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The *m*-Derivations of Analytic Vector Fields Lie Algebras

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Abstract

We consider a (real or complex) analytic manifold *M*. Assuming that *F* is a ring of all analytic functions, full or truncated with respect to the local coordinates on *M*; we study the (m ≥ 2)-derivations of all involutive analytic distributions over *F* and their respective normalizers.

Keywords: *m*-derivations; Analytic vector fields Lie algebras; Distributions; Generalized foliations; Stein manifolds; Compact holomorphic manifolds; Chevalley-Eilenberg's cohomology; Compactly supported vector fields

Introduction and Preliminary

We know several embedding theorems in differential geometry, some of them are of John F. Nash in Riemannian manifolds [1,2], of Whitney [3] in differentiable manifolds and of Grauert in analytic manifolds cf. [4]. They make easy certain study on a differentiable manifold. Here, we are interested to a real or complex analytic *n*-dimensional manifold *M* and let *F*(*M*) be the ring of all analytic functions on *M*. We know that these manifolds can be considered as smooth manifolds. But certain property on a smooth manifold cannot be true on *M*, for example the global representation of a smooth function germ theorem. Grabowski had this problem when he studied derivations of the real or complex analytic vector fields Lie algebra cf. [5] and he used Stein manifolds to avoid technical difficulties in them. Here, we examine not only the derivations but the ($m \geq 2$)-derivations (generalization of derivation's notion) of a Lie subalgebra of the real or complex analytic vector fields Lie algebra on *M*, using Lie algebra tools. In advance, we state that the considered Lie algebras have enough sections more than constant ones in the Lie algebra of all analytic vector fields. Then, we consider only Stein spaces unless expressly stated in a complex analytic case. In the real analytic one, we don't need more hypothesis because of the imbedding theorem of Grauert and Cartan theorems [6]. More precisely, any real analytic manifold can be considered as a closed submanifold of a certain $\mathbb{R}^l$ (a " real Stein manifold"). Now, an m-derivation of a Lie algebra *A* is a linear map from *A* to itself which is distributive on the brackets $[X_1,[X_2,\ldots[X_{m-1},X_m]\ldots]]$, where all X_i are in *A*. On the one hand, we have studied the *m*-derivations of polynomial vector fields Lie algebras on $\mathbb{R}^n$ in studies of 7. Randriambololondrantomalala [7], an important Lie subalgebra of analytic vector fields, we found that Lie algebras of derivations are different to those of ($m > 2$)-derivations. One can see the following example, on $\mathbb{R}^3$, the Lie $\mathbb{R}$-algebra is spanned by $\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},y\frac{\partial}{\partial x},(y)^2\frac{\partial}{\partial x}$ and let's define the $\mathbb{R}$−linear map *D* by $D\left((y)^2\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial x}$ which is zero otherwise. It's clear that *D* is not a derivation, but a 3-derivation. On the other hand, all *m*-derivations of a distribution over the full or truncated rings of smooth functions on a differentiable manifold in literature of Randriambololondrantomalala [8], are derivations. These facts lead us to ask if a distribution Lie algebra on an analytic manifold has results as the one or the other above results. So, we will divide our paper into three parts. First, we take a real or complex analytic involutive distribution Ω over *M*. That is to say, Ω is a *F*(*M*)-submodule of the analytic vector fields Lie algebra $\chi(M)$ on *M*. We can find some examples of these distributions and the interests for studying their derivations in literature of Grabowski and Cartan [5,6]. Here, we find the Ω's centralizer and the derivative ideal of Ω. We can say also that the normalizer of Ω is a Lie subalgebra of analytic vector fields. In addition, we find out that all *m*-derivations of Ω (resp. of the normalizer of Ω) are inner with respect to a normalizer's vector field (resp. are inner). Second, assuming that Ω is an involutive distribution on *M* over a subring *F* of *F*(*M*), namely an *F*-submodule of $\chi(M)$ stable by the vector fields bracket, where *F*≠ *F*(*M*). One can consider a system of commuting vector fields on *M* as in studies of Randriambololondrantomalala [8] and all distribution Lie subalgebras of the Lie algebra of analytic vector fields which commute with this system. The normalizer of Ω is an analytic vector fields Lie algebra and contains locally all constant vector fields and Euler's vector field. But in general, we can't use the reasoning by Randriambololondrantomalala [7] to characterize *m*-derivations of Ω. We make more explicit all *m*-derivations of Ω and of some of its normalizer. Whereas, in the end, we discuss the Lie algebras of holomorphic vector fields, especially when the holomorphic manifold is not a Stein one, and Lie algebras of locally polynomial vector fields on an analytic manifold *M*. Their *m*-derivations as well as their normalizers can be characterized by using some results of Randriambololondrantomalala [7].

