Research Article | **Open Access**

Stochastic Heat and Advection Cauchy

Sohalya MA*, Yassena MT and Elbaza IM

Faculty of Science, Department of Mathematics, Mansoura University, Egypt

Abstract

In this paper, the solutions of Cauchy problems for the stochastic advection and stochastic diffusion equations are obtained using the finite difference method. In the case when the flow velocity is a function of stochastic flow velocity and also, the diffusion coefficient in the stochastic heat equation is a function of stochastic diffusion coefficient, the consistency and stability of the finite difference scheme we are used need to be performed under mean square calculus.

Introduction

Diffusion and advection problems involve randomness not only from the error measurements but may be also due to material defects or from impurities. The basic motivation of this paper focuses on this fundamental difficulty: the numerical consistency and stability of finite difference discretization of a Cauchy problems for stochastic advection and stochastic diffusion equations in the forms:

$$\begin{cases} \left[p(x,\beta)U_x \right]_x = U_t & x \in R, t \geq 0 \\ U(x,0) = U_0(x) & x \in R \end{cases} \quad (1)$$

$$\begin{cases} U_t + \left[p(x,\beta)U \right]_x = 0 & x \in R, t \geq 0 \\ U(x,0) = f(x) & x \in R \end{cases} \quad (2)$$

The independent variables are t (time) and x (space). The dependent variable is U which is a function of t and x. $p(x,\beta)$ is a discrete stochastic process (s.p.) defined on a common probability space (Ω, F, P) and depending of the stochastic variable β, $(U_0(x))$, $f(x)$ are an initial deterministic data function.

In order to study the consistency and stability for the problems eqns (1) and (2) with the according difference schemes, it is necessary to use mean square calculus. Many papers have been studied the analytical solutions for stochastic differential models [1-7]. Also, in recent years numerical methods have also been applied to Cauchy problems such as heat equation with stochastic variable input [8]. Our paper is organized as follows, Section 2, deals with preliminaries of some points used in our discussing. Section 3, deals with constructing the stochastic finite difference scheme of eqns (1) and (2). Section 4, deals with the study of stability and consistency of (SFDS) of eqns (1) and (2).

Preliminaries

Definition 1: A real random variable X on a probability space (Ω, F, P) and satisfying the property that $E\left[|X|^2 \right] < \infty$ is called 2nd order random variable ($2.r.v$) where, $E[\ ]$ denotes the expectation value operator [9].

Remark 1:

If $X \in L_2(\Omega)$, Then the L_2 norm is defined as

$$\|X\|_2 = \left[E\left[|X|^2 \right] \right]^{\frac{1}{2}}. \ [9]$$

The approximations of the solution s.p. to eqns (1) and (2), [8], will

be constructed in the sense of fixed station for the time, we will work in the following Banach space $\ell 2(\Omega)$, $\|\cdot\|_{RV}$ defined by

$$\ell_2(\Omega) = \{v = (\cdots, v_{-1}, v_0, v_1, \cdots) : \| v \|_{RV} < +\infty\}$$

$$\| v \|_{RV} = \left[E\left[\left(\sup_k | v_k | \right)^2 \right] \right]^{\frac{1}{2}} \quad (3)$$

Analysis of Stochastic Finite Difference Scheme (SFDS)

The SFDS for stochastic cauchy diffusion eqn (1)

The discretization of the problem eqn (1) will lead to stochastic difference scheme where at each node the unknown represents a r.v. Let us subdivide the domain $(-\infty, \infty) \times [0, \infty]$ and define the grid cells for the space to be $\Delta x = (x_k - x_{k-1})$ for $k \geq 1$ and also, define the time steps to be $t = (t_n - t_{n-1})$ for $n \geq 1$. Consider $U_k^n = U(k\Delta x, n\Delta t)$ approximates the exact solution for the problem eqn (1) as, $U(x,t)$ at the point $(k\Delta x, n\Delta t)$. By replacing the derivatives in eqn (1) with the difference formulas, we can have the difference scheme as follows:

Using the first-order forward difference formula for approximating U_t

$$U_t(x,t) \simeq \frac{U(x,t+\Delta t) - U(x,t)}{\Delta t}$$

Then,

$$U_t(k\Delta x, n\Delta t) \simeq \frac{U_k^{n+1} - U_k^n}{\Delta t} \quad (4)$$

