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# The $\int\Gamma^{3\lambda I}$ Statistical Convergence of pre-Cauchy over the $p^-$ Metric Space Defined by Musielak Orlicz Function

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**Abstract**

In this paper we are concerned with $\int\Gamma^{3\lambda I}$ statistical convergence of pre-cauchy triple sequences. $\int\Gamma^{3\lambda I}$ statistical convergence implies $\int\Gamma^{3\lambda I}$ statistical pre-Cauchy condition and examine some properties of these concepts. We examine some properties of these concepts, if the triple entire sequence spaces is statistically convergent then statistically pre-Cauchy and also triple sequence of ideal ($I_3$)- is statistically pre-Cauchy.

**Keywords:** Analytic sequence; Double sequences; $\Gamma^3$ space; Musielak - Orlicz function; $p^-$ metric space; Ideal; Filter; $\int\Gamma^{3\lambda I}$ statistical convergence; $\int\Gamma^{3\lambda I}$ statistical pre-Cauchy

## Introduction

We introduce $\int\Gamma^{3\lambda I}$ sequence space and also discuss $\int\Gamma^{3\lambda I}$ is statistically convergent is pre-Cauchy and the ideal space is pre-Cauchy. Throughout $w$, $x$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write $w^3$ for the set of all complex triple sequences $(x_{mnk})$, where $m, n, k \in \mathbb{N}$ the set of positive integers. Then, $w^3$ is a linear space under the coordinate wise addition and scalar multiplication. We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series and interesting results are found in Apostol [1] and double sequence spaces is found in Hardy [2], Subramanian et al. [3], Deepmala et al. [4-7], Mishra et al. [8,9], Mishra and Mishra [10], Mishra [11] and many others. Later on investigated by some initial work on triple sequence spaces is found in sahiner et al. [6], Esi et al. [12-15], Savas et al. [16] , Subramanian et al. [17], Prakash et al. [18,19] and many others.

Let $(x_{mnk})$ be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence $(S_{mnk})$ is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} \, (m,n,k = 1,2,3,...) .$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if;

$$sup_{m,n,k} \left| x_{mnk} \right|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by $\Lambda^3$. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$\left| x_{mnk} \right|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty$$

The vector space of all triple entire sequences are usually denoted by $\Gamma^3$. The space $\Lambda^3$ and $\Gamma^3$ is a metric space with the metric

$$d(x,y) = sup_{m,n,k} \left\{ \left| x_{mnk} - y_{mnk} \right|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\}, \quad (1)$$

Forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in $\Gamma^3$. Let $\varphi$ = *{finite sequences}*.

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \Im_{ijq}$ for all $m, n, k \in \mathbb{N}$,

$$\Im_{ijq} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ & & & & \end{bmatrix}$$

with 1 in the $(i, j, q)^{th}$ position and zero otherwise.

A modulus function was introduced by Nakano [20]. We recall that a modulus $f$ is a function from $[0,\infty) \to [0,\infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$

(2) $f(x+y) \le f(x)+f(y,)$ for all $x \ge 0, y \ge 0,$

(3) $f$ is increasing,

(4) $f$ is continuous from the right at 0. Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from here that $f$ is continuous on $[0,\infty)$.

Let $M$ and $\phi$ are mutually complementary Orlicz functions. Then, we have:

(i) For all $u,y \ge 0$,

$uy \le M(u) + \phi(y)$, *(Young's inequality)* [3] (2)

(ii) For all $u \ge 0$, and $0 < \lambda < 1$,

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$$M(\lambda u) \le \lambda M(u) \quad (3)$$

Lindenstrauss and Tzafriri used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(|x_k|\right) < \infty, \quad for \quad some \quad \rho > 0 \right\},$$

The space $\ell_M$ with the norm

$$\|x\| = inf\left\{ \sum_{k=1}^{\infty} M\left(|x_k|\right) \le 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\ (1 \le p < \infty)$, the spaces $\ell_M$ coincide with the classical sequence space $\ell_p$.