Therefore, we have found the *m*-derivations of all distributions over a set of full or truncated analytic functions with respect to the local coordinates on *M*. When $m = 2$, we deduce from our results some [5]'s theorems. Third, we can apply our theorems on Lie algebras of real or complex analytic vector fields on *M*, of generalized foliation on *M* cf. [9], a Lie subalgebra of analytic vector fields on $\mathbb{S}^2$ and on $\mathbb{T}^2$, Riemann surfaces, etc. Relations between the Lie algebra of compactly supported vector fields and the compactness of *M* are discussed. Moreover, we emphasize the extensions of our theorems when the studied distributions are singular, by using the complexification of a real analytic manifold, Hartogs and Riemann extension theorems. Of course, in these circumstances, we can use theory of coherent sheaves

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made by Cartan [6] in a Stein case or pass into Grabowski's conjecture cf. [9]. We interpret our results in Chevalley-Eilenberg cohomology sense when $m = 2$.

Following the above notations, let M be a real or complex analytic n-dimensional manifold. In complex case, we regard a Stein manifold unless special mention. We denote by $\chi(M)$ the Lie algebra of analytic vector fields on M and $F(M)$ the ring of analytic functions on M. Throughout this paper, we take an atlas in which every chart are connected. Then, the open subset of a chart U where a non-trivial subset of $\chi(M)$ doesn't vanish, is dense on U (non-trivial means different to $\{0\}$). We can use certain results of Randriambololondrantomalala [7,8] because in the proofs of theorem of these papers we consider only analytic functions (polynomials, exponentials). In the same way, we don't need partition of the unity to make global some local results cf. [10]. In all sections of this article, we set an integer $m \geq 2$, recall that D is an m-derivation of a Lie algebra A if for $(X_i)_{1\leq i\leq m} \subset A$, we get $D[X_1,[X_2,\ldots,[X_{m-1},X_m]\ldots]] = [D(X_1),[X_2,\ldots,[X_{m-1},X_m]\ldots]] + \ldots + [X_1,[X_2,\ldots,[X_{m-1},D(X_m)]\ldots]]$. This D is said inner on a Lie algebra $\mathbb{B}$ containing A, if D is a Lie derivative with respect to an element of $\mathbb{B}$. Recall us another basic definition cf. [11].

Definition 1.1. A complex manifold M is a Stein manifold, if we have simultaneously the three following conditions: For every $x \neq y$, both in M; there is a holomorphic function f over M such that $f(x) \neq f(y)$. For all $x \in M$, it exists n holomorphic functions (f_i) over M such that df_i are linearly independent over $\mathbb{C}$ on x. If K is a compact set of M, the following set is compact (holomorphic convexity of M)

$$\left\{x \in M \,/\, |f(x)| \leq \sup_{y\in K} |f(y)|, \text{for all holomorphic functions } f \text{ over } M\right\}.$$

From these assertions, every local ring of holomorphic functions around $x \in M$ is spanned by holomorphic functions on M cf. [12].

Some results of the Lie algebra of compactly supported vector fields C_c relative to a Stein manifold are the following,

Proposition 1.2. *A complex analytic manifold M is compact iff C_c is non trivial, particularly if M is a Stein holomorphic manifold, C_c is trivial.*

Proof. It's obvious that if M is compact, then $\mathfrak{C}_c = \chi(M)$ and C_c is not trivial. Conversely, suppose that M isn't a compact set and there is $X \in C_c$ such that $K = \mathrm{Supp}(X) \neq \emptyset$. We can consider $K \neq M$ because M is not compact. Then, we have the nullity of X in the open set $\complement K \neq \emptyset$. By analyticity, X vanishes in whole M. Hence, we have a contradiction about $K \neq \emptyset$ and we obtain M is a compact set. It's clear that a Stein space is never a compact set by definition, then its Lie algebra of compactly supported vector fields is trivial.

