**Research Article**
Volume 14:3, 2020
DOI: 10.37421/glta.2020.14.304

Journal of
Generalized Lie Theory and Applications

ISSN: 1736-4337 Open Access

# Spruce Budworm Model With and Without Delay

**Dipak Maji***
*Bengal Engineering and Science University, Shibpur, India*

## Abstract

Mathematical modeling is the art of translating problems from an application area into tractable mathematical formulations whose theoretical and numerical analysis provides insight, answers, and guidance useful for the originating application.

Mathematical modeling:

- is indispensable in many applications
- gives direction for the problem solutions
- prepares the way for better designing or control system

Learning about mathematical modeling is an important step for a theoretical mathematical training to an application oriented mathematical expertise, and makes the student fit for mustering the challenges of our modern technological culture. One of the most important mathematical model is Spruce Budworm model for researchers as well as for students. Understanding the dynamic of spruce budworm is very important for the protection of spruce and fir trees of Canada and Northern Minnesota (also in recent time Indian Himalayan range forest). This model was designed to identify the (a) critical factors that affect the Budworm population dynamics (b) to evaluate the effect of budworm population on the growth and yield of wood industry and also the present loss claimed by irruptions done by Budworm (c) to formulate mathematical model and to find out the steady state and the existence of the steady state and steady state analysis. The bifurcation analysis and the hysteresis effect of the model has been discussed. Analysis of the equilibrium stability and examination of amplitudes and periodic oscillations are conducted, and the effect of Budworm control, immature population control and predation by the birds are assessed.

**Keywords:** Mathematical modeling • Dynamic • Steady state • Equilibrium

## Preface

### Mathematics and ecology

Ecology is the science, which deals with interactions between living organisms and their environment

Historically it has focused on questions such as:

- Why do we observe certain organisms in certain places and not others?
- What limits the abundances of organisms and controls their dynamics?
- What causes the observed groupings of organisms of different species, called the community, to vary across the planet?
- What are the major pathways for movement of matter and energy within and between natural systems?

The above questions serve as the focus of several distinct fields within ecology.

- Physiological ecology deals with interactions between individual organisms and external environmental forces, such as temperature, with a focus on how individual physiology and behavior varies across different environment
- Population ecology deals with the dynamics and structure (age, size, sex, etc.) of groups of organisms of the same species. Community ecology deals with the biological interactions (predator-prey, competition, mutualism, etc.), which occur between species
- Ecosystem ecology like the R.M May's model [1,2] deals with the movement of matter and energy between communities and the physical environment

Mathematics, as the language of science, allows us to carefully phrase questions concerning each of the above areas of ecology. It is through mathematical descriptions of ecological systems that we abstract out the basic principles of these systems and determine the implications of these. Ecological systems are enormously complex. A major advantage of mathematical ecology is the capability to selectively ignore much of this complexity and determine whether by doing so we can still explain the major patterns of life on the planet. Thus simple population models group together all individuals of the same species and follow only the total number in the population. By ignoring the complexity of differences in physiology, size, and age between individuals, the models attempt to compare the basic dynamics obtained from the model with observations on different species. As a next step, additional complexity, associated with introducing different age classes for example, is included. How the inclusion of such additional complexity affects the predictions of the model determines whether this additional complexity is necessary to answer the biological questions one is interested in.

Mathematical models in physiological ecology are often compartmental in form, in which the organism is assumed to be composed of several different components. For example, many plant growth models consider leaves, stem and roots as different compartments. The models then make assumptions about how different environmental factors affect the rate of change of biomass or nutrients in different compartments. These models are typically framed as systems of differential equations with one equation for each compartment. Population models are used to determine the effects of different assumptions about the age, size, or spatial structure of a population on the dynamics of the population.

### Mathematical approaches

Mathematical approaches include differential equations (both ordinary and partial), integral equations, and matrix theory. Models for communities are often framed as systems of ordinary differential equations, with separate equations for each of the interacting populations. Additional models apply graph theory to elucidate the topological structure of food webs, the links which determine who eats who in a particular community. The above has

***Address for Correspondence:** Dipak Maji, Bengal Engineering and Science University, Shibpur, India, E-mail: dipakrumahiya2015@gmail.com.*



**Received** 03 July, 2020; **Accepted** 15 July, 2020; **Published** 20 August, 2020

focused on the use of mathematics to formulate basic theory in ecology. There are also many applications of mathematical and computer models to vary practical questions arising from environmental problems. This includes the entire field of ecotoxicology, in which mathematical models predict the effects of environmental pollutants on populations and communities. The field of natural resource management uses models to help set harvest quotas for fish and game, based upon population models similar in form to those mentioned above. Conservation ecology uses models to help determine the relative effects of alternative recovery plans for endangered species, as well as aid in the design of nature preserves.