A sequence $f = (f_{mnk})$ of Orlicz function is called a Musielak -Orlicz function . A sequence $g = (g_{mnk})$ defined by

$$g_{mnk}(v) = sup\ \{|v|u - (f_{mnk})(u) : u \ge 0\}, m,n,k = 1,2,\ldots..)$$

is called the complementary function of a Musielak-Orlicz function $f$. For a given Musielak Orlicz function $f$, the Musielak-Orlicz sequence space $t_f$.

$$t_f = \{x \in w^3 : M_f(|x_{mnk}|)^{1/m+n+k} \to 0 \text{ as } m,n,k \to \infty\},$$

where $M_f$ is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(|x_{mnk}|\right)^{1/m+n+k}, x = \left(x_{mnk}\right) \in t_f.$$

We consider $t_f$ equipped with the Luxemburg metric

$$d(x,y) = sup_{m,n,k}\left\{ inf\left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mn}\left( \frac{|x_{mnk}|^{1/m+n+k}}{mnk} \right) \right) \le 1 \right\}.$$

## Definition and Preliminaries

A sequence $x = (x_{mnk})$ is said to be triple analytic if $sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty$. The vector space of all triple analytic sequences is usually denoted by $\Lambda^3$. A sequence $x = (x_{mnk})$ is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$ as $m,n,k \to \infty$. The vector space of triple entire sequences is usually denoted by $\Gamma^3$.

Let $w^3$ denote the set of all complex double sequences $x = \left(x_{mnk}\right)_{m,n,k=1}^{\infty}$ and $M : [0,\infty) \to [0,\infty)$, be an Orlicz function. Given a triple sequence, $x \in w^3$. Define the sets:

$$\Gamma^3 = \left\{ x \in w^3 : \left( M\left( \frac{|\ _{mnk}|^{m+n+k}}{} \right) \right) \to 0 \quad as \quad m,n,k \to \infty \quad for \quad some \quad > 0 \right\}$$

and

$$\Lambda_M^3 = \left\{ x \in w^3 : sup_{m,n,k \ge 1}\left( M\left( \frac{|x_{mnk}|^{\frac{1}{m+n+k}}}{\rho} \right) \right) < \infty \quad for \quad some \quad \rho > 0 \right\}.$$

The space $\Gamma_M^3$ is a metric space with the metric

$$d(x,y) = inf\left\{ \rho > 0 : sup_{m,n,k \ge 1}\left( M\left( \frac{|x_{mnk} - y_{mnk}|}{\rho} \right) \right)^{\frac{1}{m+n+k}} \le 1 \right\}$$

Let $n \in \mathbb{N}$ and $X$ be a real vector space of dimension $w$, where $n \le m$. A real valued function

$d_p(x_1,\ldots,x_n) = \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ on $X$ satisfying the following four conditions:

(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = 0$ if and and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,

(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = |\alpha| \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$, $\alpha \in \mathbb{R}$

(iv) $d_p((x_1,y_1),(x_2,y_2)\ldots(x_n,y_n)) = (d_x(x_1,x_2,\ldots x_n)^p + d_y(y_1,y_2,\ldots y_n)^p)^{1/p}$ for $1 \le p < \infty$; (or)

(v) $d((x_1,y_1),(x_2,y_2)\ldots(x_n,y_n)) := sup\{d_x(x_1,x_2,\ldots x_n), d_y(y_1,y_2,\ldots y_n)\}$,

for $x_1,x_2,\ldots x_n \in X$, $y_1,y_2,\ldots y_n \in Y$ is called the $p$ product metric of the Cartesian product of $n$ metric spaces is the $p$ norm of the $n$ - vector of the norms of the $n$ subspaces.

A trivial example of $p$ product metric of $n$ metric space is the $p$ norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the $p$ norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}(x_{mn},0))|\right) = sup\begin{pmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{mn},0) \end{pmatrix}$$

Where $x_i = (x_{i1},\ldots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\ldots n$ $i = 1,2,\cdots n$.

If every Cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $p^-$ metric. Any complete $p^-$ metric space is said to be $p^-$ Banach metric space.

**Definition**

A) Let $X$ be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if

(1) $\rho(x) \ge 0$, for all $x \in X$;

(2) $\rho(-x) = \rho(x)$, for all $x \in X$;

(3) $\rho(x+y) \le \rho(x) + \rho(y)$, for all $x,y \in X$;

(4) If $(\sigma_{mn})$ is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m,n \to \infty$ and $(x_{mn})$ be a sequence of vectors with $\rho(x_{mn} - x) \to 0$ as $m,n \to \infty$, then $\rho(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m,n \to \infty$ [18,19].

A paranorm $w$ for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X,w)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm.

