

Stability and Practically Stability of Impulsive Integro-Differential Systems by Cone-Valued Lyapunov Functions

Zhuoying Lv*, Guocheng Gao and Lailiang Zhang

Public Course Department, Shandong University of Science and Technology, Jinan, Shandong, 250031, P.R. China

Abstract

Stability and practically stability comparison criteria of impulsive integro-differential systems with fixed moments of impulse effects are established by cone-Lyapunov functions through comparing with impulsive ordinary differential equations.

Keywords: Impulsive Integro-Differential System; Cone-Lyapunov Function; Stability

Introduction

Impulsive integro-differential systems which are an important embranchment of nonlinear impulsive differential systems [1], arise from extensive applications in nature-science such as mathematic models of circuit simulation in physics and neuronal networks in biology. Consequently there are some results about stability of such systems by vector Lyapunov functions coupled with Razumikhin techniques [2,3]. However, it's difficult to choose a right vector Lyapunov function because of the restrict conditions. At the same time, the method of cone-valued Lyapunov functions is well known to be advantageous in applications [4]. Hence, the stability results for impulsive integro-differential systems could be improved via the method of cone-valued Lyapunov functions.

Adeyeye [5] considered the comparison principle by cone-valued Lyapunov functions for a class of integro-differential system without impulses. But it was not proved and also cannot be applied to impulsive integrodifferential systems. In this paper we shall firstly prove the comparison principle. Then by employing cone-valued Lyapunov functions a new comparison principle for impulsive integro-differential systems with fixed moments of impulse effects is established, which is compared with impulsive differential systems whose stability is relatively easy to solve. Finally the relevant new comparison criteria of stability and practically stability [6] of impulsive integro-differential systems are obtained too.

The remainder of this paper is organized as follows. In section 2, we describe impulsive integro-differential systems and introduce some notions and concepts. In section 3, we get some comparison results of stability and practically stability of the impulsive integro-differential systems with fixed moments of impulse effects by using the method of cone-Lyapunov functions.

Preliminaries

Consider the following impulsive integro-differential systems of the form

$$\begin{cases} x' = f(t, x, Tx), \\ x(t_k) = J_k(x(t_k^-)), \\ x(t_0^+) = x_0, \end{cases} \quad (1)$$

where N is the set of all positive integers, $0 < t_1 < t_2 < \dots < t_k < \dots$ and

$$t_k \rightarrow \infty (k \rightarrow \infty). \quad f \in C([t_k, t_{k+1}) \times S(\rho) \times R^n, R^n) (k \in N), f(t, 0, 0) = 0, \quad \text{where } S(\rho) = \{x : |x| < \rho, x \in R^n\}.$$

$$Tx = \int_{t_0}^t K(t, s, x(s)) ds, \quad \text{where } K \in C([t_k, t_{k+1}) \times [t_k, t_{k+1}) \times S(\rho), R^n), K(t, t, 0) = 0.$$

$J_k(x) : S(\rho) \rightarrow R^n, J_k(0) = 0 (\forall k \in N)$ and there exists $\rho_1 : 0 < \rho_1 \leq \rho$ such that $x \in S(\rho_1)$ implies that $J_k(x) \in S(\rho)$ for all $k \in N$.

In addition we always assume that f, J_k satisfy certain conditions such that the solution of system (1) exists on $[t_0, +\infty)$ and is unique. We denote by $x(t) = x(t, t_0, x_0)$ the solution of system (1) with initial value (t_0, x_0) . Since $f(t, 0, 0) = 0, J_k(0) = 0, k \in N$, then $x(t) = 0$ is a solution of (1), which is called the trivial solution. Note that the solutions $x(t)$ of (1) are right continuous, satisfying $x(t_k^+) = x(t_k) = J_k(x(t_k^-))$.