Mathematical ecology took start from works A. Lotka [3] and V. Volterra [4]. It accumulates big arsenal of research models of temporal regularity, recurrence in ecosystems. Recently models and methods of special structure of populations and communities research are developed.

Traditional object of eco-mathematical modeling is phytoplankton. Its cynetic processes of growth are good researched qualitatively. But the reasons of specially-temporal structuring observed in nature are not quite clear. We will analyze the phenomenon “spotting quot; of special distribution, recurrence and also will research spatial-temporal reconstruction of specific plankton association under the influence of anthropogenic factors. For the first time mathematics description in biosystems was obtained on the model scraper-victim (Lotka) [3]. It is necessary to note that mathematical modeling don’t substitute experimental researches. Quite the contrary it stimulates the accumulation of real material and specifies direction of realizing experiments.

Construction of mathematical modeling science branch needs for building dynamics prognosis of the real objects, for science predictions of various interaction consequences on the studied objects. Sometimes answers on the noted questions may be obtained by the way of laboratory modeling on physics, chemical and biological model; It is not concerned with natural ecosystems, experiments with they are very complicated and sometimes are not impossible.

Firstly such unit as population subject to mathematical modeling. As principles development and widening of knowledges in the field of population ecology models became more perfect and complicated. At the same time investigations on modeling of community of water animals and plants were developed starting with V. Volterra’s [4] works. With appearance of models realization possibility on computers the works on description by the help of mathematical models of ecosystems dynamics were began.

### Models and methods for analysis of spatial-temporal structures

Organization of ecological and biochemical systems allow to make decomposition of their mathematical modeling to quantitative description of kinetic processes of local interaction of components in space. Mathematical research apparatus of cynetics processes in local systems is theory of ordinary differential equations. Well-constructed qualitative and numerical research methods allow to study stable and oscillating regimes, equilibriums and others dynamics factors.

**Simple models of ecological processes:** For examination of composite processes in plants varying with passage of time, the descriptive mathematical models such as the differential equations (or systems of the differential equations) are applied. The equations of models are made on the basis of the physical, chemical, biological laws.

The solutions of such systems of the differential equations are functions of time and, therefore, can feature modifications in time of processes happening inside simulated objects. The models are divided on two basis-

i) With concentrated parameters-such as ordinary differential equations. These models are real for exposition of processes, which do not depend on coordinates (focused in a point)

ii) With distributed parameters-such as differential equations if the form of partial derivatives. Their solutions depend both on time, and from coordinates of area of a solution

## Introduction

Spruce budworm is a serious pest in eastern Canada and Northern Minnesota. The spruce budworm crawls upon and consumes the leaves of coniferous trees. Excessive consumption can damage and kill the host. The Budworm themselves be eaten primarily by birds who eat many other insects as well. The budworms prefer larger trees. A key factor in determining the spruce budworm population is the leaf surface area per tree. Larger trees have larger leaf surface area resulting in larger surface budworm.

The Canadians had observed that the spruce budworm population underwent irruptions approximately every 40 years.

For unknown reasons, the budworm population would explode devastating the pineries and then return to their manageable levels. The loss of timber represented a significant cost to the Canadian wood products industry and various management techniques, pesticide application. For example, we tried without success.

To understand the cycles of spruce budworm population. Several scientists of the University of British Columbia-R. Morris [5,6], D Ludwig [7-11], D. Jones [12], and C.S. Holling [13,14] studied the problem in the year 1978.

This model was designed for scientists to identify the critical factors that affect the budworm population dynamics. To evaluate the effects of the budworm population on the growth and yield of the wood industry and also to present the loss claimed by the irruptions done by the budworms respectively.

Interaction between trees, budworms, and birds-

Here are some of the pertinent features of the system

- The leaves of the trees provide food to the budworms
- The leaves of the trees also provide coverage to the budworms; essentially helping them in hiding from the birds
- The birds feed on the budworms if they can find them. If the budworms are present in a large number the birds will make short work of them. However the birds have plenty of other sources of food, so they will not critically dependent on the budworms. This means that we can consider the population of the birds to be constant

The growth rate of budworm considered here does not act instantaneously. There is a time delay for the larva to grow that becomes adults. This is called the maturation period. Other two-point: One can modify this model to introduce lag due to the gestation period (i.e. the period needed by the birds to digest budworms). Or Regeneration period (i.e. the time taken by leaves to grow, consumed by budworm to grow).