The *m*-derivations defined by distributions on *F*(*M*)

Let Ω be a non-trivial involutive analytic distribution over the analytic functions ring on M. Let N be the normalizer of Ω in $\chi(M)$, that is to say that the set of all $X \in \chi(M)$ such that $[X,\Omega] \subset \Omega$, and $\mathfrak{B} = \{x \in M \,/\, \Omega(x) \neq \{0\}\}$. We can choose a connected domain U_i of a chart. If we suppose that it exists an open set O_i in U_i where Ω vanishes, then $\Omega_{|U_i} = \{0\}$ by analyticity. Otherwise, every open set in U_i contains an element of **B**. So, $\mathbf{B} \cap U_i$ is dense over U_i. Moreover, the collection of U_i forms an atlas of M, then **B** is dense over M. The set **B** is an open analytic submanifold of M. Particularly, **B** is a Stein cf. [13]. Thus, every vector field defined over **B** admits a continuous extension on M, and if this last one is analytic, then it's necessarily an element of the normalizer of Ω. We use this last fact when we deal with extension theorems.

We know by literature of Nagano's [14] result that Ω is integrable, then it yields a generalized foliation F on M cf. [10]. So, Ω is the Lie algebra of tangent vector fields to the foliation and L_F the one of all foliation preserving vector fields. It is known that the normalizer N in $\chi(M)$ of Ω contains L_F cf. [10]. Hence, the restriction of the foliation in **B** is non singular.

Proposition 2.1. *The centralizer of Ω vanishes and the derivative ideal of Ω coincides with Ω itself.*

Proof. We say that $X \in \chi(M)$ is in the centralizer of Ω if $[X, \Omega]=\{0\}$; and the derivative ideal of denoted by $[\Omega, \Omega]$ is the Lie algebra spanned by all brackets of two elements of Ω. Suppose there is an non zero element X of the centralizer, we have $[X, f\Omega] = (X(f))\Omega = \{0\}$, for all $f \in F(M)$. It's not possible in a Stein manifold or in a real analytic manifold if X doesn't vanish identically over M and if $\Omega \neq \{0\}$. Along with this result, we can adapt the proof of Proposition 2.28 of studies of Randriambololondrantomalala [15] and assert that $[\Omega, \Omega] = \Omega$.

Let's recall an Hartogs's extension theorem and Riemann extension theorem.

Theorem 2.2. *(Hartogs [16]) Let be $t \geq 2$ and D be a bounded domain in $\mathbb{C}^t$. In addition, K be a compact subset of D such that $D - K$ is a connected set. Then all holomorphic functions f over $D - K$ can be extended holomorphically to D.*

Theorem 2.3. *(Riemann extension theorem) Let U be an open set in $\mathbb{C}$ and $z_0 \in U$. If $f : U - \{z_0\} \to \mathbb{C}$ is holomorphic function such that z_0 is a removable singularity of f, then f can be extended into an unique holomorphic function $\bar{f}$ in U where $\bar{f}(z_0) = \lim_{z\to z_0} f(z)$.*

Theorem 2.4. *In holomorphic case, all m-derivations of Ω, L_F, and of* N *are Lie derivatives with respect to elements belonging to* N. *In real analytic one, we have the same results if* **B** *= M.*

Proof. We can prove this assertion over **B** by Theorem 2.1 of studies of Randriambololondrantomalala [8] using Proposition 2.1 and partially Theorem 3.7 of literature of Randriambololondrantomalala [10]. For the corresponding extension theorem over M, we adopt the following arguments. We know that **B** is dense over M, then the restriction of **B** in each domain of a chart U is dense over U (U is a bounded set). The complement of this $\mathbf{B} \cap U$ in U can be considered as a compact set of the chart such that $\mathbf{B} \cap U$ is connected. In holomorphic case, when $n \geq 2$, we use Hartogs's theorem in a domain of the chart, so the extension theorem over M holds. If $n = 1$, we know by the isolated zeros principle that the domain of chart contains only a finite number of zeros in the corresponding restriction of **B**. By continuity at these zeros, which are removable singularities, the Riemann extension theorem can be used. Of course, if **B** = M in real analytic situation, the extension theorem is applicable.

Definition 2.5. The complexification of a real analytic manifold M is a holomorphic manifold $\mathbb{H}$ such that there is a real analytic embedding $f: M \to \mathbb{H}$ where $\mathbb{H}$ has a holomorphic atlas $(U_i, \varphi_i)_i$ and $\varphi_i(f(M) \cap U_i) = \varphi_i(U_i) \cap \mathbb{R}^n$. We have a Stein complexification if $\mathbb{H}$ is Stein.

The next theorem is due by Grauert cf. [4,12].

Theorem 2.6. *Every real analytic manifold has a Stein complexification and can be analytically properly embedded into an Euclidean space $\mathbb{R}^N$.*

The following complexification of a Lie subalgebra $\mathbb{G}$ of the real analytic vector fields Lie algebra of M is in the following sense: if M can

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be embedded in a holomorphic manifold $\mathbb{H}$, the complexification $\overline{\mathbb{G}}$ of $\mathbb{G}$ in $\mathbb{H}$ is such that $\overline{\mathbb{G}}_{|M} = \mathbb{G}$.

