Stability of Convergence Theorems of the Noor Iteration Method for an Enumerable Class of Continuous Hemi Contractive Mapping in Banach Spaces

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Abstract

The purpose of this to study the Noor iteration process for the sequence \( \{x_n\} \) converges to a common fix point for enumerable class of continuous hemi contractive mapping in Banach Spaces.

Keywords: Stability; Noor iterations; Hemicontractive mapping; Convergence theorem; Continuous pseudocontractive mapping

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Introduction

Let \( E \) be a real Banach space and let \( J \) denote the normalized duality mapping from \( E \) to \( E^* \) defined by

\[
J(x) = \{ f \in E^* : \langle f, x \rangle = \|x\| \|f\| \text{ for all } x \in E \}
\]

Where \( E^* \) denotes the dual space of \( E \) and \( \langle \cdot, \cdot \rangle \) denotes the generalization duality pairing.

It is well known that if \( E^* \) is strictly convex then \( J \) is single-valued. In the sequel, we shall denote the single-valued duality mapping by \( j \). Let \( K \) be a nonempty closed convex subset of Banach space \( E \). Let \( T: K \rightarrow K \) be a self-mapping of \( K \).

Definition 3.1: (i) A mapping \( T \) with domain \( D(T) \) and range \( R(T) \) in a Banach space is called pseudocontractive mapping, if for all \( x, y \in D(T) \), there exists \( j(x-y) \in J(x-y) \) such that

\[
\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2
\]

(ii) A mapping \( T \) with domain \( D(T) \) and range \( R(T) \) in \( E \) is called a hemi-contraction if \( F(T) \neq \emptyset \) and for all \( x \in D(T), x' \in F(T) \), such that,

\[
\langle Tx - T x', j(x - x') \rangle \leq \|x - x'\|^2
\]

(iii) A mapping \( T: K \rightarrow K \) is called L-Lipschitzian there exists \( L > 0 \) such that

\[
\|Tx - Ty\| \leq L \|x - y\| \quad \text{for all } x, y \in K
\]

Definition 3.2: If \( \{x_n\}_{n=0}^\infty \) and \( \{\beta_n\}_{n=0}^\infty \) are sequences of real numbers in \( [0, 1] \) [2]. For arbitrary \( x \in E \), Let \( \{x_n\}_{n=0}^\infty \) be a Noor iteration defined by,

\[
x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n
\]

\[
g_n = (1 - \beta_n)x_n + \beta_n T r_n
\]

\[
r_n = (1 - \beta_n)x_n + \beta_n T r_n
\]

Lemma 3.4: Let \( \delta \) be a number satisfying \( 0 \leq \delta < 1 \) and \( \{\epsilon_n\} \) a positive sequence satisfying \( \lim_{n \to \infty} \epsilon_n = 0 \) [4,5]. Then, for any positive sequence \( \{u_n\} \) satisfying \( u_{n+1} \leq \delta u_n + \epsilon_n \). It follows that \( \lim_{n \to \infty} u_n = 0 \).

Results

Theorem 4.1: Let \( \{T_n\}_{n=0}^\infty \) be defined as above and \( E \) be a Banach space, \( T: E \rightarrow E \) a self-map of \( E \) with a fixed point \( p \), satisfying the contractive condition

\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2
\]

For \( x \in E \).

Then, for any \( T \) be defined as above and let \( F(T) \neq \emptyset \) and \( \{\epsilon_n\}_{n=0}^\infty \) be a real uniformly convex Banach space.

\[
\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2
\]

For \( x \in E \).

Then, for any \( \{T_n\}_{n=0}^\infty \) converges strongly to a common fixed point of \( \{T_n\}_{n=0}^\infty \) if and only if \( \lim_{n \to \infty} (T_n, F(p)) = 0 \).

Proof: Let \( p \in F \) and \( n \geq 1 \) be a fixed point of \( T \) such that

\[
\|x_{n+1} - p\|^2 = \langle x_{n+1} - p, j(x_{n+1} - p) \rangle
\]

\[
\|x_n - p\|^2 \leq \|x_{n+1} - (1 - \alpha_n)x_n - \alpha_n T q_n + ((1 - \alpha_n)x_n + \alpha_n T q_n - p)\| + \|x_{n+1} - (1 - \alpha_n)x_n - \alpha_n T q_n + ((1 - \alpha_n)x_n + \alpha_n T q_n - p)\|
\]

\[
= \epsilon_n + \|x_{n+1} - p\|^2 + \|x_n - p\| + \|x_n + \alpha_n T q_n - p\|
\]

\[
= \epsilon_n + \|x_{n+1} - p\|^2 + \|x_n - p\| + \|x_n + \alpha_n T q_n - p\|
\]

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For the estimate of in (1) we get
\[
\beta_n\leq\beta_n - \beta_n + \beta_n - \beta_n - \beta_n
\]
Substituting (2) into (1) gives
\[
\|x_{n+1} - p\| \leq \epsilon + \alpha(1 - \epsilon) \beta_n \|x_n - p\| + \alpha \beta_n \|a\| \|T_n - p\|
\]
For \(\beta_n - \beta_n\) in (3) we have,
\[
\|\beta_n - \beta_n\| = \|1 - \gamma_n\| \|x_n + \gamma_n T_n - p\|
\]
Substituting (4) into (3) and using lemma 3.3
\[
\|x_{n+1} - p\| = \epsilon_n + (1 - \alpha) \|x_n - p\| + \alpha \beta_n \|a\| \|T_n - p\|
\]
Taking infimum over all \(p \in F\), we have,
\[
d(x_n, F) = \frac{\alpha_m}{p \in F} \|x_n - p\| \leq \frac{\alpha_m}{p \in F} \|x_{n+1} - p\| = d(x_{n+1}, F).
\]
Thus \(\lim_{n \to \infty} d(x_n, F)\) exist we finally prove (iii) suppose that \(x_n \to p \in F\) from (ii) and
\[
d(x_n, F) \leq \|x_n - p\| \to 0, \quad \text{We have} \quad \lim_{n \to \infty} d(x_n, F) = 0 \quad \text{for} \quad n, m \in \mathbb{N}
\]
and \(p \in F\); it follows
\[
(1.3) \text{From (1.3) that}
\]
\[
\|x_{n+1} - x_n\| \leq \|x_n - p\| + \|x_n - p\| \leq \|x_n - p\|
\]
Consequently,
\[
\|x_{n+1} - x_n\| \leq 2 \|x_n - F\| \to 0
\]
Therefore \(\{x_n\}\) is a Cauchy sequence. Suppose \(\lim_{n \to \infty} x_n = u\) for some \(u \in E\). Then
\[
d(u, F) = \lim_{n \to \infty} d(x_n, F) = 0
\]
Since \(F\) is closed set, \(u \in F\)

\[\text{So, Noor iteration process is } T\text{-stable.}\]

\[\text{Conclusion}\]

Thus, the stability of Noor iteration considerable for finding fixed point for enumerable class of continuous hemi contractive mapping in Banach spaces.

References