## The Mathematical Tools

### The existence and uniqueness theorem

Consider the initial value problem

$X=F(X), X(t_0)=X_0$

where $X_0 \in I\Re^n$ suppose that $F : I\Re^n \to I\Re^n$ is C′ then, first of all, there exists a solution of this initial value problem and, secondly this is only such solution. More precisely, there emits an a>0 and a unique solution

$$X : (t_0 - a, t_0 + a) \to I\Re^2$$

of this differential equation satisfying the initial condition $X(t_0)=X_0$

### The phase portraits or phase plane

The phase portraits is a picture of a collection of representation solution curves of the system stem in $I\Re^2$ which is called phase plane (Figure 1).

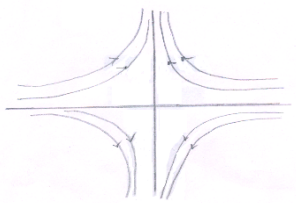

**Figure 1.** Saddle phase plane for $x^{l}=x$ and $y^{l}=y$.

**Bifurcation**

Bifurcation is the study of qualitative change of dynamics. Particular example- let us consider the dynamics of the system $x=r + x^2$ when r=0

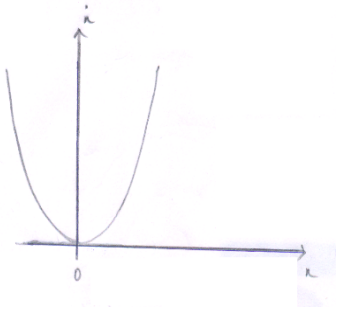


**Figure 2.** r=0.

when r>0

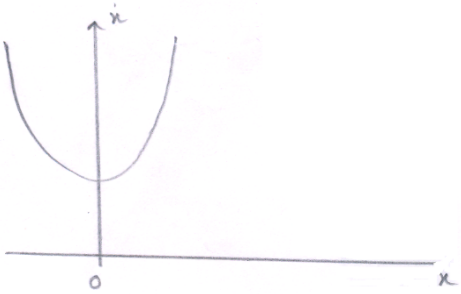


**Figure 3.** r>0.

When r<0 then

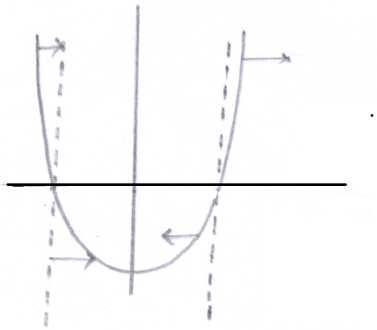

**Figure 4.** r<0.

Here the controlling parameter is r. Depending on different values of r (r>0, r=0, or<0) there are no find point when r=0 as depicted in Figures 2-4. There is only one find point when r>0 and to find r<0. Those finding point appears and disappeared as the values of r passes from positive (+) to negative (-) and to positive respectively (+), this phenomenon is well represented by the following Figure 5.

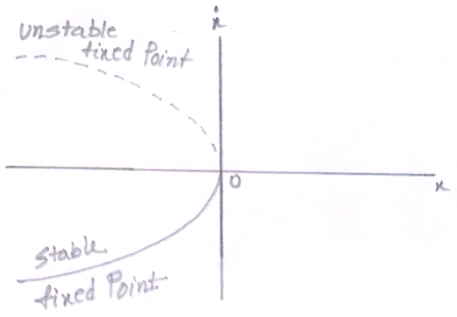


**Figure 5.** Find points.

## Saddle node bifurcation

Let us consider $x^{'} = fa(x)$ is a first order differential equation for which-

$$fa_0(x_0) = 0;$$

$$f_{a_0}(x_0) = 0;$$

$$f^{''}_{a_0}(x_0) \neq 0;$$

$$\frac{\partial fa_0}{\partial x_0}(x_0) \neq 0;$$

then this differential equation undergoes a saddle node bifurcation at $a=a_0$.