Theorem 2.7. *If the complexification of Ω in a Stein space T is still an involutive distribution, then the first result of the Theorem 2.4 holds in real analytic case.*

Proof. We use the complexification of M on a Stein space T. Consequently, let be $\overline{\Omega}$ the complexification over T of Ω. Recall that $\overline{\Omega}$ is an involutive distribution over T where its normalizer on $\chi(T)$ is denoted by N_0. So, all m-derivations of Ω, of L_F, and of N have their complexified m-derivations over T on respectively $\overline{\Omega}$, the Lie algebra of all foliation preserving vector fields $\overline{L_{\mathfrak{F}}}$ on T and N_0. By the results of Theorem 2.4, these last m-derivations are Lie derivatives with respect to elements belonging to N_0. We can affirm that $\mathfrak{N}_{0|M} = \mathfrak{N}$ and $\overline{L_{\mathfrak{F}}}_{|M} = L_{\mathfrak{F}}$ by $\overline{\Omega}_{|M} = \Omega$. Thus, we have the same result as in the first part of Theorem 2.4.

By definition, the first space of Chevalley-Eilenberg's cohomology of a Lie algebra $\mathcal{A}$ denoted by $H^1(\mathcal{A})$ is $\mathrm{Der}(\mathcal{A})/\mathrm{ad}_{\mathcal{A}}$ with $\mathrm{Der}(\mathcal{A})$ the Lie algebra of all derivations of $\mathcal{A}$ and $\mathrm{ad}_{\mathcal{A}}$ the set of inner derivations of $\mathcal{A}$.

Throughout this paper, we suppose that all hypothesis of Theorem 2.7 are satisfied or $\mathfrak{B} = M$, in real analytic case.

Following ideas of Theorem 3.7, Corollary 2.14 and Remark 2.15 of literature of Randriambololondrantomalala [10], we state

Corollary 2.8. *The first space of Chevalley-Eilenberg's cohomology of Ω, $L_{\mathfrak{F}}$, and of $\mathfrak{N}$ is respectively isomorphic to the following respective Lie algebras, $\mathfrak{N}/\Omega$, $\mathfrak{N}/L_{\mathfrak{F}}$, $\{0\}$.*

Remark 2.9. *By Theorem 2.4, we deduce Theorem 3.2 and 4.1 of studies of Grabowski* [5].

The *m*-derivations associated to a distribution over a subring of *F*(*M*)

Let be an atlas of M such that Ω is locally spanned by $\left(\frac{\partial}{\partial x^i}\right)_{1\leq i\leq n}$ over the ring F_0 of real or complex functions depending only on $(x^j)_{1\leq j\leq k}$ with respect to the atlas (where $1 \leq k < n$). We can consider Ω to be a Lie algebra which commutes with a system S of commuting vector fields by the usual bracket. That is to say, $S = \{X_1, \ldots, X_q\}$ such that $[X_i, X_j] = 0$ and S is locally of rank $n - k$ ($0 < q \leq n$). It is easy to check that $[\Omega, \Omega] = \Omega$ because of Randriambololondrantomalala's [8] result. So with the same reason, every m-derivation of Ω is local. Moreover, the normalizer N of Ω is locally isomorphic to $\Omega \oplus gl(n-k, \mathbb{R} \text{ or } \mathbb{C})$ as a vector space. We consider the closed 1−differential forms α^i and ω^i over a $(n - k)$-dimensional distinguished connected chart of the generalized foliation generated by S, where $i = k+1, \ldots, n$ and an m-derivation of Ω, $D^{(\alpha,\omega)} = \sum_{k+1\leq j\leq n}(\alpha^j + \omega^j) \otimes \frac{\partial}{\partial x^j}$ such that $ker(\omega^j) \supset F_0(U)\left\langle \frac{\partial}{\partial x^i} \right\rangle_{1\leq i\leq k}$ and $ker(\alpha^j) \supset F_0(U)\left\langle \frac{\partial}{\partial x^i} \right\rangle_{k+1\leq i\leq n}$ (S in this chart is $\{\frac{\partial}{\partial x^j}\}_{k+1\leq j\leq n}$, $F\langle A \rangle$ denotes a module spanned by A over a ring F) cf. [8]. We have omitted all singular charts of the foliation because the open set R of all regular points is dense over M cf. [10], we have no problem for the extension of our results from R towards M as in the previous section. By adapting Theorem 3.12 of literature of Randriambololondrantomalala [8], we obtain easily

