

Research Article

The Generalized Burnside Theorem in Noncommutative Deformation Theory*

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Abstract Let A be an associative algebra over a field k , and let \mathcal{M} be a finite family of right A -modules. A study of the noncommutative deformation functor $\text{Def}_{\mathcal{M}}$ of the family \mathcal{M} leads to the construction of the algebra $\mathcal{O}^A(\mathcal{M})$ of observables and the generalized Burnside theorem, due to Laudal (2002). In this paper, we give an overview of aspects of noncommutative deformations closely connected to the generalized Burnside theorem.

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1 Introduction

Let k be a field and let A be an associative k -algebra. For any right A -module M , there is a commutative deformation functor $\text{Def}_M : \mathbb{1} \rightarrow \text{Sets}$ defined on the category $\mathbb{1}$ of local Artinian commutative k -algebras with residue field k . We recall that for an algebra R in $\mathbb{1}$, a deformation of M to R is a pair (M_R, τ) , where M_R is an R - A bimodule (on which k acts centrally) that is R -flat, and $\tau : k \otimes_R M_R \rightarrow M$ is an isomorphism of right A -modules.

Let \mathfrak{a}_r be the category of r -pointed Artinian k -algebras for $r \geq 1$, the natural noncommutative generalization of $\mathbb{1}$. We recall that an algebra R in \mathfrak{a}_r is an Artinian ring, together with a pair of structural ring homomorphisms $f : k^r \rightarrow R$ and $g : R \rightarrow k^r$ with $g \circ f = \text{id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. Any algebra R in \mathfrak{a}_r has r simple right modules of dimension one, the natural projections $\{k_1, \dots, k_r\}$ of k^r .

In [2], a noncommutative deformation functor $\text{Def}_{\mathcal{M}} : \mathfrak{a}_r \rightarrow \text{Sets}$ of a finite family $\mathcal{M} = \{M_1, \dots, M_r\}$ of right A -modules was introduced, as a generalization of the commutative deformation functor $\text{Def}_M : \mathbb{1} \rightarrow \text{Sets}$ of a right A -module M . In the case $r = 1$, this generalization is completely natural, and can be defined word for word as in the commutative case. The generalization to the case $r > 1$ is less obvious and has further-reaching consequences, but is still very natural. A deformation of \mathcal{M} to R is defined to be a pair $(M_R, \{\tau_i\}_{1 \leq i \leq r})$, where M_R is an R - A bimodule (on which k acts centrally) that is R -flat, and $\tau_i : k_i \otimes_R M_R \rightarrow M_i$ is an isomorphism of right A -modules for $1 \leq i \leq r$. We remark that M_R is R -flat if and only if

$$M_R \cong (R_{ij} \otimes_k M_j) = \begin{pmatrix} R_{11} \otimes_k M_1 & R_{12} \otimes_k M_2 & \cdots & R_{1r} \otimes_k M_r \\ R_{21} \otimes_k M_1 & R_{22} \otimes_k M_2 & \cdots & R_{2r} \otimes_k M_r \\ \vdots & \vdots & \ddots & \vdots \\ R_{r1} \otimes_k M_1 & R_{r2} \otimes_k M_2 & \cdots & R_{rr} \otimes_k M_r \end{pmatrix},$$

considered as a left R -module, and that a deformation in $\text{Def}_{\mathcal{M}}(R)$ may be thought of as a right multiplication $A \rightarrow \text{End}_R(M_R)$ of A on the left R -module M_R that lifts the multiplication $\rho : A \rightarrow \bigoplus_i \text{End}_k(M_i)$ of A on the family \mathcal{M} .

There is an obstruction theory for $\text{Def}_{\mathcal{M}}$, generalizing the obstruction theory for the commutative deformation functor. Hence there exists a formal moduli (H, M_H) for $\text{Def}_{\mathcal{M}}$ (assuming a mild condition on \mathcal{M}). We consider the algebra of observables $\mathcal{O}^A(\mathcal{M}) = \text{End}_H(M_H) \cong (H_{ij} \widehat{\otimes}_k \text{Hom}_k(M_i, M_j))$ and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathcal{O}^A(\mathcal{M}) \\ & \searrow \rho & \downarrow \pi \\ & & \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i) \end{array}$$

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given by the versal family $M_H \in \text{Def}_{\mathcal{M}}(H)$. The algebra $B = \mathcal{O}^A(\mathcal{M})$ has an induced right action on the family \mathcal{M} extending the action of A , and we may consider \mathcal{M} as a family of right B -modules. In fact, \mathcal{M} is the family of simple B -modules since π can be identified with the quotient morphism $B \rightarrow B/\text{rad } B$.