## Limit cycle

A closed trajectory of a dynamical system is called an orbit of the system. The motion along the orbit is periodic. An orbit is said to be limit cycle if every trajectory that starts at a point closed to the orbit converge towards the orbit at the end as shown in Figure 6.

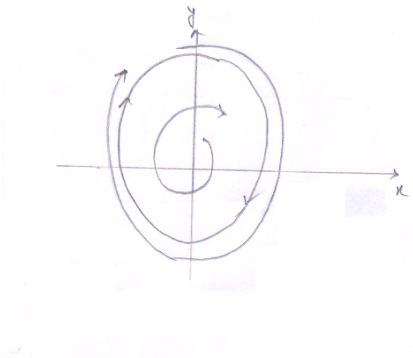


**Figure 6.** Limit cycle.

## Periodic point

The point x is called fixed point for f if f (x)=x. The point x is a periodic point of period n if $f^n(x)=x$. The last positive integer n for which $f^n(x)=x$, is called the prime period on minimal period.

# The Spruce Budworm Model and its Analysis

## Assumption and formulation of mathematical model

i) In absence of predation, the population satisfies the logistic growth (P.F. Verhulst's model [15]).

ii) i.e $\frac{dN}{dt} = rN(1-\frac{N}{K}) - p(N)$

iii) The effect of predators saturates at high prey density i.e. there is an upper limit to the rate of budworm mortality due to predation

iv) If the spruce budworm density is low it will opt for some other prey which mostly lives in the other parts of the trees

So $\lim_{N\to\infty}$ p(N)=B>0, $\lim_{n\to\infty}\frac{p(N)}{N}=0$

or p(n) ≈ bN² near holing type-III model (Figure 7).

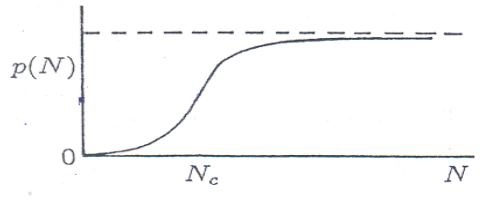


**Figure 7.** Holing type-III model.

$$\therefore \frac{dN}{dt} = PN(1-\frac{N}{K}) - \frac{BN^2}{A^2+N^2} \quad (1)$$

Where

r: The natural growth rate , as in the logistic model

K: The caring capacity

D: is a measure of predation efficiency. If the birds are good at catching spruce budworm this number will be larger than when the birds miss the budworm they are attacking.

C: is the bird's population, considered a constant in this model.

A: is called the switching value that is the population, which predates begins showing increased interest in here costing budworm.

Through observation it had been observed that out of all the model parameters A and K are directly dependent on the average leaf surface area per tree. Letting the average leaf surface area then A=0.55, N=45.

As the finishing touch of this modeling denotes B=CD

$$\therefore \frac{dN}{dt} = PN(1-\frac{N}{K}) - \frac{BN^2}{A^2+N^2} \quad (2)$$

Now here the terms N+N/(A+A+N+N ) is called a Holing type-III predation function. It measures how intensively the bird will select spruces budworms for predation. The idea is that birds are lazy, they will go where food density is high allowing them to consume much while expending minimal energy. If the spruce budworms density is low, birds will opt for some other prey which most likely are present in other parts of trees.

## Nondimentionalization

It is often convenient to change variables when analyzing models. In this case the model equation.

$$\frac{dN}{dt} = rN(1-\frac{N}{K}) - \frac{BN^2}{A^2+N^2}$$

contains four parameters, we introduce new variables

$u = \frac{N}{A}$ and $\tau = \frac{Bt}{A}$

then from (2)

$$\frac{1}{B}\frac{dN}{dt} = \frac{rN}{B}(1-\frac{N}{K}) - \frac{N^2}{A^2+N^2}$$

$$\Rightarrow \frac{d(N/A)}{d(Bt/A)} = \frac{Ar}{B}\frac{N}{A}\left(1-\frac{N/A}{K/A}\right) - \frac{N^2/A^2}{1+N^2/A^2}$$

Now this implies

$$\frac{du}{d\tau} = \frac{Ar}{B}u\left(1-\frac{u}{K/A}\right) - \frac{u^2}{1+u^2} \quad (3)$$

We assume $\frac{Ar}{B} = R$ and $Q = K/A$ are called "lumped constants" then equation (3) reduces to

$$\frac{du}{d\tau} = u\left[R\left(1-u/Q\right) - \frac{u}{1+u^2}\right] \quad (4)$$

We are not required to show this, but we might want to do computation any away.