Using the first-order backward difference formula for approximating $p'(x,\beta)$, U_x and the second-order centred difference formula for approximating U_{xx}

$$\left[p(x,\beta)U_x \right]_x \simeq p(k)\left[\frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{(\Delta x)^2} \right] + \left[\frac{U_k^n - U_{k-1}^n}{\Delta x} \right]\left[\frac{p(k) - p(k-1)}{\Delta x} \right]$$

$$\simeq \frac{p(k)U_{k+1}^n - p(k)U_k^n - p(k-1)U_k^n + p(k-1)U_{k-1}^n}{(\Delta x)^2} \quad (5)$$

Substituting eqns (4) and (5) into eqn (1), we can motivate the scheme

$$\frac{p(k)U_{k+1}^n - p(k)U_k^n - p(k-1)U_k^n + p(k-1)U_{k-1}^n}{(\Delta x)^2} = \frac{U_k^{n+1} - U_k^n}{\Delta t}$$

Then,

$$
\begin{aligned}
U_k^{n+1} &= U_k^n + \frac{\Delta t}{(\Delta x)^2}\left[p(k)U_{k+1}^n - p(k)U_k^n - p(k-1)U_k^n + p(k-1)U_{k-1}^n \right] \\
&= [1 - rp(k) - rp(k-1)]U_k^n + rp(k)U_{k+1}^n + rp(k-1)U_{k-1}^n, \quad r = \frac{\Delta t}{(\Delta x)^2}
\end{aligned}
\tag{6}
$$

The SFDS for stochastic cauchy advection eqn (2)

Now, we will construct the stochastic finite difference scheme by the same approach we use in the diffusion equation. Using eqn (4) for approximating U_t and first-order backward difference formula for approximating $p'(x,ß)$ and U_x.

$$
\begin{aligned}
\left[p(x,\beta)U \right]_x &\simeq p(k)\left[\frac{U_k^n - U_{k-1}^n}{\Delta x} \right] + U_k^n\left[\frac{p(k) - p(k-1)}{\Delta x} \right] \\
&\simeq \frac{2p(k)U_k^n - p(k)U_{k-1}^n - p(k-1)U_k^n}{\Delta x}
\end{aligned}
\tag{7}
$$

Substituting eqns (4) and (7) into eqn (2), we get

$$\frac{U_k^{n+1} - U_k^n}{\Delta t} + \frac{2p(k)U_k^n - p(k)U_{k-1}^n - p(k-1)U_k^n}{\Delta x} = 0$$

Then,

$$U_k^{n+1} = [1 - 2rp(k) + rp(k-1)]U_k^n + rp(k)U_{k-1}^n, \quad r = \frac{\Delta t}{\Delta x} \tag{8}$$

Mean Square Consistency and Stability of SFDS

Consistency and stability of SFDS eqn (6)

The consistency of numerical scheme is determined by studying the truncation error of the scheme. A finite difference scheme is consistent with the stochastic partial differential equation if the difference between the finite difference equation and the stochastic partial differential equation vanishes as the size of the grid spacing goes to zero independently. If this condition is met, then the numerical scheme is said to be consistent. Consistency of finite difference schemes is determined by using the technique of modified stochastic partial differential equation (MSPDE). This MSPDE is determined by expressing each term in the finite difference equation in a Taylor series at some base point. If the grid points approaches zero and the MSPDE changes back to original SPDE, then the scheme is consistent.

Definition 2: The stochastic finite difference scheme (SFDS)

$$U^{n+1} = QU^n + (\Delta t)G^n$$

is said to be mean square $\|\cdot\|_{RV}$ –consistent with the stochastic partial differential equation (RPDE) $Lu=F$, if the solution stochastic process (s.p.) of (RPDE), U, satisfies

$$U^{n+1} = QU^n + (\Delta t)G^n + (\Delta t)\tau^n \tag{9}$$

and $\| \tau^n \|_{RV} \to 0$ as $\Delta t, \Delta x \to 0$

Theorem 1: Let us consider the stochastic cauchy problem (1), and assume that

(1) $p(x)$ and $p'(x)$ are functions of the 2-r.v. ß such that $0 < m < p(x,\omega) < M$; m,M are positive constants.