When A is an algebra of finite dimension over an algebraically closed field k and \mathcal{M} is the family of simple right A -modules, Laudal proved the *generalized Burnside theorem* in [2], generalizing the structure theorem for semi-simple algebras and the classical Burnside theorem. Laudal's result is stated in the following form.

Theorem (The generalized Burnside theorem). *Let A be a finite-dimensional algebra over a field k , and let $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ be the family of simple right A -modules. If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\eta : A \rightarrow \mathcal{O}^A(\mathcal{M})$ is an isomorphism. In particular, η is an isomorphism when k is algebraically closed.*

Let A be an algebra of finite dimension over an algebraically closed field k and let \mathcal{M} be any finite family of right A -modules of finite dimension over k . Then the algebra $B = \mathcal{O}^A(\mathcal{M})$ has the property that $\eta_B : B \rightarrow \mathcal{O}^B(\mathcal{M})$ is an isomorphism, or equivalently, that the assignment $(A, \mathcal{M}) \mapsto (B, \mathcal{M})$ is a closure operation. This means that the family \mathcal{M} has exactly the same module-theoretic properties, in terms of (higher) extensions and Massey products, considered as a family of modules over B as over A .

2 Noncommutative deformations of modules

Let k be a field. For any integer $r \geq 1$, we consider the category \mathfrak{a}_r of r -pointed Artinian k -algebras. We recall that an object in \mathfrak{a}_r is an Artinian ring R , together with a pair of structural ring homomorphisms $f : k^r \rightarrow R$ and $g : R \rightarrow k^r$ with $g \circ f = \text{id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. The morphisms of \mathfrak{a}_r are the ring homomorphisms that commute with the structural morphisms. It follows from this definition that $I(R)$ is the Jacobson radical of R , and therefore that the simple right R -modules are the projections $\{k_1, \dots, k_r\}$ of k^r .

Let A be an associative k -algebra. For any family $\mathcal{M} = \{M_1, \dots, M_r\}$ of right A -modules, there is a noncommutative deformation functor $\text{Def}_{\mathcal{M}} : \mathfrak{a}_r \rightarrow \text{Sets}$, introduced by Laudal [2]; see also Eriksen [1]. For an algebra R in \mathfrak{a}_r , we recall that a deformation of \mathcal{M} over R is a pair $(M_R, \{\tau_i\}_{1 \leq i \leq r})$, where M_R is an R - A bimodule (on which k acts centrally) that is R -flat, and $\tau_i : k_i \otimes_R M_R \rightarrow M_i$ is an isomorphism of right A -modules for $1 \leq i \leq r$. Moreover, $(M_R, \{\tau_i\}) \sim (M'_R, \{\tau'_i\})$ are equivalent deformations over R if there is an isomorphism $\eta : M_R \rightarrow M'_R$ of R - A bimodules such that $\tau_i = \tau'_i \circ (1 \otimes \eta)$ for $1 \leq i \leq r$. We may prove that M_R is R -flat if and only if

$$M_R \cong (R_{ij} \otimes_k M_j) = \begin{pmatrix} R_{11} \otimes_k M_1 & R_{12} \otimes_k M_2 & \cdots & R_{1r} \otimes_k M_r \\ R_{21} \otimes_k M_1 & R_{22} \otimes_k M_2 & \cdots & R_{2r} \otimes_k M_r \\ \vdots & \vdots & \ddots & \vdots \\ R_{r1} \otimes_k M_1 & R_{r2} \otimes_k M_2 & \cdots & R_{rr} \otimes_k M_r \end{pmatrix},$$

considered as a left R -module, and a deformation in $\text{Def}_{\mathcal{M}}(R)$ may be thought of as a right multiplication $A \rightarrow \text{End}_R(M_R)$ of A on the left R -module M_R that lifts the multiplication $\rho : A \rightarrow \oplus_i \text{End}_k(M_i)$ of A on the family \mathcal{M} .

Let us assume that \mathcal{M} is a *swarm*, that is, $\text{Ext}_A^1(M_i, M_j)$ has finite dimension over k for $1 \leq i, j \leq r$. Then $\text{Def}_{\mathcal{M}}$ has a pro-representing hull or a formal moduli (H, M_H) ; see Laudal [2, Theorem 3.1]. This means that H is a complete r -pointed k -algebra in the pro-category $\hat{\mathfrak{a}}_r$, and that $M_H \in \text{Def}_{\mathcal{M}}(H)$ is a family defined over H with the following versal property: for any algebra R in \mathfrak{a}_r and any deformation $M_R \in \text{Def}_{\mathcal{M}}(R)$, there is a homomorphism $\phi : H \rightarrow R$ such that $\text{Def}_{\mathcal{M}}(\phi)(M_H) = M_R$. The formal moduli (H, M_H) is unique up to non-canonical isomorphism. However, the morphism ϕ is not uniquely determined by (R, M_R) .