B) A family $I \subset 2^{Y \times YY}$ of subsets of a non empty set $Y$ is said to be an ideal in $Y$ if;

(1) $\varphi \in I$

(2) $A, B, C \in I$ simply $A \bigcup B \bigcup C \in I$

(3) $A, B \in I$, $C \subset A$ imply $C \in I$.

while an admissible ideal $I$ of $Y$ further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. A sequence $(x_{mnk})_{m,n,k \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ in $X$ is said to be $I$- convergent to $0 \in X$, if for each $\varepsilon > 0$ the set

$A(\varepsilon) = \left\{ m,n,k \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|(d_1(x_1,0),\ldots,d_n(x_n,0)) - 0\|_p \ge \varepsilon \right\}$ belongs to $I$.

C) A non-empty family of sets $F \subset 2^{X \times XX}$ is a filter on $X$ if and only if

(1) $\varphi \in F$

(2) for each $A, B, C \in F$, we have imply $A \bigcap B \bigcap C \in F$

(3) each $A, B \in F$ and each $B \subset C$, we have $C \in F$

D) An ideal $I$ is called non-trivial ideal if $I \ne \varphi$ and $X \notin I$. Clearly $I \subset 2^{X \times XX}$ is a non-trivial ideal if and only if $F = F(1) = \{X - A : A \in I\}$ is a filter on $X$.

E) A non-trivial ideal $I \subset 2^{\wedge\{X\times X\times X\}}$ is called (i) admissible if and only if $\{\{x\}:x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing $I$ as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}: A \text{ is a finite subset}\}$. Then $I_f$ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \delta(A) = 0\}$ where $\delta(A)$ denote the asymptotic density of the set $A$. Then $I_\delta$ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

F) A sequence space $E$ is said to be monotone if $E$ contains the canonical pre-images of all its step spaces.

**Remark**

Let $\mu = (\lambda_{rsu})$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_{111} = 1$ and $_{r+1,s+1,u+1} \leq \lambda_{rsu} + 1$ for all $r,s,u \in \mathbb{N}$.

The generalized de la Vallee-Poussin means is defined by

$$t_{rsu}(x) = \frac{1}{\alpha\beta\gamma}\sum_{(p,q,t)\in I_{rsu}} \left|x_{mnk} - x_{abc}\right|^{1/m+n+k}$$

Where $I_{rsu} = [r,s,u - \lambda_{rsu} + 1, rsu]$. A sequence $x = (x_{mnk})$ of complex numbers is said to be $(V_3,\lambda)$– summable to a number if $t_{rsu}(x) \to L$ as $r,s,u \to \infty$.

## Some New Integrated Statistical Convergence Sequence Spaces of pre-Cauchy

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_{mnk})$ be a sequence of positive real numbers for all $m,n,k \in \mathbb{N}$. $f = (f_{mnk})$ be a sequence of Musielak-Orlicz function, $(X,\|(d(x_1, 0), d(x_2, 0),\ldots d(x_{n-1}, 0))\|_p)$ be a $p$– metric space, and $(\alpha\beta\gamma)$ be a sequence of non-zero scalars and $\mu_{mnk}(X) = d(t_{rsu},0)$ be a sequence of pre-Cauchy, we define the following sequence spaces as follows:

**Definition**

Let $f$ is a Musielak Orlicz function and a triple sequence $(x_{mnk})_{m,n,k\in\mathbb{N}}$ is said to be $I$– statistically convergent if, for any ε > 0 and δ > 0,

$$\left[\Gamma^{3q}_{f\mu}, \left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right]^{I_3} = lim_{r,s,u\to\infty}$$
$$\left\{\left|\left\{(m,n,k)\in I_{rsu} : \left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq \varepsilon\right\}\right|\right\} \in I_3,$$

where

$$\mu_{mnk}(x) = d\left(t_{rsu}(x),0\right) = d\left(\frac{1}{\alpha\beta\gamma}\sum_{(p,q,t)\in I_{rsu}} \left|x_{mnk}\right|^{1/m+n+k},0\right).$$

## Main Results

**Theorem**

Let $f$ is a Musielak Orlicz function and if $\left[\Gamma^{3q}_{f\mu}, \left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right]$ statistically convergent then

$\left[\ {}_{f}\ , \left\|\left(d(x_1,0),d(x_2,0),\ \ ,d(x_{n\ 1},0)\right)\right\|\ \right]$ statistically pre-Cauchy.

