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# The Generalization of the Stalling's Theorem

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**Abstract**

In this paper, we present a relative version of the concept of lower marginal series and give some isomorphisms among $\mathcal{V}$G-marginal factor groups. Also, we conclude a generalized version of the Stalling's theorem. Finally, we present a sufficient condition under which the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of its factor groups.

**Keywords:** Schur-Baer variety; Pair of groups; $\mathcal{V}$G-marginal series

## Introduction

There exists a long history of interaction between Schur multipliers and other mathematical concepts. This basic notion started by Schur [1], when he introduced multipliers in order to study projective representations of groups. It was known later that the Schur multiplier had a relation with homology and cohomology of groups. In fact, if $G$ is a finite group, then $M(G)\cong H^2(G,\mathbb{C}^*)\cong H_2(G,\mathbb{Z})$, where $M(G)$ is the Schur multiplier of $G$, $H^2(G, \mathbb{C}^*)$ is the second cohomology of $G$ with coefficient in $\mathbb{C}^*$ and $H_2(G, \mathbb{Z})$ is the second internal homology of $G$ [2]. Hopf [3] proved that $M(G)\cong(R\cap F^2)/[R,F]$. He also proved that the Schur multiplier of $G$ is independent of the free presentation of $G$. Let $(G, N)$ be a pair of groups, where $N$ is a normal subgroup in Ellis [4] defined the Schur multiplier of the pair $(G, N)$ to be the abelian group $M(G, N)$ appears in the following natural exact sequence

$$H_3(G)\to H_3(G,N)\to M(G,N)\to M(G)\to M(G/N)$$
$$\to G/[N,G]\to (G)_{ab}\to (G/N)_{ab}\to 1,$$

where $H_3(-)$ denote the third homology of a group with integer coefficients. He also proved that if the normal subgroup $N$ possess a complement in $G$, then for each free presentation $1\to R\to F\to G\to 1$ of $G$, $M(G, N)$ is isomorphic with the factor group $(R\cap[S,F])/[R,F]$, where $S$ is a normal subgroup of $F$ such that $S/R\cong N$. In particular, if $N=G$ then the Schur multiplier of $(G, N)$ will be $M(G)=(R\cap[F,F])/[R,F]$.

We assume that the reader is familiar with the notions of the *verbal subgroup* $V(G)$, and the *marginal subgroup*

$V^*(G)$, associated with a variety of groups $\mathcal{V}$ and a group $G$ [5] for more information on varieties of groups). Let $F_\infty$ be the free group freely generated by the countable set $X=\{x_1,x_2,\ldots\}$ and $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws $\mathcal{V}$ and $\mathcal{W}$, respectively. Let $N$ be a normal subgroup of a group $G$, then we define $[NV^*G]$ to be the subgroup of $G$ generated by the elements of the following set:

$$\{v(g_1,g_2,\ldots,g_in,\ldots,g_r)v(g_1,g_2,\ldots,g_r)^{-1}\mid 1\le i\le r, v\in V, g_1,\ldots,g_r\in G, n\in N\}.$$

It is easily checked that $[NV^*G]$ is the least normal subgroup $T$ of $G$ such that $N/T$ is contained in $V^*(G/T)$ [6].

The first to create the generalization of the Schur multiplier to any variety of groups was Baer [7]. It is well known fact that the recent concept is useful in classifying groups into isologism classes. Leedham-Green and McKay [8] introduced the following generalized version of the Baer-invariant of a group with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$.

Let $G$ be an arbitrary group in $\mathcal{W}$ with a free presentation $1\to R\to F\to G\to 1$, in which $F$ is a free group. Clearly, $1=W(G)=W(F)R/R$ and hence $W(F)\subseteq R$, therefore,

$$1\to R/W(F)\to F/W(F)\to G\to 1$$

is a $\mathcal{W}$-free presentation of the group $G$. We call

$$\mathcal{WV}M(G)=\frac{R/W(F)\cap V(F/W(F))}{[R/W(F)V^*(F/W(F))]}=\frac{W(F)(R\cap V(F))}{W(F)[RV^*F]}$$

the *generalized Baer-invariant* of the group $G$ in $\mathcal{W}$ with respect to the variety $\mathcal{V}$. Now if $N$ is a normal subgroup of the group $G$ for a suitable normal subgroup $S$ of the free group $F$, we have $N\cong S/R$. Then we can define the generalized Baer-invariant of the pair of groups with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$ as follows:

$$\mathcal{WV}M(G,N)=\frac{R/W(F)\cap[S/W(F)V^*(F/W(F))]}{[R/W(F)V^*(F/W(F))]}=\frac{W(F)(R\cap[SV^*F])}{W(F)[RV^*F]}.$$

One may check that $\mathcal{WV}M(G, N)$ is always abelian and independent of the free presentation of $G$. In particular, if $\mathcal{W}$ is the variety of all groups and $N=G$ then the generalized Baer-invariant of the pair $(G, N)$ will be

$$\mathcal{V}M(G,G)=\frac{R\cap V(F)}{[RV^*F]}=\mathcal{V}M(G),$$

which is the usual Baer-invariant of $G$ with respect to $\mathcal{V}$ [8].