Let $t_0 = 0$; the following sets are introduced:

$$G_k = \{(t, x) \in R_+ \times S(\rho) : t_{k-1} < t < t_k\}, G = \bigcup_{k=1}^{\infty} G_k$$

For convenience, we define the following classes of functions:

$K = \{a \in C[R_+, R_+] : \text{strictly increasing and } a(0) = 0\}$ strictly increasing and $a(0) = 0$;

$$CK = \{a \in C[R_+, R_+] : \text{for every } t \in R_+, a(t, s) \in K\}$$

$$S(h, \rho) = \{(t, x) \in R_+ \times R^n : h(t, x) < \rho\}$$

In addition, we introduce some definitions as follows:

Definition 1: Let $Z \subseteq R^n$ be a cone, that is, Z is closed, convex with $\lambda Z \subset Z, \lambda \geq 0$ and $Z \cap (-Z) = \{0\}$ with interior $Z^0 \neq \emptyset$. For any $x, y \in R^n$, we let $x \leq y$ if $y - x \in Z$ and for any functions $u, v : R_+ \rightarrow R^n, u \leq v$ if $u(t) \leq v(t)$ on R_+ . Also let $Z^* = \{\varphi \in R^n, \varphi(x) \geq 0, \text{ for all } x \in Z\}$ and $Z_0^* = Z^* - \{0\}$,

*Corresponding author: Zhuoying Lv, Public Course Department, Shandong University of Science and Technology, Jinan, Shandong, 250031, P.R. China, E-mail: fancygreen@126.com

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where $\varphi(x) = \sum_{i=1}^n \varphi_i x_i$

Definition 2: A function $F: R^n \rightarrow R^n$ is said to be quasi-monotone nondecreasing relative to the cone $Z \subseteq R^n$ if $x \leq y$ and $\varphi(y-x) = 0$ for all $\varphi \in Z_0^*$ implies $\varphi(F(y)-F(x)) \geq 0$.

Definition 3: We shall say that the function $V_0: R_+ \times R^n \rightarrow Z$ belongs to the class of cone-Lyapunov if \

i. $V \in V_0$ is continuous on $G_k \times R^n$ and the following limits exist

$$\lim_{(t,y) \rightarrow (t_0,x)} V(t,y), k \in N$$

ii. $V(t,x) \in V_0$ is locally Lipschitz in x relative to Z.

Derivative of $V(t,x) \in V_0$ along system(1) is defined:

$$D^-V(t,x(t)) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, x(t) + hf(t, x(t))) - V(t, x(t))]$$

Definition 4: The trivial solution of (1) is said to be

i. stable, if for any $\varepsilon > 0$, every $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| \leq \varepsilon$ for all $t \geq t_0$;

ii. uniformly stable, if δ in (i) is independent on t_0 ;

iii. uniformly asymptotically stable, if it is uniformly stable, and there exists a δ such that for any and every $t_0 \in R_+$ there is a $T = T(\varepsilon) > 0$ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| \leq \varepsilon$ for all $t \geq t_0 + T$

Definition 5: The trivial solution of (1) is said to be

i. practically stable, if for given number pair (λ, A) with $0 < \lambda < A$, we have $|x_0| < \lambda$ implies $|x(t)| < A, t \geq t_0$ for some $t_0 \in R_+$;

ii. uniformly practically stable, if(i)holds for every $t_0 \in R_+$;

iii. practically quasi-stable, if for given number $(\lambda, B, T) > 0$ and some $t_0 \in R_+$, we have $|x_0| < \lambda$ implies $|x(t)| < B$ for $t \geq t_0 + T$;

iv. uniformly practically quasi-stable, if (iii) holds for every $t_0 \in R_+$;

v. strongly practically stable, if (i) and (iii) hold together;

vi. strongly uniformly practically stable, if (ii) and (iv) hold together.

We also consider the comparison differential system:

$$\begin{cases} u' = g(t, u), & t \neq t_k \\ u(t_k) = \Psi_k(u(t_k^-)) & k \in N \\ u(t_0^+) = u_0 \end{cases} \quad (2)$$

where $g \in C([t_k, t_k + 1) \times Z, R^n), k \in N$.