(2) $U_{xxx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t) \in [x_k-$

$\delta, x_k+\delta] \times [t_n-\delta, t_n+\delta]$. Then, the SFDS (6) is mean square $\|\cdot\|_{RV}$ –consistent.

Proof: Assuming that for each $\omega \in \Omega$, the solution $U(x,t)(\omega)$ of (1) admits taylor expansion of the form

$$U_{k+1}^n(\omega) = U_k^n(\omega) + \Delta x\left(U_x\right)_k^n(\omega) + \frac{(\Delta x)^2}{2}\left(U_{xx}\right)_k^n(\omega) + \mathcal{O}\left((\Delta x)^3\right) \tag{10}$$

$$U_{k-1}^n(\omega) = U_k^n(\omega) - \Delta x\left(U_x\right)_k^n(\omega) + \frac{(\Delta x)^2}{2}\left(U_{xx}\right)_k^n(\omega) - \mathcal{O}\left((\Delta x)^3\right) \tag{11}$$

$$U_k^{n+1}(\omega) = U_k^n(\omega) + \Delta t\left(U_t\right)_k^n(\omega)\mathcal{O}\left((\Delta t)^2\right) \tag{12}$$

According to expression (9) with $G=0$, and using the expansions (10)-(12), we have

$$
\begin{aligned}
\left(U^{n+1} - QU^n\right)_k &= U_k^{n+1} - [1 - rp(k) - rp(k-1)]U_k^n - rp(k)U_{k+1}^n - rp(k-1)U_{k-1}^n \\
&= U_k^n(\omega) + \Delta t\left(U_t\right)_k^n(\omega) + \mathcal{O}\left((\Delta t)^2\right) - [1 - rp(k) - rp(k-1)]U_k^n(\omega) \\
&\quad - rp(k)\left[U_k^n(\omega) + \Delta x\left(U_x\right)_k^n(\omega) + \frac{(\Delta x)^2}{2}(U_{xx})_k^n(\omega) + \mathcal{O}\left((\Delta x)^3\right) \right] \\
&\quad - rp(k-1)\left[U_k^n(\omega) - \Delta x\left(U_x\right)_k^n(\omega) + \frac{(\Delta x)^2}{2}(U_{xx})_k^n(\omega) - \mathcal{O}\left((\Delta x)^3\right) \right] \\
&= \Delta t(U_t)_k^n(\omega) + \mathcal{O}\left((\Delta t)^2\right) - \frac{\Delta t}{\Delta x}p(k)(U_x)_k^n(\omega) - \frac{\Delta t}{2}p(k)(U_{xx})_k^n(\omega) + \mathcal{O}(\Delta t(\Delta x)) \\
&\quad + \frac{\Delta t}{\Delta x}p(k-1)(U_x)_k^n(\omega) - \frac{\Delta t}{2}p(k-1)(U_{xx})_k^n(\omega) + \mathcal{O}(\Delta t(\Delta x)) \\
&= \Delta t\left\{ (U_t)_k^n(\omega) - \left[\frac{p(k) - p(k-1)}{\Delta x}(U_x)_k^n(\omega) + \frac{p(k) + p(k-1)}{2}(U_{xx})_k^n(\omega) \right] \right\} \\
&\quad + \mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}(\Delta t(\Delta x))
\end{aligned}
$$

Since $p(x,\omega)$ is mean square differentiable, then we have

$$
\begin{aligned}
\frac{p(k)(\omega) - p(k-1)(\omega)}{\Delta x} &= p'(x_k)(\omega) + \mathcal{O}(\Delta x) \\
\frac{p(k)(\omega) + p(k-1)(\omega)}{2} &= p(x_k)(\omega) + \mathcal{O}(\Delta x)
\end{aligned}
\tag{13}
$$