It is interesting to know the connection between the Baer-invariant of a pair of finite groups $(G, N)$ and its factor groups with respect to the Schur-Baer variety $\mathcal{V}$. In the next section, we show that under some circumstances there are some isomorphisms among $\mathcal{V}_G$-marginal factor groups (Theorem 2.2). Also, a sufficient condition will be given such that the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of the pair of its factor groups (Theorem 2.5).

Variety $\mathcal{V}$ is called a *Schur-Baer* variety if for any group $G$ in which

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the marginal factor group $G / V^*(G)$ is finite, then the verbal subgroup $V(G)$ is also finite. Schur [9] proved that the variety of abelian groups is a Schur-Baer variety and Baer [10] showed that a variety defined by outer commutator words carries this property. In 2002, Moghaddam et al. [11] proved that for a finite group $G$, $\mathcal{V}M(G)$ is finite with respect to a Schur-Baer variety $\mathcal{V}$. In the following lemma we prove similar result for the $\mathcal{WV}M(G, N)$ and $\mathcal{WV}M(G)$ with using another technique.

**Lemma 1.1.** *Let $\mathcal{V}$ be a Schur-Baer variety and $G$ be a finite group in $\mathcal{W}$ with a normal subgroup $N$. Then there exists a group $H$ with a normal subgroup $K$ such that*

$$|[NV^*G]||\mathcal{WV}M(G,N)|=|[KV^*H]|<\infty.$$

*In particular,* $|V(G)||\mathcal{WV}M(G)|=|V(H)|<\infty.$

*Proof.* Let $G = F / R$ be a free presentation for the group $G$ and $S$ be a normal subgroup of the free group $F$ such that $N \cong S / R$, then

$$\frac{R}{W(F)[RV^*F]}\subseteq V^*\left(\frac{F}{W(F)[RV^*F]}\right).$$

Let $H=F/W(F)[RV^*F]$ and $K=S/W(F)[RV^*F]$, then $|\frac{H}{V^*(H)}|<|G|<\infty$ and $|[KV^*H]|\le|V(H)|<\infty$. But

$$|[KV^*H]|=|\frac{W(F)[SV^*F]}{W(F)[RV^*F]}|=|\frac{W(F)[SV^*F]}{W(F)(R\cap[SV^*F])}||\frac{W(F)(R\cap[SV^*F])}{W(F)[RV^*F]}|.$$

Also, $[NV^*G]=\frac{[SV^*F]R}{R}=\frac{W(F)[SV^*F]R}{R}\cong\frac{W(F)[SV^*F]}{W(F)(R\cap[SV^*F])}$. Thus the result holds.

## Stallings' Theorem

In the following lemma we present some exact sequences for the generalized Baer-invariant of a pair of groups and its factor groups.

**Lemma 2.1.** *Let $G$ be a group with a free presentation $1\to R\to F\to G\to 1$ and $S$, $T$ be normal subgroups of the free group $F$ such that $T\subseteq S$, $S/R\cong N$ and $T/R\cong K$. Then the following sequences are exact:*

$$\text{(i)}\ 1\to\frac{W(F)(R\cap[TV^*F])}{W(F)[RV^*F]}\to\mathcal{WV}M(G,N)\to\mathcal{WV}M(G/K,N/K)\to\frac{K\cap[NV^*G]}{[KV^*G]}\to 1;$$

$$\text{(ii)}\ \mathcal{WV}M(G,N)\to\mathcal{WV}M(G/K,N/K)\to\frac{K}{[KV^*G]}\to\frac{N}{[NV^*G]}\to\frac{N}{[NV^*G]K}\to 1;$$

(iii) Moreover, if $K$ is contained in $V^*(G)$, then the following sequence is exact:

$$1\to\frac{R\cap[SV^*F]}{W(F)[TV^*F]\cap[SV^*F]}\to\mathcal{WV}M(G/K,N/K)\to K\to\frac{N}{[NV^*G]}\to\frac{N}{[NV^*G]K}\to 1.$$

*Proof.* Considering the definition mentioned above we can conclude:

$$\mathcal{WV}M(G/K,N/K)=\frac{W(F)(T\cap[SV^*F])}{W(F)[TV^*F]}\qquad\frac{K\cap[NV^*G]}{[KV^*G]}=\frac{(T\cap[SV^*F])R}{[TV^*F]R},$$

$$\mathcal{WV}M(G,N)=\frac{W(F)(R\cap[SV^*F])}{W(F)[RV^*F]}.$$

Now one can easily check that the sequences (i) and (ii) are exact.