Proof: For any ε > 0 and  > 0,

$$A = lim_{r,s,u\to\infty}\left\{\left|\left\{m,n,k\in I_{rsu} : \left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq \frac{\varepsilon}{2}\right\}\right| \geq \delta\right\} \in I_3.$$

Then

$$\left|\left\{\left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq \frac{\varepsilon}{2}\right\}\right| < \delta,$$

that is,

$$\left|\left\{\left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} < \frac{\varepsilon}{2}\right\}\right| > 1-\delta,$$

for all $(m,n,k) \in A^c$, where $c$ stands for the complement of the set $A$. Writing

$B = \left\{\left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} < \frac{\varepsilon}{2}\right\}$, we observe that $m,n,k,a,b,c \in B$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}$$
$$\leq \left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} +$$
$$\left[f_{mnk}\left(\left\|\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Therefore

$$B\times B\times B \subset \left\{\left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} < \varepsilon\right\}$$

which implies

$$\left[|B|\right]^3 \leq \left\{\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} < \varepsilon\right\}.$$

Hence

$$\left\{\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} < \varepsilon\right\} \geq \left[|B|\right]^3 > (1-\delta)^3,$$

that is

$$\left\{\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} < \varepsilon\right\} < 1-(1-\delta)^3$$

for all $(m,n,k) \in A^c$. Let $\delta_{111} > 0$ be given. Choosing $\delta > 0$ so that $1-(1-\delta)^3 < \delta_{111}$, we see that every $(m,n,k) \in A^c$

$$\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq \varepsilon\right| < \delta_{111}$$

and so

$$\left\{\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq \varepsilon\right| \geq \delta_{111}\right\} \subset A.$$

Since $A \in I_3$, we obtain

$$\left\{\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq \varepsilon\right| \geq \delta_{111}\right\} \subset I_3.$$

**Theorem**

Let $f$ is a Musielak Orlicz function and a triple sequence $x = (x_{mnk})$ is $I_3$ – statistically pre-cauchy if and only if

$$I_3 - lim_{r,s,u\to\infty}\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} = 0.$$

Proof: We assume that

$$I_3 - lim_{r,s,u\to\infty}\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}$$
$$\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} = 0.$$

Given ε > 0 and we have

$$lim_{r,s,u\to\infty}\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq$$
$$\varepsilon.\left(\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}} \geq \varepsilon\right|\right).$$

Therefore for any δ > 0,

$$\left\{\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\varepsilon\right|\geq\delta\right\}\subset$$

$$\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\right|\geq\varepsilon\delta.$$

Since,

$$I_3-lim_{r,s,u\to\infty}\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}$$

$$\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}=0.$$

Hence,

$$\left\{\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\varepsilon\right|\geq\delta\right\}\in I_3.$$

Hence $x$ is $I_3$ statistically pre-Cauchy.

Conversely assume that $x$ is $I_3$, where $I_3$ is triple sequence of ideal is statistically pre-cauchy and given ε > 0. Since $x$ is analytic there exists an integer $M$ such that $|x_{mnk}|^{1/m+n+k}\leq M$ for all m,n,k ∈ℕ,

$$\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\right|\leq$$

$$\frac{\varepsilon}{2}+2M\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\right|.$$

Since $x$ is $I_3$ − statistically pre-Cauchy, for δ > 0.

$$A=lim_{r,s,u\to\infty}\left\{\left|\left\{(m,n,k)\in I_{rsu}:\left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\frac{\varepsilon}{2}\right\}\right|\geq\delta\right\}\in I_3.$$

Then for $(m,n,k)\in A^c$

$$\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\frac{\varepsilon}{2}\right|<\delta \text{ and so}$$

$$\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\right|$$

$$\leq\frac{\varepsilon}{2}+2M\delta.$$

Let $\delta_1>0$ be given. Then choosing ε, δ > 0 so that $\frac{\varepsilon}{2}+2M\delta_1$ therefore every $(m,n,k)\in A^c$

$$\sum_{(m,n,k)\in I_{rsu}}\sum_{(a,b,c)\in I_{rsu}}$$

$$\left|\left[f_{mnk}\left(\left\|\mu_{mnk}(x)-\mu_{abc}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\right|<\delta_1,$$

that is

$$lim_{r,s,u\to\infty}$$

$$\left\{\left|\left\{(m,n,k)\in I_{rsu}:\left[f_{mnk}\left(\left\|\mu_{mnk}(x),\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\right]^{q_{mnk}}\geq\frac{\varepsilon}{2}\right\}\right|\geq\delta_1\right\}\subset A\in I_3.$$

This completes the proof.

## Conclusion

We examine some properties of these concepts, if the triple entire sequence spaces is statistically convergent then statistically pre-Cauchy and also triple sequence of ideal is statistically pre-Cauchy. In future developed by rough statistical convergence on triple sequence and also rough sets in statistical convergence of fractional order of triple sequence of Γ.

### Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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