When \mathcal{M} is a swarm with formal moduli (H, M_H) , right multiplication on the H - A bimodule M_H by elements in A determines an algebra homomorphism

$$\eta : A \longrightarrow \text{End}_H(M_H).$$

We write $\mathcal{O}^A(\mathcal{M}) = \text{End}_H(M_H)$ and call it the *algebra of observables*. Since M_H is H -flat, we have that $\text{End}_H(M_H) \cong (H_{ij} \widehat{\otimes}_k \text{Hom}_k(M_i, M_j))$, and it follows that $\mathcal{O}^A(\mathcal{M})$ is explicitly given as the matrix algebra

$$\begin{pmatrix} H_{11} \widehat{\otimes}_k \text{End}_k(M_1) & H_{12} \widehat{\otimes}_k \text{Hom}_k(M_1, M_2) & \cdots & H_{1r} \widehat{\otimes}_k \text{Hom}_k(M_1, M_r) \\ H_{21} \widehat{\otimes}_k \text{Hom}_k(M_2, M_1) & H_{22} \widehat{\otimes}_k \text{End}_k(M_2) & \cdots & H_{2r} \widehat{\otimes}_k \text{Hom}_k(M_2, M_r) \\ \vdots & \vdots & \ddots & \vdots \\ H_{r1} \widehat{\otimes}_k \text{Hom}_k(M_r, M_1) & H_{r2} \widehat{\otimes}_k \text{Hom}_k(M_r, M_2) & \cdots & H_{rr} \widehat{\otimes}_k \text{End}_k(M_r) \end{pmatrix}.$$

Let us write $\rho_i : A \rightarrow \text{End}_k(M_i)$ for the structural algebra homomorphism defining the right A -module structure on M_i for $1 \leq i \leq r$, and

$$\rho : A \longrightarrow \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i)$$

for their direct sum. Since H is a complete r -pointed algebra in $\hat{\mathfrak{a}}_r$, there is a natural morphism $H \rightarrow k^r$, inducing an algebra homomorphism

$$\pi : \mathcal{O}^A(\mathcal{M}) \longrightarrow \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i).$$

By construction, there is a right action of $\mathcal{O}^A(\mathcal{M})$ on the family \mathcal{M} extending the right action of A , in the sense that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathcal{O}^A(\mathcal{M}) \\ & \searrow \rho & \downarrow \pi \\ & & \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i) \end{array}$$

commutes. This makes it reasonable to call $\mathcal{O}^A(\mathcal{M})$ the algebra of observables.

3 The generalized Burnside theorem

Let k be a field and let A be a finite-dimensional associative k -algebra. Then the simple right modules over A are the simple right modules over the semi-simple quotient algebra $A/\text{rad}(A)$, where $\text{rad}(A)$ is the Jacobson radical of A . By the classification theory for semi-simple algebras, it follows that there are finitely many non-isomorphic simple right A -modules.

We consider the noncommutative deformation functor $\text{Def}_{\mathcal{M}} : \mathfrak{a}_r \rightarrow \text{Sets}$ of the family $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ of simple right A -modules. Clearly, \mathcal{M} is a swarm, hence $\text{Def}_{\mathcal{M}}$ has a formal moduli (H, M_H) , and we consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathcal{O}^A(\mathcal{M}) \\ & \searrow \rho & \downarrow \pi \\ & & \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i). \end{array}$$

By a classical result, due to Burnside, the algebra homomorphism ρ is surjective when k is algebraically closed. This result is conveniently stated in the following form.

