9	2.07747461 × 10^{-2}	7.66189679 × 10^{-1}

Table 2: The results of example 2 for $u_4(x)$ with h =-1.

And

$$g(x) = -\cos(2\pi x) - 2\pi\sin(2\pi x) - \frac{1}{2}\sin(4\pi x) \tag{26}$$

We have from (8), $u_0(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$. Using initial condition and $L_1[u_0(x)] = g(x)$ gives us that, $\alpha_0 = 1, \alpha_1 = -1, \alpha_2 = \frac{-16\pi^3 - 8\pi^2}{8\pi}, \alpha_3 = \frac{2}{3}\pi^2$.

Now, if we set H(x)=1 then we obtain from (10),

$$u_{k+1}(x) = (1+h)u_k(x) - h - h\left(\frac{-8\pi - 1 - 4\sin(2\pi x) + 8\cos(2\pi x)\pi + \cos(4\pi x)}{8\pi}\right)$$
$$-h\int_0^x [u_k(t) + \int_0^1 \sin(4\pi t + 2\pi s)u_k(s)ds]dt, \; k = 0,1... \tag{27}$$

The obtained result for the 4th iteration was shown in Table 2 and 3 and compare with obtained results of CAS wavelet method [10]; also the valid region of h i.e. R_h was presented in Figure 2.

Example 3: we consider the Linear Fred holmintegro-differential of the second kind solution u(x)=sin x as the following:

$$u''(x) = -\sin x + x - \int_0^{\frac{\pi}{2}} xtu(t)dt, \; 0 \le x \le 1, \tag{28}$$

$$u(0) = 0, u'(0) = 1 \tag{29}$$

According to PIM procedure we set;

$$L_1[u(x)] = u'(x) \tag{30}$$

And

$$g(x)=x-\sin(x) \tag{31}$$

Then we have from (8) $u_0(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$, (25), (26) gives us that

$$\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 0, \alpha_3 = -\frac{1}{6}$$

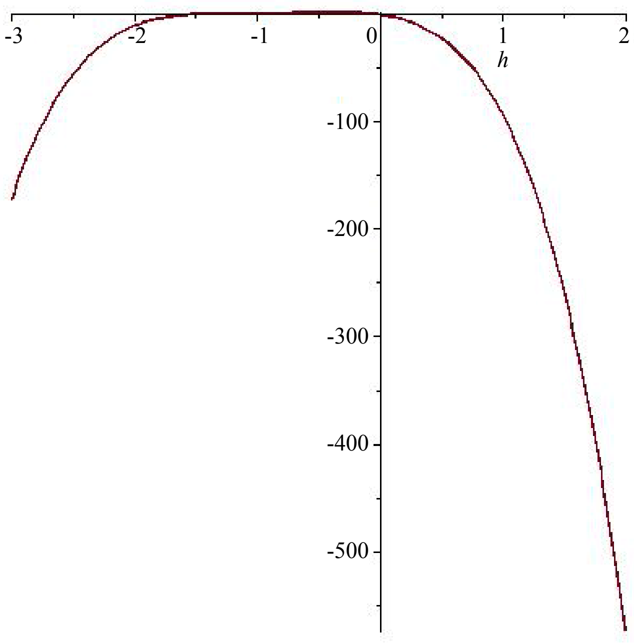

Figure 2: The valid region of h; for example 2 shows -1.5 ≤ h ≤ -0.2.

x	u_{exact}	u_{12}
0.000	0.0000000000	0.0000000000
0.125	0.1416435566	0.1416435566
0.250	0.3210063542	0.3210063550
0.375	0.5456217806	0.5456217803
0.5000	0.8243606355	0.8243606345
0.625	1.167653723	1.167653723
0.750	1.587750013	1.587750005
0.875	2.099015882	2.099015873
1.000	2.718281828	2.718281828

Table 3: The results of example 3 for $u_{12}(x)$ with h =-1.