Theorem 3.1. *All m-derivations of Ω (resp. of $\mathfrak{N}$) are a sum of a Lie derivative with respect to one element of $\mathfrak{N}$ and a derivation $D^{(\alpha,w)}$ as denoted before (resp. are similar to m-derivations of Ω).*

Hence, adopting the arguments of Theorem 3.19 of studies of Ravelonirina [17], we hold the following

Corollary 3.2. *When the rank of S is a positive constant $n - k$, the first spaces of Chevalley-Eilenberg's cohomology of Ω and of N are both isomorphic to $\left(H_R(B)\right)^{n-k} \times \left(\mathbb{R}^{(n-k)^2+n-k} \text{ or } \mathbb{C}^{(n-k)^2+n-k}\right)$ with $H_R(B)$ is the de Rham cohomology of foliation basic forms of M.*

As we know, we can split the above Ω into a semi-direct sum of Lie algebras Ω_S^1 and Ω_S^2 as in studies of Randriambololondrantomalala [8], where they are modules on the ring $F_0(M)$ of constant functions over the leaves relative to the above generalized foliation. We notice that Ω_S^2 is spanned by S on $F_0(M)$. We can reason on a distinguished chart U with the coordinates $(x^i)_{1\leq i\leq n}$. The $F_0(U)$ is the set of all analytic functions depending only on $(x^i)_{1\leq i\leq k}$, $\Omega^1_{S|U}$ is spanned by $\left(\frac{\partial}{\partial x^i}\right)_{1\leq i\leq k}$ and $\Omega^2_{S|U}$ by $\left(\frac{\partial}{\partial x^i}\right)_{k+1\leq i\leq n}$ on $F_0(U)$.

Now, we discuss the m-derivations of Ω_S^1. The normalizer N^1 of this Lie algebra can be written as a direct sum of Lie algebras $\mathfrak{N}^1 = [\Omega_S^1, \Omega_S^1] \oplus [\mathcal{Q}, \mathcal{Q}]$, where $\mathcal{Q}$ is the centralizer of Ω_S^1 in $\chi(M)$ and the center of Ω_S^1 is zero ($\mathcal{Q}$ is locally spanned by $\left(\frac{\partial}{\partial x^i}\right)_{k+1\leq i\leq n}$ on the ring of all analytic functions depending only on $(x^i)_{k+1\leq i\leq n}$). By a similar argument of Nakanishi [18], we deduce that all m-derivations of N^1 are a direct sum of those of Ω_S^1 and of $\mathcal{Q}$. By Theorem 2.4, it's clear that

Theorem 3.3. *Each m-derivation of Ω_S^1 (resp. of $\mathfrak{N}^1$) is a Lie derivative with respect to an element of* N^1.

The normalizer of Ω_S^2 is locally the sum of the $F_0(U)$ -module spanned by $\underset{1\leq i\leq n}{\frac{\partial}{\partial x^i}}$ and a vector space generated by $\underset{k+1\leq i,\, l\leq n}{x^l\frac{\partial}{\partial x^i}}$. That is to say, its normalizer is $\mathfrak{N}$. In addition, its centralizer is Ω_S^2 itself. Because of $[\Omega_S^2, \Omega_S^2] = \{0\}$ or Ω_S^2 is nilpotent of order 1, we obtain easily

Theorem 3.4. *Every endomorphism of Ω_S^2 is an m-derivation of Ω_S^2.*

So, it's immediate that

Corollary 3.5. *The first space of Chevalley-Eilenberg's cohomology of Ω_S^1, Ω_S^2 and of* N^1 *are respectively isomorphic to the following respective Lie algebras,* $\mathfrak{N}^1/\Omega_S^1$, $\mathrm{End}(\Omega_S^2)/\Omega_S^2$, $\{0\}$.

Let's consider Ω_S^3 the Lie subalgebra of Ω, spanned by X_i over a ring $F \subseteq F(M)$. That is to say, F is locally the set of all analytic functions depending only on $(x^i)_{l\leq i\leq n}$ where $0 < l < k + 1$ (resp. $k + 1 < l < n + 1$). When $F = F(M)$ (resp. $F = F_0(M)$), it is a special case of Lie algebras defined in Theorem 2.4 (resp. in Theorem 3.1) when the submodule is generated by X_i. In the distinguished local coordinates, Ω_S^3 is spanned by $\left(\frac{\partial}{\partial x^j}\right)_{k+1\leq j\leq n}$ over F. The normalizer $\mathfrak{N}^3$ of Ω_S^3 coincides with the sum of Ω_S^3 and Ω_S^4 where the element of this last one is locally the following analytic vector fields $\sum_{l\leq i\leq k} f^i(x^t, l\leq t\leq k)\frac{\partial}{\partial x^i} + \sum_{1\leq j\leq l-1} g^j(x^t, 1\leq t\leq k)\frac{\partial}{\partial x^j}$