Assuming, $U_{xxx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t) \in [x_k-\delta, x_k+\delta] \times [t_n-\delta, t_n+\delta]$ Then, from eqn (13) we have

$$
\begin{aligned}
\left(U^{n+1} - QU^n\right)_k &= \Delta t\left\{ (U_t)_k^n(\omega) - \left[p'(x_k)(U_x)_k^n(\omega) + p(x_k)(U_{xx})_k^n(\omega) \right] \right\} \\
&\quad + \mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}(\Delta t(\Delta x))
\end{aligned}
$$

Then, from eqn (9) we get

$$\tau_k^n = (U_t)_k^n(\omega) - \left[p'(x_k)(U_x)_k^n(\omega) + p(x_k)(U_{xx})_k^n(\omega) \right] + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)$$

Then,

$$
\begin{aligned}
\| \tau_k^n \|^2 &= \int_R \left[(U_t)_k^n(\omega) - \left[p'(x_k)(U_x)_k^n(\omega) + p(x_k)(U_{xx})_k^n(\omega) \right] \right]^2 f_\beta(\beta)d\beta \\
&\quad + \int_R |\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)|^2 f_\beta(\beta)d\beta \to 0, \quad as \quad \Delta t, \Delta x \to 0
\end{aligned}
$$

$f_ß(ß)$ is the probability density function of the stochastic variable. Hence, the SFDS (6) is mean square consistent.

A finite difference scheme is said to be stable if it produces a bounded solution for a stable stochastic partial differential equation and it is unstable if it produces an unbounded solution of the stochastic partial differential equations. There are several methods used to analyse stability of numerical schemes, for example the discrete perturbation method, the Von-Neumann method and the matrix method. In this work, we use the mean square calculus for stability analysis of numerical scheme.

Definition 3: A stochastic difference scheme $L_k^n u_k^n = G_k^n$ is mean square stable, if there exist some positive constants ε, δ, non-negative constants l, m and u^0 is an initial data such that:

$$E\left[\sup_{k}\left|u_{k}^{n+1}\right|^{2}\right]\leq le^{mt}E\left[\sup_{k}\left|u_{k}^{0}\right|^{2}\right] \tag{14}$$

for all, $0\leq t=(n+1)\Delta t,\ 0<\Delta x\leq\varepsilon$ and $0<\Delta t\leq\delta$

Theorem 2: The SFDS (6) that according to the problem (1) is mean square stable under the condition: $0<\Delta t\leqslant\dfrac{(\Delta x)^{2}}{2M}$, $0<m<p(x,\omega)<M$ where m,M are positive constants.

Proof: Since, our scheme is

$$U_{k}^{n+1}=\left[1-rp(k)-rp(k-1)\right]U_{k}^{n}+rp(k)U_{k+1}^{n}+rp(k-1)U_{k-1}^{n},\quad r=\frac{\Delta t}{(\Delta x)^{2}}$$

$$\leqslant E\left[\sup_{k}\left[\left|(1-rp(k)-rp(k-1))U_{k}^{n}\right|+\left|rp(k)U_{k+1}^{n}\right|+\left|rp(k-1)U_{k-1}^{n}\right|\right]^{2}\right]$$

$$\leqslant E\left[\sup_{k}\left[\left|(1-rp(k)-rp(k-1))U_{k}^{n}\right|+\left|rp(k)U_{k+1}^{n}\right|+\left|rp(k-1)U_{k-1}^{n}\right|\right]^{2}\right]$$

$$=E\left[\sup_{k}\left[(1-rp(k)-rp(k-1))^{2}\left|U_{k}^{n}\right|^{2}+(rp(k))^{2}\left|U_{k+1}^{n}\right|^{2}+(rp(k-1))^{2}\left|u_{k-1}^{n}\right|^{2}\right.\right.$$

$$+2r\left|p(k-1)(1-rp(k)-rp(k-1))\right|\left\|U_{k-1}^{n}U_{k}^{n}\right\|+2r^{2}\left|p(k)p(k-1)\right|\left\|U_{k+1}^{n}U_{k-1}^{n}\right\|$$

$$\left.\left.+2r\left|p(k)(1-rp(k)-rp(k-1))\right|\left\|U_{k}^{n}U_{k+1}^{n}\right\|\right]\right]$$