In addition we always assume that g, Ψ_k satisfy certain conditions such that the solution of system(2) exists on $[t_0 + \infty]$ and is unique. We denote by $u(t) = u(t, t_0, u_0)$ the solution of system(2) with initial value (t_0, u_0) . Note that the solutions $u(t)$ of (2) are right continuous, satisfying $u(t_k^+) = u(t_k) = \Psi_k(u(t_k^-))$.

Definition 6: The trivial solution of (2) is said to be

i. ϕ_0 -stable, if for any $\varepsilon > 0$, every $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $(\phi_0, u_0) < \delta$ implies $(\phi_0, u(t)) < \varepsilon$, for all

$t \geq t_0$; where $\phi_0 \in Z_0^*$;

ii. ϕ_0 -uniformly stable, if δ in (i) is independent on t_0 ;

iii. ϕ_0 -attractive, if for any $\varepsilon > 0$, every $t_0 \in R_+$, there exists a $\delta = \delta(t_0) > 0, T = T(t_0, \varepsilon) > 0$, such that $(\phi_0, u_0) < \delta$ implies for all $t \geq t_0 + T$, where $\phi_0 \in Z_0^*$;

iv. ϕ_0 -uniformly attractive, if δ, T in(iii)is independent on t_0 ;

v. ϕ_0 -asymptotically stable, if (i)and(iii) hold together;

vi. ϕ_0 -uniformly asymptotically stable, if (ii) and (iv) hold together.

Definition 7: The trivial solution of (2) is said to be

i. ϕ_0 -practically stable, if for given number pair (λ, A) with $0 < \lambda < A$, we have $(\phi_0, u_0) < \lambda$ implies $(\phi_0, u(t)) < A, t \geq t_0$ for some $t_0 \in R_+, \phi_0 \in Z_0^*$, where $\phi_0 \in Z_0^*$;

ii. ϕ_0 -uniformly practically stable, if(i)holds for every $t_0 \in R_+$;

iii. ϕ_0 -practically quasi-stable, if for given number $(\lambda, B, T) > 0$ and some $t_0 \in R_+, \phi_0 \in Z_0^*$ we have $(\phi_0, u_0) < \lambda$ implies $(\phi_0, u(t)) < B$, for all $t \geq t_0 + T$;

iv. ϕ_0 -uniformly practically quasi-stable, if (iii) holds for every $t_0 \in R_+$;

v. ϕ_0 -strongly practically stable, if (i) and (iii) hold together;

vi. ϕ_0 -strongly uniformly practically stable, if (ii) and (iv) hold together.

Main Results

Lemma 1: Assume that

i. $g \in C[R_+ \times Z, R^n], g(t, u)$ is quasi-monotone non decreasing in u for each fixed t on Z and $r(t, t_0, u_0)$ is the maximal solution of the

$$\begin{cases} u'(t) = g(t, u), \\ u(t_0) = u_0; \end{cases}$$

ii. $V \in C[R_+ \times R^n, Z], V(t, x)$ is locally Lipschitz in x relative to the cone Z and $D^-V(t, x(t)) \leq_Z$

$$g(t, V(t, x(t))) (t \geq t_0) \text{ for any solution } x(t) = x(t, t_0, x_0) \text{ of}$$

$$\begin{cases} x' = f(t, x, Tx), \\ x(t_0) = x_0. \end{cases}$$

Then $V(t_0, x_0) \leq_Z u_0$ implies $V(t, x(t)) \leq_Z r(t, t_0, u_0), t \geq t_0$.

Proof: Let $x(t) = x(t, t_0, x_0)$ is any solution of system in (ii), satisfying $V(t_0, x_0) \leq_Z u_0$.

Set $m(t) = V(t, x(t))$, for small enough $h < 0$; from (ii) $V(t, x)$ is locally Lipschitz in x relative to the cone Z, therefore $m(t+h) - m(t) \leq_Z L \|x(t+h) - x(t) - hf(t, x, Tx)\| + V(t+h, x + hf(t, x, Tx)) - V(t, x(t))$, when $h \rightarrow 0$ we have $D^-m(t) \leq_Z D^-V(t, x(t)) \leq_Z g(t, m(t))$.