Theorem 1 (Burnside’s theorem). *If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then ρ is surjective. In particular, ρ is surjective when k is algebraically closed.*

Proof. There is a factorization $A \rightarrow A/\text{rad}(A) \rightarrow \bigoplus_i \text{End}_k(M_i)$ of ρ . If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $A/\text{rad}(A) \rightarrow \bigoplus_i \text{End}_k(M_i)$ is an isomorphism by the classification theory for semi-simple algebras. Since $\text{End}_A(M_i)$ is a division ring of finite dimension over k , it is clear that $\text{End}_A(M_i) = k$ whenever k is algebraically closed. \square

Let us write $\bar{\rho} : A/\text{rad} A \rightarrow \bigoplus_i \text{End}_k(M_i)$ for the algebra homomorphism induced by ρ . We observe that ρ is surjective if and only if $\bar{\rho}$ is an isomorphism. Moreover, let us write $J = \text{rad}(\mathcal{O}^A(\mathcal{M}))$ for the Jacobson radical of $\mathcal{O}^A(\mathcal{M})$. Then we see that

$$J = (\text{rad}(H)_{ij} \hat{\otimes}_k \text{Hom}_k(M_i, M_j)) = \ker(\pi).$$

Since $\rho(\text{rad} A) = 0$ by definition, it follows that $\eta(\text{rad} A) \subseteq J$. Hence there are induced morphisms

$$\text{gr}(\eta)_q : \text{rad}(A)^q / \text{rad}(A)^{q+1} \rightarrow J^q / J^{q+1}$$

for all $q \geq 0$. We may identify $\text{gr}(\eta)_0$ with $\bar{\rho}$, since $\mathcal{O}^A(\mathcal{M})/J \cong \bigoplus_i \text{End}_k(M_i)$. The conclusion in Burnside’s theorem is therefore equivalent to the statement that $\text{gr}(\eta)_0$ is an isomorphism.

Theorem 2 (The generalized Burnside theorem). *Let A be a finite-dimensional algebra over a field k , and let $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ be the family of simple right A -modules. If $\text{End}_A(M_i) = k$ for $1 \leq i \leq r$, then $\eta : A \rightarrow \mathcal{O}^A(\mathcal{M})$ is an isomorphism. In particular, η is an isomorphism when k is algebraically closed.*

Proof. It is enough to prove that η is injective and that $\text{gr}(\eta)_q$ is an isomorphism for $q = 0$ and $q = 1$, since A and $\mathcal{O}^A(\mathcal{M})$ are complete in the $\text{rad}(A)$ -adic and J -adic topologies. By Burnside's theorem, we know that $\text{gr}(\eta)_0$ is an isomorphism. To prove that η is injective, let us consider the kernel $\ker(\eta) \subseteq A$. It is determined by the obstruction calculus of $\text{Def}_{\mathcal{M}}$; see Laudal [2, Theorem 3.2] for details. When A is finite-dimensional, the right regular A -module A_A has a decomposition series

$$0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = A_A$$

with F_p/F_{p-1} a simple right A -module for $1 \leq p \leq n$. Namely, A_A is an *iterated extension* of the modules in \mathcal{M} . This implies that η is injective; see Laudal [2, Corollary 3.1]. Finally, we must prove that $\text{gr}(\eta)_1 : \text{rad}(A)/\text{rad}(A)^2 \rightarrow J/J^2$ is an isomorphism. This follows from the Wedderburn-Malcev theorem; see Laudal [2, Theorem 3.4], for details. \square

4 Properties of the algebra of observables

Let A be a finite-dimensional algebra over a field k , and let $\mathcal{M} = \{M_1, \dots, M_r\}$ be any family of right A -modules of finite dimension over k . Then \mathcal{M} is a swarm, and we denote the algebra of observables by $B = \mathcal{O}^A(\mathcal{M})$. It is clear that

$$B/\text{rad}(B) \cong \bigoplus_i \text{End}_k(M_i)$$

is semi-simple, and it follows that \mathcal{M} is the family of simple right B -modules. In fact, we may show that \mathcal{M} is a swarm of B -modules, since B is complete and $B/(\text{rad } B)^n$ has finite dimension over k for all positive integers n .

Proposition 3. *If k is an algebraically closed field, then $\eta_B : B \rightarrow \mathcal{O}^B(\mathcal{M})$ is an algebra isomorphism.*

Proof. Since \mathcal{M} is a swarm of A -modules and of B -modules, we may consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\eta^A} & B = \mathcal{O}^A(\mathcal{M}) & \xrightarrow{\eta^B} & C = \mathcal{O}^B(\mathcal{M}) \\ & \searrow \rho & \downarrow & \swarrow & \\ & & \bigoplus_{1 \leq i \leq r} \text{End}_k(M_i) & & \end{array}$$

The algebra homomorphism η^B induces maps $B/\text{rad}(B)^n \rightarrow C/\text{rad}(C)^n$ for all $n \geq 1$. Since k is algebraically closed and $B/\text{rad}(B)^n$ has finite dimension over k , it follows from the generalized Burnside theorem that $B/\text{rad}(B)^n \rightarrow C/\text{rad}(C)^n$ is an isomorphism for all $n \geq 1$. Hence η^B is an isomorphism. \square

In particular, the proposition implies that the assignment $(A, \mathcal{M}) \mapsto (B, \mathcal{M})$ is a closure operation when k is algebraically closed. In other words, the algebra $B = \mathcal{O}^A(\mathcal{M})$ has the following properties:

- (1) the family \mathcal{M} is the family of the simple B -modules;
- (2) the family \mathcal{M} has exactly the same module-theoretic properties, in terms of (higher) extensions and Massey products, considered as a family of modules over B as over A .