$$\leq E\left[\sup_{k}\left[(1-2rM)^{2}\left|U_{k}^{n}\right|^{2}+(rM)^{2}\left|U_{k+1}^{n}\right|^{2}+(rM)^{2}\left|u_{k-1}^{n}\right|^{2}+2r\left|M(1-2rM)\right|\left\|U_{k-1}^{n}U_{k}^{n}\right\|\right.\right.$$

$$\left.\left.+2r^{2}M^{2}\left|U_{k+1}^{n}U_{k-1}^{n}\right|+2r\left|M(1-2rM)\right|\left\|U_{k}^{n}U_{k+1}^{n}\right\|\right]\right]$$

$$=E\left[(1-2rM)^{2}\sup_{k}\left|U_{k}^{n}\right|^{2}+r^{2}M^{2}\sup_{k}\left|U_{k}^{n}\right|^{2}+r^{2}M^{2}\sup_{k}\left|U_{k}^{n}\right|^{2}+2r\left|M(1-2rM)\right|\sup_{k}\left|U_{k}^{n}\right|^{2}\right.$$

$$\left.+2r^{2}M^{2}\sup_{k}\left|U_{k}^{n}\right|^{2}\right]+2r\left|M(1-2rM)\right|\sup_{k}\left|U_{k}^{n}\right|^{2}\right]$$

$$=E\left[\left[(1-2rM)^{2}+4r^{2}M^{2}+4r\left|M(1-2rM)\right|\right]\sup_{k}\left|u_{k}^{n}\right|^{2}\right]$$

Under the hypotheses $r=\dfrac{\Delta t}{(\Delta x)^{2}}$, $0<\Delta t\leq\delta$ and $0<m<p(x,\omega)<M$, we have $0<rM\leq\dfrac{1}{2}$ Then,

$$0<\Delta t\leq\frac{(\Delta x)^{2}}{2M}, \tag{15}$$
$$0<p(x,\beta)\leq M$$

Hence,

$$\left|M(1-2rM)\right|=M(1-2rM)$$

Then,

$$E\left[\sup_{k}\left|U_{k}^{n+1}\right|^{2}\right]\leq E\left[\sup_{k}\left|U_{k}^{n}\right|^{2}\right]$$

Finally, we have:

$$E\left[\sup_{k}\left|U_{k}^{n+1}\right|^{2}\right]\leq E\left[\sup_{k}\left|U_{k}^{n}\right|^{2}\right]\leq E\left[\sup_{k}\left|U_{k}^{n-1}\right|^{2}\right]\leq\cdots\leq E\left[\sup_{k}\left|U_{k}^{0}\right|^{2}\right]$$

Therefore,

$$E\left[\sup_{k}\left|U_{k}^{n+1}\right|^{2}\right]\leq E\left[\sup_{k}\left|U_{k}^{0}\right|^{2}\right]$$

Hence, the SFDS (6) is mean square stable with $l=1$, $m=0$ and under the conditions eqn (15).

Consistency and stability of SFDS (8)

Taking into account definition eq (2), we can state that theorem:

Theorem 3: The SFDS (8) is mean square $\|\cdot\|_{RV}$-consistent Let us consider the random cauchy problem (2), and assume that

(1) $p(x)$ and $p'(x)$ are functions of the 2-r.v. ß such that $0<m<p(x,\omega)<M$; m,M are positive constants.

(2) $U_{xxx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t)\in[x_{k}-\delta,x_{k}+\delta]\times[t_{n}-\delta,t_{n}+\delta]$

Proof: Assuming that for each $\omega\in\Omega$, the solution $U(x,t)(\omega)$ of (2) admits taylor expansion of the form

$$U_{k}^{n+1}(\omega)=U_{k}^{n}(\omega)+\Delta t\left(U_{t}\right)_{k}^{n}(\omega)+\mathcal{O}\left((\Delta t)^{2}\right) \tag{16}$$

$$U_{k-1}^{n}(\omega)=U_{k}^{n}(\omega)-\Delta x\left(U_{x}\right)_{k}^{n}(\omega)+\mathcal{O}\left((\Delta x)^{2}\right) \tag{17}$$