$\left(\text{resp.}\ \underset{1\le i\le k}{f^i(x^t, 1\le t\le k)\frac{\partial}{\partial x^i}} + \underset{k+1\le j\le l-1}{h^j(x^t, l\le t\le n)x^j\frac{\partial}{\partial x^j}}\right)$. In the first case, we can adapt Theorem 2.4 because all analytic functions depending on (x^j) where $k+1\le j\le n$, are in the base ring of Ω_S^3. In the following case, it is easy to see that Theorem 3.1 can be adapted to Ω_S^3. Thus

Theorem 3.6. *In the first case, every m-derivation D of Ω_S^3 is a Lie derivative with respect to a $\mathfrak{N}^3$'s element; in the second, it is a sum of a Lie derivative of an element of $\mathfrak{N}^3$ and a $D^{(\alpha,w)}$ analogous to that of Theorem 3.1. In addition, the corresponding extension theorems hold.*

Corollary 3.7. *The first space of Chevalley-Eilenberg's cohomology of Ω_S^3 is respectively isomorphic to Ω_S^4 in the first case;*

$$\Omega_S^4 \oplus \left(\left(\mathrm{H}_R(\mathrm{B}')\right)^{l-k-1} \times \left(\mathbb{R}^{(l-k)(l-k-1)}\ ou\ \mathbb{C}^{(l-k)(l-k-1)}\right)\right)$$

in the other one if *S* has a constant rank ($\oplus$ is a module direct sum and *B′* is the set of the corresponding foliation basic forms of *M*).

When we regard all the above normalizers on a distinguished chart, they contain locally all constant fields and Euler vector field. So, we ask one question: could we adapt Theorem 3.6 and Theorem 3.9 in [7] to these normalizers? The following remark shows us that this argument is false.

Remark 3.8. On $\mathbb{R}^3$, we consider the Lie $\mathbb{R}$-algebra *A* spanned by $\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},\quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x},(y)^2\frac{\partial}{\partial x}, e^y\frac{\partial}{\partial x}, ye^y\frac{\partial}{\partial x}, y^2e^y\frac{\partial}{\partial x},\dots$. Lemma 2.3 of literature of Randriambololondrantomalala [10] is not true for *A*, so the arguments proposed in the proof of Theorem 3.6 of Princy [7] don't hold in this situation.

Whereas, let P be a Lie subalgebra consisting locally of polynomial vector fields in $\chi(M)$, where the Euler vector field and all constant vector fields are locally in P. Especially, *M* is not supposed to be a Stein in the holomorphic case. Let's recall a well known theorem,

Theorem 3.9. *(The maximum principle)* [12] *Let be M a connected holomorphic manifold and f a holomorphic function on M such that $|f(z)|\le|f(z_0)|$, where $z_0\in M$ for all $z\in M$; then f is a constant function.*

One consequence of the maximum principle is the following, if the holomorphic manifold *M* is compact, every holomorphic function on *M* is constant in every connected component of *M*. We know that *M* is locally connected, then each function over *M* is locally constant. Therefore, it's clear that if *M* is a compact and connected holomorphic manifold, the ring of all holomorphic functions on *M* is the complex constant functions ring. It's confirm that results of the following theorem complete our study about an involutive analytic distribution when *F*(*M*) is reduced to $\mathbb{C}$.

By adapting Randriambololondrantomalala's [7] theorems and taking account that the vector field found in the proof of Theorem 3.6 of Princy [7] is analytic, it follows that

Theorem 3.10. *When m is even, all m-derivations of $\mathfrak{P}$ (resp. of the normalizer $\mathcal{N}$ of $\mathfrak{P}$ in $\chi(M)$) are a Lie derivative with respect to one and only one vector field belonging to $\mathcal{N}$ (resp. to the normalizer of $\mathcal{N}$ in $\chi(M)$). If m is an odd number, they are sum of a Lie derivative with respect to one element of $\mathcal{N}$ (resp. of the normalizer of $\mathcal{N}$) and an m-derivation of local homogeneous degree -2 of $\mathfrak{P}$.*

So, taking into account: the vanishing of the centralizer of $\mathfrak{P}$ cf. [19] p.91; both the proofs of Theorem 2.12 of Ravelonirina [19], Corollary 3.12 of Randriambololondrantomalala [7] and Theorem 3.7 in literature of Randriambololondrantomalala [10], we obtain

Corollary 3.11. *The first space of Chevalley-Eilenberg's cohomology of $\mathfrak{P}$, of $\mathcal{N}$ and of* N *is respectively isomorphic to the following respective Lie algebras $\mathcal{N}/\mathfrak{P}$,* N / $\mathcal{N}$, {0}, *where* N *is the normalizer of $\mathcal{N}$.*

Illustrations

Some illustrations of our theorems are given in this section.