In the above formulation u, τ, R and Q are all dimensionless quantities

Where u: Represents the population of spruce budworm

τ: Represents time

R: Represents the reproduction rate

Q: Represents the caring capacity

The advantages of reformulation in (4) are two-fold

(i) First the number of parameters required are reduced to two from four parameter

(ii) Secondly, now both parameters appear in the expression R, which is linear in 'u 'and more complicated expression $u/1+u^2$ is parameter free

### Steady state

**Analytical approaches:** Now from the equation (4) we see that the steady state are detune by

$$u\left[R\left(1-u/Q\right) - \frac{u}{1+u}\right] = 0 \quad (5)$$

This implies that the trivial equilibrium is u=0. The remaining equilibrium points are calculated by setting the remaining factors of equation (5) equal to zero

i.e $R\left(1-u/Q\right) - \frac{u}{1+u^2} = 0$ (6)

$$R\left(1-u/Q\right) - \frac{u}{1+u^2}$$

$$\Rightarrow R\left(1-u/Q\right)(1+u^2) = u$$

$$\Rightarrow R\left(1-u/Q\right) + R\left(1+u/Q\right)u^2 - u = 0$$

$$\Rightarrow -R/Q\,u^3 + Ru^2 - \frac{Ru}{Q} - u + R = 0$$

$$\Rightarrow -R/Q\left\{u^3 - Qu^2 + \left(1+Q/R\right)u - Q\right\} = 0$$

$$\Rightarrow u^3 - Qu^2 + \left(1+Q/R\right)u - Q = 0 \quad (7)$$

which is in the form of $u^3+Qu^2+Bu+C=0$

here the roots of these equations are

$$\left.\begin{aligned} u_1 &= 2\alpha^{1/2}\sin\phi - a \\ u_2 &= -2\alpha^{1/2}\sin(\pi/3+\phi) - a \\ u_3 &= 2\alpha^{1/2}\sin(\pi/3-\phi) - a \end{aligned}\right] \quad (8)$$

Where $\phi = \left(1/3\right)\sin^{\tau}\left\{\beta/\left[2\,\alpha^{3/2}\right]\right\}$

$-\pi/6 \le \phi \le \pi/6$

A=3a= $a = 1/3 - Q \Rightarrow a = -Q/3$

$b = 1/3(1 + Q/R)$ and C=-Q

$\alpha = a^2 - b$ and $\beta = 2a^3 - 3ab + c$ (9)

**Existence of steady state by graphical approaches:** This eventually leads to a rather intractable solution of a cubic polynomial. So an alternating strategy, that of determining where

$$R(1-u/Q) = \frac{u}{1+u^2} \quad (10)$$

The analysis continues by plotting graphs of the left and right hand sides of equation (10) versus 'u' and looking for point of intersection.

The following Figure 8 shows the intersection set for two values of R and one value of u.

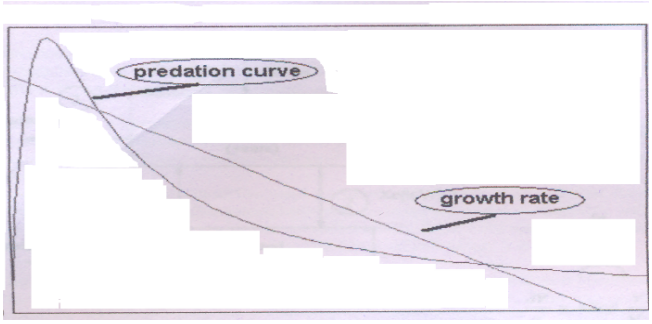


**Figure 8.** Predation and growth curves plotted vs. budworm density.

Now the positive equilibrium is given by the intersection of straight line and the curves.

We have plotted the straight line, the left hand side of equation (10) and the function of right hand side of equation (10) the existence of one, three or again, one solution as R increases to find Q as in the Figure 8 or also happen for affixed R and roaring Q . When R is in opposite range, which depends on Q.