According to expression (10) with $G=0$, and using the expansions (16) and (17), we have

$$\begin{aligned}\left(U^{n+1}-QU^{n}\right)_{k}&=U_{k}^{n+1}-\left[1-2rp(k)+rp(k-1)\right]U_{k}^{n}-rp(k)U_{k-1}^{n}\\&=U_{k}^{n}(\omega)+\Delta t(U_{t})_{k}^{n}(\omega)+\mathcal{O}\left((\Delta t)^{2}\right)-\left[1-2rp(k)+rp(k-1)\right]U_{k}^{n}(\omega)\\&\quad-rp(k)\left[U_{k}^{n}(\omega)-\Delta x(U_{x})_{k}^{n}(\omega)+\mathcal{O}\left((\Delta x)^{2}\right)\right]\\&=\Delta t(U_{t})_{k}^{n}(\omega)+\mathcal{O}\left((\Delta t)^{2}\right)+rp(k)U_{k}^{n}(\omega)-rp(k-1)U_{k}^{n}(\omega)+\Delta tp(k)(U_{x})_{k}^{n}(\omega)+\mathcal{O}(\Delta t(\Delta x))\\&=\Delta t\left\{(U_{t})_{k}^{n}(\omega)+\left[\frac{p(k)-p(k-1)}{\Delta x}U_{k}^{n}(\omega)+p(k)(U_{x})_{k}^{n}(\omega)\right]\right\}+\mathcal{O}\left((\Delta t)^{2}\right)+\mathcal{O}(\Delta t(\Delta x))\end{aligned}$$

Since $p(x,\omega)$ is mean square differentiable, then we have

$$\frac{p(k)(\omega)-p(k-1)(\omega)}{\Delta x}=p'(x_{k})(\omega)+\mathcal{O}(\Delta x) \tag{18}$$

Assuming, $U_{xxx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t)\in[x_{k}-\delta,x_{k}+\delta]\times[t_{n}-\delta,t_{n}+\delta]$ Then, from eqn (18) we have

$$\left(U^{n+1}-QU^{n}\right)_{k}=\Delta t\left\{(U_{t})_{k}^{n}(\omega)+\left[p'(x_{k})(\omega)+p(k)(U_{x})_{k}^{n}(\omega)\right]\right\}+\mathcal{O}\left((\Delta t)^{2}\right)+\mathcal{O}(\Delta t(\Delta x))$$

Then, from eqn (9) we get

$$\tau_{k}^{n}=\left(U_{t}\right)_{k}^{n}(\omega)+\left[p'(x_{k})(\omega)+p(k)(U_{x})_{k}^{n}(\omega)\right]+\mathcal{O}(\Delta t)+\mathcal{O}(\Delta x)$$

Then,

$$\|\tau_{k}^{n}\|^{2}=\int_{R}\left[\left(U_{t}\right)_{k}^{n}(\omega)+\left[p'(x_{k})(\omega)+p(k)(U_{x})_{k}^{n}(\omega)\right]\right]^{2}f_{\beta}(\beta)d\beta$$

$$+\int_{R}\left|\mathcal{O}(\Delta t)+\mathcal{O}(\Delta x)\right|^{2}f_{\beta}(\beta)d\beta\to0,\quad as\quad \Delta t,\Delta x\to0$$

$f_{\beta}(ß)$ is the probability density function of the stochastic variable. Hence, the SFDS (8) is mean square consistent.

Now, taking into account definition (3), we can state that theorem

Theorem 4: The SFDS (8) that according to the problem (2) is mean square stable under the condition: $0<\Delta t\leqslant\dfrac{(\Delta x)}{M}$, $0<m<p(x,\omega)<M$ where m,M are positive constants.

Proof: Since, our scheme is

$$U_{k}^{n+1}=\left[1-2rp(k)+rp(k-1)\right]U_{k}^{n}+rp(k)U_{k-1}^{n},\quad r=\frac{\Delta t}{\Delta x}$$

$$E\left[\sup_{k}\left|U_{k}^{n+1}\right|^{2}\right]=E\left[\sup_{k}\left[\left|(1-2rp(k)+rp(k-1))U_{k}^{n}+rp(k)U_{k-1}^{n}\right|\right]^{2}\right]$$

$$\leqslant E\left[\sup_{k}\left[\left|(1-2rp(k)+rp(k-1))U_{k}^{n}\right|+\left|rp(k)U_{k-1}^{n}\right|\right]^{2}\right]$$