For small enough $\forall \varepsilon > 0$, consider system $\begin{cases} u'(t) = g(t, u) + \varepsilon \eta, \\ u(t_0) = u_0, t_0 \in R_+, \end{cases}$ where $\eta \in Z$.

The solution of it is, then we have.

To prove the conclusion we only need to prove

$$m(t) \leq_Z u(t, \varepsilon), t \geq t_0. \quad (3.1)$$

If it is not true, there exists $t_1 > t_0$, such that $u(t_1, \varepsilon) - m(t_1) \in \partial Z$, and $u(t, \varepsilon) - m(t) \in Z^0, t \in [t_0, t_1]$.

From (i) $g(t, u)$ is quasi-monotone non decreasing in u for each fixed t on Z , so there exists $\varphi \in Z_0^+$, such that

$$\varphi(u(t_1, \varepsilon) - m(t_1)) = 0 \text{ and } \varphi(g(t_1, u(t_1, \varepsilon)) - g(t_1, m(t_1))) \geq 0.$$

Set $\bar{u}(t) = \varphi(u(t, \varepsilon) - m(t)), t \in [t_0, t_1]$. Obviously $\bar{u}(t) > 0, t \in [t_0, t_1]$ and $\bar{u}(t_1) = 0$.

Therefore, $D^- \bar{u}(t_1) < 0$

But

$$\begin{aligned} D^- \bar{u}(t_1) &= \varphi(D^- u(t_1, \varepsilon) - D^- m(t_1)) \\ &> \varphi(g(t_1, u(t_1, \varepsilon)) + \varepsilon \eta - g(t_1, m(t_1))) \\ &> \varphi(g(t_1, u(t_1, \varepsilon)) - g(t_1, m(t_1))) \\ &\geq 0, \end{aligned}$$

a contradiction So (3.1) holds, thus lemma 1 holds

Lemma 2: Assume that

i. $g \in C[[t_k, t_{k+1}] \times Z, R_+^n], k \in N, g(t, u)$ is quasi-monotone non decreasing in u for each fixed t on the cone Z and $r(t) = r(t, t_0, u_0)$ is the maximal solution of system(2) on Z ;

ii. $\Psi : Z \rightarrow Z (k \in N)$ is strictly increasing on Z ;

iii. for any solution $x(t) = x(t, t_0, x_0)$ of system(1) and $V \in V_0$,

$$D^- V(t, x(t)) \leq_Z g(t, V(t, x(t))), t \neq t_k, k \in N$$

iv. $V(t_k, J_k(x(t_k^-))) \leq_Z \Psi_k(V(t_k^-, x))$

Then implies

Proof: For any $t_0 \in R_+$, and $t_0 \in [t_k, t_{k+1}]$, for some $k \geq 1$, we designate $t_i = t_{k+i}, i = 1, 2, \dots$, for convenience, then for $t \in [t_0, t_1]$ from lemma 1 we have $V(t, x(t)) \leq_Z r_1(t, t_0, u_0)$ where $r_1(t, t_0, u_0)$ is the maximal solution of system(2) on $[t_0, t_1]$ such that $r_1(t_0^+, t_0, u_0) = u_0$.

$$\text{So } V(t_1^-, x) \leq_Z r_1(t_1^-, t_0, u_0)$$

Thus from (iii) we have

$$\begin{aligned} V(t_1, x(t_1, t_0, x_0)) &= V(t_1, J_1(x(t_1^-))) \\ &\leq_Z \Psi_1(V(t_1^-, x)) \\ &\leq_Z \Psi_1(r_1(t_1^-, t_0, u_0)) \\ &= r_1(t_1^-, t_0, u_0) \doteq r_1 \end{aligned}$$

Again from lemma 1, for $t \in [t_1, t_2], V(t; x(t; t_0; x_0)) \leq_Z r_2(t; t_1; r_1)$; where $r_2(t; t_1; r_1)$ is the maximal solution of system (2) on $[t_1, t_2]$ such that $r_2(t_1^+, t_1, r_1) = r_1$