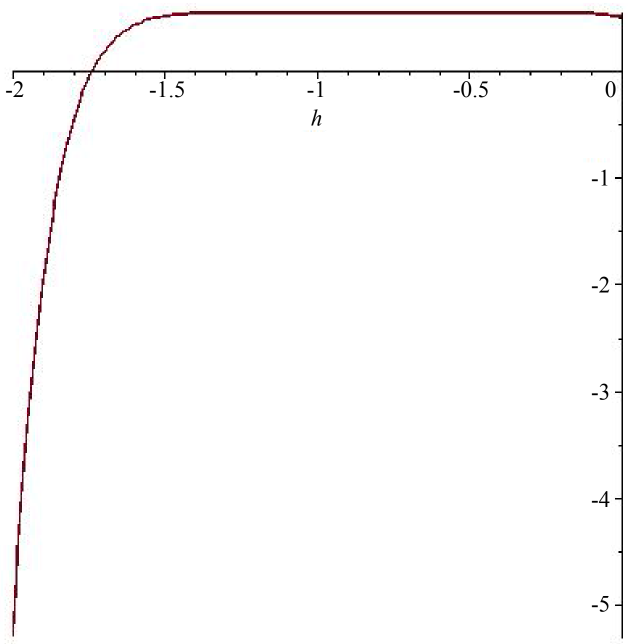

Figure 3: The valid region of h; for example 3 shows -1.4 ≤ *h* ≤ -0.2.

Now similar to example1, the iteration scheme is as the following:

$$u_{k+1}(x)=(1+h)u_k(x)-h\left[\left(\frac{1}{6}x^3+\sin x\right)-\int_0^x\left[(x-t)\int_0^{\frac{\pi}{2}}tsu_k(s)\right]\right]dt,\ \ k=0,1,\ldots \quad (32)$$

The results of example 2 for 10^{th} are available in Table 2. $R_h$for Equation (23) is presented in Figure 2 and 3.

Results and Figure

In this section we present the results of examples 1, 2 in two tables and plot the h-curve to determine R_h.All the computations have been done with Maple 13.

Conclusion

In this article, parametric iteration method was applied to solve the Linear Fredholmintegro-differentialequations. In order to illustrate the method we solve three examples. PIM results compared to CAS waveletmethod shows that the former is easier in practice and more accurate for LFIDEs. Also we show the highaccuracy of PIM in example 3 with comparing the obtained results with exact solution. Further the convergence of PIM for solving LFIDEs in the valid region of h (R_h) was presented. Additionally, if weincrease the number of iterations by PIM scheme it seems the results will have more accuracy in solutions.

References

1. Xu L, He JH, Liu Y (2007) Electro spun nonporous spheres with Chinese drug. IJNSNS 8: 199-202.
2. Sun FZ, Gao M, Lei SH (2007) The fractal dimension of the fractal model of drop wise condensation and its experimental study, IJNSNS 8: 211-222.
3. Jerri AJ (1985) Introduction to Integral Equations with Applications. Marcel Dekker, New York.
4. Nadjafi JS, Tamamgar M (2008) The Variational iteration method: A highly promising method for solving the system of integro-differential equations. Comput Math Appl 56: 346-351.
5. El-Sayed SM, Abdel-Aziz MR (2003) Comparison of domains decomposition method and wavelet-Galerk in method for solving integro-differential equations. Appl Math Comput 136: 151-159.
6. Maleknejad K, Sohrabi S, Rostami Y (2007) Numerical Solution of Linear Fredholm Integral Equations of the Second Kind by Using Chebyshev Polynomials. Applied Mathematics & Computation (AMC) 188: 123-128.
7. Maleknejad K, Hashemizade E, Ezzati R (2011) A new approach to the numerical solution of Fredholm integral equations by using Bernstein's approximation. Communications in Linear Science and Numerical Simulation 16: 647-655.
8. Gorbani A, Morteza G, Jafar SN (2010) A modified Parametric iteration Method for solving Linear second order BVPS. Comput Appl Math 30.
9. Liao S (2003) Beyond perturbation: Introduction to the Homotopy Analysis Method, Boca Raton, Chapman & Hall/CRC Press.
10. Danfu H, Xufeng S (2007) Numerical solution of integro-differential equations by applying CAS Wavelet operational matrix of integration, App Math Comput 194: 460-466.