$$=E\left[\sup_{k}\left[(1-2rp(k)+rp(k-1))^{2}\left|U_{k}^{n}\right|^{2}+(rp(k))^{2}\left|U_{k-1}^{n}\right|^{2}\right.\right.$$

$$\left.\left.+2r\left|p(k)(1-2rp(k)+rp(k-1))\right|\left\|U_{k}^{n}U_{k-1}^{n}\right\|\right]\right]$$

$$\leq E\left[\sup_{k}\left[(1-rM)^{2}\left|U_{k}^{n}\right|^{2}+r^{2}M^{2}\left|U_{k-1}^{n}\right|^{2}+2r\left|M(1-rM)\right|\left\|U_{k}^{n}U_{k-1}^{n}\right\|\right]\right]$$

$$= E\left[(1-rM)^2 \sup_k \left|U_k^n\right|^2 + r^2 M^2 \sup_k \left|U_k^n\right|^2 + 2r\left|M(1-rM)\right| \sup_k \left|U_k^n\right|^2 \right]$$

$$= E\left[\left[(1-rM)^2 + r^2 M^2 + 2r\left|M(1-2rM)\right|\right] \sup_k \left|u_k^n\right|^2 \right]$$

Under the hypotheses $r = \dfrac{\Delta t}{\Delta x}$, $0<\Delta t \leq \delta$ and $0<m<p(x,\omega)<M$, we have for $0<rM\leq 1$ Then,

$$0 < \Delta t \leq \frac{\Delta x}{M},$$
$$0 < p(x,\beta) \leq M \tag{19}$$

Hence,

$$\left|M(1-rM)\right| = M(1-rM)$$

Then,

$$E\left[\sup_k \left|U_k^{n+1}\right|^2 \right] \leq E\left[\sup_k \left|U_k^n\right|^2 \right]$$

Finally, we have:

$$E\left[\sup_k \left|U_k^{n+1}\right|^2 \right] \leq E\left[\sup_k \left|U_k^n\right|^2 \right] \leq E\left[\sup_k \left|U_k^{n-1}\right|^2 \right] \leq \cdots \leq E\left[\sup_k \left|U_k^0\right|^2 \right]$$

Therefore,

$$E\left[\sup_k \left|U_k^{n+1}\right|^2 \right] \leq E\left[\sup_k \left|U_k^0\right|^2 \right]$$

Hence, the SFDS (8) is mean square stable with $l=1$, $m=0$ and under the conditions eqn (19).

References

1. Kloeden PE, Platen E (1995) Higher-order implicit strong numerical schemes for stochastic differential equations. Journal of Statistical Physics 66: 283-314.

2. Yassen MT, Sohaly MA, Elbaz I (2016) Random Crank-Nicolson Scheme for Random Heat Equation in Mean Square Sense. American Journal of Computational Mathematics 6: 66.

3. Cortés JC, Sevilla-Peris P, Jódar L (2005) Analytic-numerical approximating processes of diffusion equation with data uncertainty. Computers & Mathematics with Applications 49: 1255-1266.

4. El-Tawil MA, Sohaly MA (2011) Mean square numericabl methods for initial value random differential equations. Open Journal of Discrete Mathematics 1: 66.

5. Bishehniasar M, Soheili AR (2013) Approximation of stochastic advection-diffusion equation using compact finite difference technique. Iranian Journal of Science and Technology 37: 327-333.

6. Wan X, Xiu D, Karniadakis GE (2004) Stochastic solutions for the two-dimensional advection-diffusion equation. SIAM Journal on Scientific Computing 26: 578-590.

7. Villafuerte L, Cortés JC (2013) Solving Random Differential Equations by Means of Differential Transform Methods. In Advances in Dynamical Systems and Applications. Research India Publications 8: 413-425.

8. Navarro-Quiles A, Romero JV, Roselló MD, Sohaly MA (2016) Approximating the solution stochastic process of the random Cauchy one-dimensional heat model. Abstract and Applied Analysis 2016: 5391368.

9. Soong TT, Bogdanoff JL (1974) Random Differential Equations in Science and Engineering. Journal of Applied Mechanics 41: 1148.