Therefore we have

$$V(t, x(t, t_0, x_0)) \leq_Z r_k(t, t_{k-1}, r_{k-1}),$$

where $r_k(t, t_{k-1}, r_{k-1})$ is the maximal solution of system(2) on (t_{k-1}, t_k) such that $r_k(t_{k-1}^+, t_{k-1}, r_{k-1}) = r_{k-1}$

So if we define

$$u^*(t) = \begin{cases} u_0 & t = t_0, \\ r_1(t, t_0, u_0), & t \in [t_0, t_1), \\ r_2(t, t_1, r_1), & t \in [t_1, t_2), \\ \dots\dots & \dots\dots \\ r_k(t, t_{k-1}, r_{k-1}), & t \in [t_{k-1}, t_k), k = 3, 4, \dots \end{cases}$$

Then $u^*(t)$ is the solution of system (2), and $V(t, x(t)) \leq_Z u^*(t)$; since $r(t; t_0; u_0)$ is the maximal solution of system(2) on Z , we get immediately

$$V(t, x(t)) \leq_Z z(t, t_0, u_0), t \geq t_0.$$

Theorem 1: Assume that $\exists a, b \in K$ such that:

i) $b(|x|) \leq (\mathcal{O}_0, V(t, x)) \leq a(|x|), x \in S(\rho)$;

ii) $g \in C[[t_k, t_{k+1}] \times Z, R_+^n], k \in N, g(t, u)$ is quasi-monotone nondecreasing in u for each fixed t on the cone Z and $g(t, 0) \equiv 0, r(t) = r(t, t_0, u_0)$ is the maximal solution of system(2) on Z ;

iii) $\Psi_k : Z \rightarrow Z (k \in N)$ is strictly increasing on $Z \Psi_k(0) \equiv 0$;

(iv) for any solution $x(t) = x(t, t_0, x_0) \in S(\rho)$ of system (1) and $V \in V_0$ we have $D^- V(t, x(t)) \leq_Z g(t, V(t, x(t))), t \neq t_k, k \in N$;

v) $V(t_k, J_k(x(t_k^-))) \leq_Z \Psi_k(V(t_k^-, x)), x \in S(\rho_1)$.

Then the \mathcal{O} stability properties of the trivial solution of system (2) imply the corresponding stability properties of the trivial solution of system (1).

Proof: For any $\varepsilon: 0 < \varepsilon < \rho_1$, every set $V(t) = V(t, x(t))$.

Let the trivial solution of system (2) is \mathcal{O} -stable then for $b(\varepsilon) > 0$ every $t_0 \in R_+$, there

Exists $\delta_1 = \delta_1(t_0, \varepsilon)$ such that $0 \leq (\mathcal{O}_0, u_0) < \delta_1$ implies $(\mathcal{O}_0, u(t, t_0, u_0)) < b(\varepsilon), t \geq t_0$.

Set $\delta_2 = \delta_2(\varepsilon)$ satisfying $a(\delta_2) < \delta_1$.

We define $\delta = \min\{\delta_1, \delta_2\}$.

Next, we claim that $|x_0| < \delta$ implies

$$|x(t)| < \varepsilon, t \geq t_0. \tag{3.2}$$

If it is not true, there exists a solution $x(t)$ of system(1) such that $x(t_0, t_0, x_0) = 0$ with $|x_0| < \delta$, then there exists $t^* > t_0$ such that $t_k \leq t^* < t_{k+1} (k \in N)$ satisfying $|x(t^*)| \geq \varepsilon$ and $|x(t)| < \varepsilon, t_0 \leq t < t_k$.

From $0 < \varepsilon < \rho_1$ we have $|x(t_k)| = |J_k(x(t_k^-))| < \rho$, then there exists $\bar{t} \in [t_k, t^*]$ satisfying $\varepsilon \leq |x(\bar{t})| < \rho$. From (iv) we have $D^- V(t) \leq_Z g(t, V(t)), t \in [t_0, t]$.

Set $u_0 = V(t_0)$ thus from lemma 2 we get that $t \in [t_0, t]$ implies $V(t) \leq_Z r(t, t_0, u_0)$, where $r(t, t_0, u_0)$ is the maximal solution of system (2) such that $u(t_0, t_0, u_0) = u_0$.