(iii) Using the assumption, we have $W(F)[TV^*F]\subseteq R$. Therefore, one can easily check that the following sequence is exact:

$$1\to\frac{R\cap[SV^*F]}{W(F)[TV^*F]\cap[SV^*F]}\to\frac{W(F)(T\cap[SV^*F])}{W(F)[TV^*F]}\to T/R\to\frac{S}{[SV^*F]R}\to\frac{S}{[SV^*F]T}\to 1.$$

Let $N$ be a normal subgroup of a group $G$. Then we define a series of normal subgroups of $N$ as follows:

$$N=V_0(N,G)\supseteq V_1(N,G)\supseteq V_2(N,G)\supseteq\cdots\supseteq V_n(N,G)\supseteq\cdots,$$

where $V_i(N,G)=[V_{i-1}(N,G)V^*G]$ for all $n\geq 1$. We call such a series the *lower $\mathcal{V}_G$-marginal series* of $N$ in $G$. One may also define the *upper $\mathcal{V}_G$-marginal series* as in studies of Moghaddam et al. [11].

We say that the normal subgroup $N$ of $G$ is *$\mathcal{V}_G$-nilpotent* if it has a finite *lower $\mathcal{V}_G$-marginal series*. The shortest length of such series is called the class of *$\mathcal{V}_G$-nilpotency* of $N$ in $G$. If $N = G$, then this is called *lower $\mathcal{V}$-marginal series* of $G$. The group $G$ is said to be *$\mathcal{V}$-nilpotent* iff $V_n(G) = 1$, for some positive integer $n$ [12].

Now, we want to show that under some circumstances there are some isomorphisms among $\mathcal{V}_G$-marginal factor groups. By using Lemma 2.1, we have the following Theorem, which generalizes 7.9.1 of literature of Hilton and Stammbach [13].

**Theorem 2.2.** *Let $f : G \to H$ be a group homomorphism and $N$ be a normal subgroup of $G$ and $K$ be a normal subgroup of $H$ such that $f(N)\subseteq K$. Suppose $f$ induces isomorphisms $f_0:G/N\to H/K$ and $\overline{f}_1:N/[NV^*G]\to K/[KV^*H]$, and that $f_*:\mathcal{WV}M(G,N)\to\mathcal{WV}M(H,K)$ is an epimorphism. Then $f$ induces isomorphisms $f_n:G/V_n(N,G)\overset{\simeq}{\to}H/V_n(K,H)$ and $\overline{f}_n:N/V_n(N,G)\overset{\simeq}{\to}K/V_n(K,H)$ for all $n\geq 0$.*

*Proof.* At first, we want to mention a point that for making it easier to draw the following diagrams, we would like to introduce $P_n=V_n(N,G)$ and $Q_n=V_n(K,H)$. We proceed by induction. For $n = 0$ the assertion is trivial. For $n = 1$, consider the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N/[NV^*G] & \longrightarrow & G/[NV^*G] & \longrightarrow & G/N & \longrightarrow & 1\\
 & & \downarrow{\scriptstyle \overline{f}_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\
1 & \longrightarrow & K/[KV^*H] & \longrightarrow & H/[KV^*H] & \longrightarrow & H/K & \longrightarrow & 1.
\end{array}$$

By the hypothesis $\overline{f}_1$ and $f_0$ are isomorphism, hence $f_1$ is an isomorphism. Assume that $n\geq 2$. By consedering Lemma 2.1(ii), we can conclude the following communicative diagram:

$$\begin{array}{ccccccccccc}
\mathcal{WV}M(G,N) & \to & \mathcal{WV}M(G/P_{n-1},N/P_{n-1}) & \to & P_{n-1}/P_n & \to & N/[NV^*G] & \to & N/[NV^*G]P_{n-1} & \to & 1\\
\downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_3} & & \downarrow{\scriptstyle\alpha_4} & & \downarrow{\scriptstyle\alpha_5} & & \\
\mathcal{WV}M(H,K) & \to & \mathcal{WV}M(H/Q_{n-1},K/Q_{n-1}) & \to & Q_{n-1}/Q_n & \to & K/[KV^*H] & \to & K/[KV^*H]Q_{n-1} & \to & 1
\end{array}$$

Note that the naturality of the map $f$ induces homomorphisms $\alpha_i$, $i = 1,2,\ldots,5$ such that $(*)$ is commutative. By hypothesis $\alpha_1$ is an epimorphism and $\alpha_4$, $\alpha_5$ are isomorphisms. Also, by considering the induction hypothesis and definition of the Baer-invariant of the pair of groups, $\alpha_2$ is an isomorphism. Hence by five lemma of Rotman's studies [14] $\alpha_3$ is an isomorphism. Now consider the following diagram and in the same way, $f_n$ is an isomorphism.