Moreover, these properties characterize the algebra $B = \mathcal{O}^A(\mathcal{M})$ of observables.

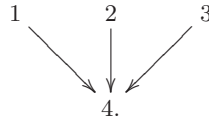
5 Examples: representations of ordered sets

Let k be an algebraically closed field, and let Λ be a finite ordered set. Then the algebra $A = k[\Lambda]$ is an associative algebra of finite dimension over k . The category of right A -modules is equivalent to the category of presheaves of vector spaces on Λ , and the simple A -modules correspond to the presheaves $\{M_\lambda : \lambda \in \Lambda\}$ defined by $M_\lambda(\lambda) = k$ and $M_\lambda(\lambda') = 0$ for $\lambda' \neq \lambda$. The following results are well known:

- (1) if $\lambda > \lambda'$ in Λ and $\{\gamma \in \Lambda : \lambda > \gamma > \lambda'\} = \emptyset$, then $\text{Ext}_A^1(M_\lambda, M_{\lambda'}) \cong k$;
- (2) if $\{\gamma \in \Lambda : \lambda \geq \gamma \geq \lambda'\}$ is a simple loop in Λ , then $\text{Ext}_A^2(M_\lambda, M_{\lambda'}) \cong k$;
- (3) in all other cases, $\text{Ext}_A^1(M_\lambda, M_{\lambda'}) = \text{Ext}_A^2(M_\lambda, M_{\lambda'}) = 0$.

5.1 A hereditary example

Let us first consider the following ordered set. We label the elements by natural numbers, and write $i \rightarrow j$ when $i > j$:



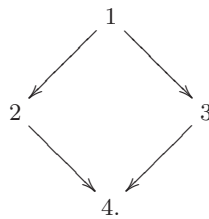
In this case, the simple modules are given by $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$, and we can easily compute the algebra $\mathcal{O}^A(\mathcal{M})$ of observables since $\text{Ext}_A^2(M_i, M_j) = 0$ for all $1 \leq i, j \leq 4$. We obtain

$$\mathcal{O}^A(\mathcal{M}) = (H_{ij} \widehat{\otimes}_k \text{Hom}_k(M_i, M_j)) \cong H \cong \begin{pmatrix} k & 0 & 0 & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}.$$

It follows from the generalized Burnside theorem that $\eta : A \rightarrow \mathcal{O}^A(\mathcal{M})$ is an isomorphism. Hence we recover the algebra $A \cong \mathcal{O}^A(\mathcal{M}) \cong H$.

5.2 The diamond

Let us also consider the following ordered set, called *the diamond*. We label the elements by natural numbers, and write $i \rightarrow j$ when $i > j$:



In this case, the simple modules are given by $\mathcal{M} = \{M_1, M_2, M_3, M_4\}$. Since $\text{Ext}_A^2(M_1, M_4) \cong k$, we must compute the cup-products

$$\begin{aligned} \text{Ext}_A^1(M_1, M_2) \cup \text{Ext}_A^1(M_2, M_4) &\longrightarrow \text{Ext}_A^2(M_1, M_4), \\ \text{Ext}_A^1(M_1, M_3) \cup \text{Ext}_A^1(M_3, M_4) &\longrightarrow \text{Ext}_A^2(M_1, M_4) \end{aligned}$$

in order to compute H . These cup-products are non-trivial; see Laudal [2, Remark 3.2] for details. Hence we obtain

$$\mathcal{O}^A(\mathcal{M}) = (H_{ij} \widehat{\otimes}_k \text{Hom}_k(M_i, M_j)) \cong H \cong \begin{pmatrix} k & k & k & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Note that H_{14} is two-dimensional at the tangent level and has a relation. Also in this case, it follows from the generalized Burnside theorem that $\eta : A \rightarrow \mathcal{O}^A(\mathcal{M})$ is an isomorphism. Hence we recover the algebra $A \cong \mathcal{O}^A(\mathcal{M}) \cong H$.

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