And from (i) $(\mathcal{O}_0, u_0) = (\mathcal{O}_0, V(t_0)) \leq a(|x_0|) < a(\delta) < \delta_1$ so from the -stability of the trivial solution of system(2) we have $(\mathcal{O}_0, r(t, t_0, u_0)) < b(\varepsilon), t \geq t_0$. Therefore $b(\varepsilon) \leq b(|x(\bar{t})|) \leq (\mathcal{O}_0, V(\bar{t})) < (\mathcal{O}_0, r(\bar{t}, t_0, u_0)) < b(\varepsilon)$, a contradiction Then (3.2)

holds, thus the trivial solution of system (1) is stable. If the trivial solution of system (2) is \mathcal{O}_0 -uniformly stable then it is clear that \mathcal{D} will be independent of t_0 and thus we get the uniform stability of the trivial solution of system (1). Assume that the trivial solution of system (2) is \mathcal{O}_0 -symptotically stable, consequently we get that the trivial solution of system(1) is stable, then for $\epsilon = \rho_1$, there exists $\delta_0^* = \delta(t_0, \rho_1)$, such that $|x_0| < \delta_0^*$ implies $\|x(t)\| < \rho_1, t \geq t_0$. From (iv) we have $D^-V(t) \leq z g(t, V(t)), t \geq t_0$.

For any $\epsilon: 0 < \epsilon < \rho_1$ every $t_0 \in R_+$, from the \mathcal{O}_0 -attractivity of the trivial solution

of system (2) we can get that for $b(\epsilon) > 0$, every $t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$, such that $0 \leq (\mathcal{O}_0, u_0) < \delta_0$ implies $(\mathcal{O}_0, u(t, t_0, u_0)) < b(\epsilon), t \geq t_0 + T$.

Set $\delta_0 = \min\{\delta_0^*, a^{-1}(\delta_0)\}$.

For $|x_0| < \delta_0$, set $u_0 = V(t_0)$ then from lemma 2 we have $V(t) \leq z r(t, t_0, u_0), t \geq t_0$

And from (i) $(\mathcal{O}_0, u_0) = (\mathcal{O}_0, V(t_0)) \leq a(|x_0|) < a(\delta_0) < \delta_0$ so from the \mathcal{O}_0 -attractivity

Of the trivial solution of system (2) we get that $(\mathcal{O}_0, r(t, t_0, u_0)) < b(\epsilon), t \geq t_0 + T$ therefore $b(|x(t)|) \leq (\mathcal{O}_0, V(t)) \leq (\mathcal{O}_0, r(t, t_0, u_0)) > b(\epsilon), t \geq t_0 + T$, thus $\|x(t)\| < \epsilon, t \geq t_0 + T$ so the trivial solution of system(1) is attractive.

Then the trivial solution of system (1) is asymptotically stable. If the trivial solution of system (2) is \mathcal{O}_0 -uniformly asymptotically stable, then it is clear that δ_0 T will be independent of t_0 , and thus we get the uniform asymptotically stability of the trivial solution of system (1).

Theorem 2: Assume that theorem1 (i)-(v) hold and we have:

(vi) for $0 < \lambda < A \leq \rho_1$ with $a(\lambda) < b(A)$.

Then the \mathcal{O}_0 -practical stability properties of the trivial solution of system (2) with respect to $(a(\lambda), b(A))$ imply the corresponding practical stability properties of the trivial solution of system (1) with respect to (λ, A) .

Proof: Set $V(t) = V(t, x(t))$, suppose that the trivial solution of system (2) is \mathcal{O}_0 practically

Stable with respect to $a(\lambda), b(A)$ then exists $t_0 \in R_+$, such that $0 \leq (\mathcal{O}_0, u_0) < a(\lambda)$ implies $(\mathcal{O}_0, u(t, t_0, u_0)) < b(A), t \geq t_0$.