Example 4.1. It's clear that Theorem 2.4 works well on the Lie algebra of all analytic vector fields $\chi(M)$, that is to say, all *m*-derivations of $\chi(M)$ are Lie derivatives by elements of $\chi(M)$. We can define the Lie algebra of compactly supported real analytic vector fields C and this theorem holds for this last one. In particular, $\mathrm{H}^1(\mathfrak{C})=\chi(M)/\mathfrak{C}$ for a non-trivial $\mathfrak{C}$. More, $\mathrm{H}^1(\mathfrak{C})=\{0\}$ if and only if *M* is compact. Obviously, we can use the above cited theorem on the Lie algebras of vector fields relative to a generalized foliation over M. We can cite some well known Stein spaces, $\mathbb{C}^n$, an open poly-disc in $\mathbb{C}^n$, non-compact Riemann surfaces, ... and build our results in these.

Example 4.2. Let be $\mathbb{S}^2$ a holomorphic compact connected manifold. It's not a Stein manifold nor a submanifold of $\mathbb{C}^u$ for any *u*, it's a compact Riemann surface. We choose the modified stereographic coordinates over this manifold. The $\mathbb{S}^2$ has an atlas composed by two charts (U, z_1) and (V, z_2) with the overlap map $\psi(z)=z^{-1}$ in $U\cap V$. We remark that the Lie algebra $\mathbb{L}$ of vector fields on *M* spanned over $\mathbb{C}$ by Y_1, Y_2 and Y_3 is the one of all polynomial vector fields in $\mathbb{S}^2$, where

$$Y_1:\begin{cases}\dfrac{\partial}{\partial z^1} & \text{in } U\\ -(z^2)^2\dfrac{\partial}{\partial z^2} & \text{in } V\end{cases},\qquad Y_2:\begin{cases}-(z_1)^2\dfrac{\partial}{\partial z^1} & \text{in } U\\ \dfrac{\partial}{\partial z^2} & \text{in } V\end{cases},$$

$Y_3:\begin{cases}-z^1\dfrac{\partial}{\partial z^1} & \text{in } U\\ z^2\dfrac{\partial}{\partial z^2} & \text{in } V\end{cases}$. By Theorem 3.10, all *m*-derivations of $\mathbb{L}$ are Lie derivatives with respect to a vector field in $\mathbb{L}$ itself. That is to say, if *D* is an *m*-derivation of $\mathbb{L}$ defined by $D(Y_1)=\alpha^iY_i$, $D(Y_2)=\beta^iY_i$, $D(Y_2)=\gamma^iY_i$; we have $D=\mathrm{L}_{-\gamma^1Y_1+\gamma^2Y_2+\alpha^1Y_3}$.

When we look at $\mathbb{S}^2$ as a real analytic manifold, we set the charts $(U,(x^1,x^2))$ and $(V,(y^1,y^2))$ with the overlap map

$$\phi(x^1,x^2)=\left(y^1=\frac{x^1}{(x^1)^2+(x^2)^2}, y^2=\frac{x^2}{(x^1)^2+(x^2)^2}\right).$$

We set the real analytic vector field $Y_3:\begin{cases}-x^1\dfrac{\partial}{\partial x^1}-x^2\dfrac{\partial}{\partial x^2} & \text{in } U\\ y^1\dfrac{\partial}{\partial y^1}+y^2\dfrac{\partial}{\partial y^2} & \text{in } V\end{cases}$ on $\mathbb{S}^2$

and the Lie algebra A of real analytic vector fields which commute with Y_3. This $\mathfrak{A}$ consists of real analytic vector fields *Y* such that

$$Y:\begin{cases}F_1\left(\dfrac{x^2}{x^1}\right)x^1\dfrac{\partial}{\partial x^1}+F_2\left(\dfrac{x^1}{x^2}\right)x^2\dfrac{\partial}{\partial x^2} & \text{in } U\\ F_3\left(\dfrac{y^2}{y^1}\right)y^1\dfrac{\partial}{\partial y^1}+F_4\left(\dfrac{y^1}{y^2}\right)y^2\dfrac{\partial}{\partial y^2} & \text{in } V\end{cases}$$