Now we obtain the following corollary.

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$$\begin{tikzcd}
1 \arrow[r] & P_{n-1}/P_n \arrow[r] \arrow[d, "\alpha_3"] & N/P_n \arrow[r] \arrow[d, "\bar{f}_n"] & N/P_{n-1} \arrow[r] \arrow[d, "\bar{f}_{n-1}"] & 1 \\
1 \arrow[r] & Q_{n-1}/Q_n \arrow[r] & K/Q_n \arrow[r] & K/Q_{n-1} \arrow[r] & 1
\end{tikzcd}$$

By the above discussion $\alpha_3$ is an isomorphism and by induction of hypothesis $\bar{f}_{n-1}$ is an isomorphism, therefore, $\bar{f}_n$ is an isomorphism. Finally, by the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & N/P_n \arrow[r] \arrow[d, "\bar{f}_n"] & G/P_n \arrow[r] \arrow[d, "f_n"] & G/N \arrow[r] \arrow[d, "f_1"] & 1 \\
1 \arrow[r] & K/Q_n \arrow[r] & H/Q_n \arrow[r] & H/K \arrow[r] & 1
\end{tikzcd}$$

And the same way, $f_n$ ia an isomorphism.

Now we obtain the following collary.

**Corollary 2.3.** *Let* $(f, f|):(G,N)\to(H,K)$ *are group homomorphisms satisfy the hypotheses of Theorem 2.2. Suppose further that N and K are* $\mathcal{V}_G$*-nilpotent and* $\mathcal{V}_H$*-nilpotent, respectively. Then f and f | are isomorphisms.*

*Proof.* The assertion follows from Theorem 2.2 and the remark that there exists $n \geq 0$ such that $V_n(N,G) = \{1\}$ and $V_n(K,H) = \{1\}$.

Now, we have the following theorem, which is a generalization of Stalling's theorem [15].

**Theorem 2.4.** *Let* $\mathcal{V}$ *be a variety of groups and* $f : G \to H$ *be an epimorphism. Let N be a* $\mathcal{V}_G$*-nilpotent normal subgroup of G and K be a normal subgroup of H such that f (N) = K. If* $\ker f \subseteq [NV^*G]$ *and* $\mathcal{WVM}(H, K)$ *is trivial, then f and f | are isomorphisms.*

*Proof.* Put $M = \ker f$, then $\frac{N}{[NV^*G]} \cong \frac{K}{[KV^*H]}$, $\frac{G}{N} \cong \frac{H}{K}$ and $\frac{V_n(N,G)M}{M} = V_n(K,H)$ for all $n \geq 0$. Now the result follows from Corollary 2.3.

Finally, a sufficient condition will be given such that the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of the pair of its factor groups with respect to two varieties of groups. Let $\psi : E \to G$ be an epimorphism such that $\ker\psi \subseteq V^*(E)$. We denote by $(WV^*)^*(G)$ the intersection of all subgroups of the form $\psi(V^*(E))$. Clearly, $(WV^*)^*(G)$ is a characteristic subgroup of $G$ which is contained in $V^*(G)$. In particular, if $\mathcal{W}$ is the variety of all groups and $\mathcal{V}$ is a variety of abelian groups then this subgroup is denoted by $Z^*(G)$ as in literature of Karpilovsky [2].

Now using the above concept we have the following Theorem.

**Theorem 2.5.** *Let K be a normal subgroup of G contained in* $N \cap (WV^*)^*(G)$. *Then*

$|\mathcal{WVM}(G,N)|$ *divides* $|\mathcal{WVM}(G/K, N/K)|$.

*Proof.* By theorem 3.2 of Neumann [5], natural homomorphism $\mathcal{WVM}(G) \to \mathcal{WVM}(G/K)$ will be a monomorphism. Now the following commutative diagram

$$\begin{tikzcd}
WVM(G,N) \arrow[r, "\subseteq"] \arrow[d] & WVM(G) \arrow[d] \\
WVM(G/K, N/K) \arrow[r, "\subseteq"] & WVM(G/K)
\end{tikzcd}$$

implies that the natural homomorphism $\mathcal{WVM}(G,N) \to \mathcal{WVM}(G/K, N/K)$ is also a monomorphism. Thus Lemma 1.2 (i) implies that $\mathcal{WVM}(G, K)$ is trivial. Now we have $|\mathcal{WVM}(G/K, N/K))| = |K \cap [NV^*G]||\mathcal{WVM}(G,N)|$, which completes the result.

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