For above $t_0 \in R_+$ next we will prove that $|x_0| < \lambda$ implies $|x(t)| < A, t \geq t_0$. If it is not true, there exists a solution $x(t)$ of system (1) such that $x(t_0, t_0, x_0) = 0$ with $|x_0| < \delta$, then there exists $t^* > t_0$ such that $t_k > t^* < t_{k+1} (k \in N)$ satisfying $|x(t^*)| \geq A$ and $|x(t)| < A, t_0 \leq t < t_k$.

Since $0 < A < \rho_1$ we have $|x(t_k)| = |J_k(x(t_k^-))| < \rho$ then exists $\bar{t} \in [t_k, t^*]$ satisfying $A \leq |x(\bar{t})| < \rho$.

From (iv) we have $D^-V(t) \leq z g(t, V(t)), t \in [t_0, \bar{t}]$.

Set $u_0 = V(t_0)$ then from lemma 2 we get that $t \in [t_0, \bar{t}]$ implies $V(t) \leq z r(t, t_0, u_0)$, where

$r(t, t_0, u_0)$ is the maximal solution of system(2) such that $u(t_0, t_0, u_0) = u_0$. And from $(\mathcal{O}_0, u_0) = (\mathcal{O}_0, V(t_0)) \leq a(|x_0|) < a(\lambda)$, so from the \mathcal{O}_0 -practical stability we have $(\mathcal{O}_0, r(t_0, t_0, u_0)) < b(A), t \geq t_0$.

Then $b(A) \leq b(|x(\bar{t})|) \leq (\mathcal{O}_0, V(\bar{t})) < (\mathcal{O}_0, r(\bar{t}, t_0, u_0)) < b(A)$ a contradiction thus (3.3) holds, so the trivial solution of system(1) is practically stable. Suppose that the trivial solution of system (2) is \mathcal{O}_0 -uniformly practically stable with respect to $(a(\lambda), b(A))$ then it is clear that t_0 will be independent of t_0 , and thus we get the trivial solution of system (1) is uniformly practically stable with respect to (λ, A) . Suppose that the trivial solution of system (2) is \mathcal{O}_0 -strongly practically stable with respect To $(a(\lambda), b(A), b(B), T)$, consequently we get that the trivial solution of system (1) is practically stable with respect to (λ, A) then exists $t_0 \in R_+$, such that $|x_0| < \lambda$ implies $|x(t)| < A \leq \rho, t \geq t_0$. From (iv) $D^-V(t) \leq z g(t, V(t)), t \geq t_0$. And since the trivial solution of system(2) is \mathcal{O}_0 -practically quasi-stable with respect to $(a(\lambda), b(B), T)$, we have that $0 \leq (\mathcal{O}_0, u_0) < a(\lambda)$ implies $(\mathcal{O}_0, u(t, t_0, u_0)) < b(B), t \geq t_0 + T$.

For $|x_0| < \lambda$ set $u_0 = V(t_0)$ then from lemma 2 we get $V(t) \leq z r(t, t_0, u_0), t \geq t_0$.

And from (i) $(\mathcal{O}_0, u_0) = (\mathcal{O}_0, V(t_0)) \leq a(|x_0|) < a(\lambda)$, so from the \mathcal{O}_0 practically quasistability of the trivial solution of system (2) we have $(\mathcal{O}_0, r(t, t_0, u_0)) < b(B), t \geq t_0 + T$ then $b(|x(t)|) \leq (\mathcal{O}_0, V(t)) \leq (\mathcal{O}_0, r(t, t_0, u_0)) < b(B), t \geq t_0 + T$ thus $|x(t)| < (B), t \geq t_0 + T$ so the trivial solution of system (1) is practically quasi-stable with respect to (λ, B, T) .

Therefore the trivial solution of system (1) is strongly practically stable with respect to (λ, A, B, T) .

If the trivial solution of system (2) is \mathcal{O}_0 -strongly uniformly practically stable with respect to (λ, A, B, T) then it is clear that the above proof establish for every t_0 , therefore we get the trivial solution of system (1) is strongly uniformly practically stable with respect to (λ, A, B, T) .

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