where in the $U\cap V$,

$$F_3\left(\frac{y^2}{y^1}\right)=\frac{1}{(y^1)^2+(y^2)^2}\left(\left(-2F_2\left(\frac{y^1}{y^2}\right)+F_1\left(\frac{y^2}{y^1}\right)\right)(y^2)^2-F_1\left(\frac{y^2}{y^1}\right)(y^1)^2\right)$$ and

$$F_4\left(\frac{y^1}{y^2}\right)=\frac{1}{(y^1)^2+(y^2)^2}\left(\left(-2F_1\left(\frac{y^2}{y^1}\right)+F_2\left(\frac{y^1}{y^2}\right)\right)(y^1)^2-F_2\left(\frac{y^1}{y^2}\right)(y^2)^2\right)$$

with F_k are arbitrary convenient functions of one variable. So, we can apply all theorems in Section 3 to $\mathfrak{A}$. Particularly $\mathrm{H}^1(\mathbb{L})=\{0\}$ and $\mathrm{H}^1(\mathfrak{A})=H^1_R(B)\times\mathbb{R}^2$.

Example 4.3. Indeed, Theorem 3.10 can be applied to a polynomial vector fields Lie algebra on the real analytic manifold $\mathbb{R}^n$ or the Stein manifold $\mathbb{C}^n$ having the corresponding hypothesis.

Example 4.4. We set the Lie algebra $\mathbb{A}$ over the Stein manifold $\mathbb{C}^3$ spanned over $\mathbb{C}$ by $\frac{\partial}{\partial z^1},\frac{\partial}{\partial z^2},\frac{\partial}{\partial z^3},z^1\frac{\partial}{\partial z^1}+z^2\frac{\partial}{\partial z^2}+z^3\frac{\partial}{\partial z^3},z^2\frac{\partial}{\partial z^1},(z^2)^2\frac{\partial}{\partial z^1}$. The normalizer of $\mathbb{A}$ is $\mathcal{N}_0=\mathbb{A}+R$, where R is the space over $\mathbb{C}$ generated by $z^1\frac{\partial}{\partial z^1},z^2\frac{\partial}{\partial z^2},z^3\frac{\partial}{\partial z^3},z^2\frac{\partial}{\partial z^3},z^3\frac{\partial}{\partial z^1}$. It is permit to use Theorem 3.10 and when m is even, every m-derivation of $\mathbb{A}$ is inner on $\mathcal{N}_0$. If m is odd, the m-derivation is a sum of an inner derivation on $\mathcal{N}_0$ with a $\mathbb{C}$-linear map D defined by $D\left((z^2)^2\frac{\partial}{\partial z^1}\right)=\alpha\frac{\partial}{\partial z^1}$ which is zero otherwise *(α ∈ ℂ)*. Moreover, all m-derivations of $\mathcal{N}_0$ are inner for all $m\geq 2$. So, $\mathrm{H}^1(\mathbb{A})\cong R\ominus\left\langle z^1\frac{\partial}{\partial z^1}+z^2\frac{\partial}{\partial z^2}+z^3\frac{\partial}{\partial z^3}\right\rangle$ and $\mathrm{H}^1(\mathcal{N}_0)\cong\{0\}$.

Remark 4.5. In the following example, Theorem 3.10 cannot be applied. We take the 2-torus $\mathbb{T}^2=\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, which is a holomorphic connected compact manifold cf. [20], it's not a Stein. All overlap maps are translations, that is to say, they are holomorphic. We can define globally the Lie algebra of all constant vector fields $\mathfrak{Q}$ on M and find that $\mathfrak{Q}$ is the Lie algebra of all holomorphic vector fields over M. All endomorphisms of each Lie subalgebra of $\mathfrak{Q}$, which is inevitably nilpotent of order 1, are m-derivations of this subalgebra. The normalizer of this subalgebra or its centralizer is the Lie algebra of all vector fields over $\mathbb{T}^2$. But $\mathrm{H}^1(\mathfrak{Q})\cong\mathrm{End}(\mathfrak{Q})/\mathrm{ad}_{\mathfrak{Q}}$ and $\mathrm{H}^1(\chi(M))\cong\mathrm{End}(\chi(M))/\mathrm{ad}_{\chi(M)}$ since $\mathrm{H}^1(\chi(M))=\{0\}$ in smooth cases.

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