## Stability analysis

**For trivial equilibrium:** We get the trivial equilibrium point u=0. We analyze whether it is stable or unstable. From equation (2) we have

$$\frac{du}{d\tau} = u\left[R\left(1 - u/Q\right) - \frac{u}{1+u^2}\right] \quad (11)$$

Let us consider $u(\tau) = \bar{u} + V(\tau)$ (12)

$$\frac{dV}{d\tau} = \frac{du}{d\tau}$$

By putting the values of equation (12) in equation (11) we get

$$\frac{dv}{d\tau} = R\bar{u} - \frac{\bar{u}^2}{Q} - \frac{\bar{u}v}{Q} + Rv - (\bar{u} + v) \quad (13)$$

After linearizing to neglect the higher and second order terms we get

$$\frac{dv}{d\tau} = R\bar{u} - \frac{\bar{u}^2}{Q} - \frac{\bar{u}v}{Q} + Rv - (\bar{u} + v)$$

Substituting $\bar{u} = 0$ then we get

$$\frac{dv}{d\tau} = Rv - v$$

$$\frac{dv}{d\tau} = v(R-1)$$

$$\Rightarrow \frac{dV}{V} = (R-1)d\tau$$

Consider (R-1)=P

**Stability of non-trivial equilibrium:** As shown in Figure 9. We see that u=0, u=$u_2$ are line only unstable. Since $\frac{\partial f}{\partial u} > 0 \ at\, u = 0 \ and \ u_2$. Where

$$f(u;r,q) = ru(1-u/q) - \frac{u_2}{1+u_2}$$

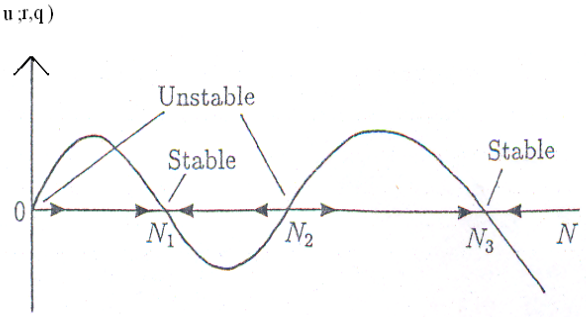


**Figure 9.** There one 3 non-trivial steady state with f (u; r; q) are typically shown.

Where u=0, u=$u_3$ are stable steady state as shown in Figure 10. Since at this $\frac{\partial f}{\partial u} < 0 \ at\, u = u_1, u = u_2$.

There is a domain in R, Q parameter space where the root of equation (10) exists (by the existence and uniqueness the Orem ).

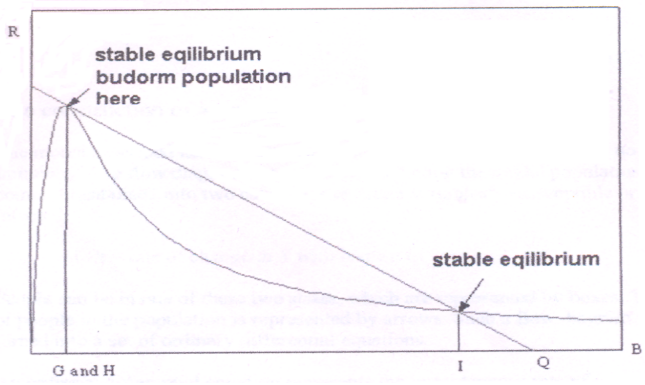


**Figure 10.** Stable steady state.

# The Bifurcation Point and Hysteresis Effect

## The bifurcation points

We have seen that sometimes when we increase R a new equilibrium point suddenly appears and Min divides into two, or two equilibrium point coalesces into one and then disappear. The point (R, Q) for which this occurs are called bifurcation point.

At the bifurcation points the graphs of the functions $R(1-u/Q)$ and $u/1+u^2$ intersect tangentially the curve intersecting means that

$$R(1-u/Q) = \frac{u}{1+u^2} \quad (14)$$

In fact the curve here shows same tangent that means

$$\frac{d}{du}\{R(1-u/Q)\} = \frac{d}{du}\left\{\frac{u}{1+u^2}\right\} \quad (15)$$

from equation (15) we have

$-R/Q = \frac{1-u^2}{(1+u)^2}$ ; now again from equation (14) we get

$$\Rightarrow R + \frac{(1+u^2)u}{(1+u^2)^2} = \frac{u}{1+u^2}$$

$$\Rightarrow R+\frac{(1+u^2)u}{(1+u^2)^2}=\frac{u}{1+u^2}$$

$$\left.\begin{aligned}\Rightarrow R+\frac{2u^3}{(1+u^2)^2}\\ and\ Q=\frac{2u^3}{u^2-1}\end{aligned}\right] \quad (16)$$

Now from the bifurcation point we get $Q=K/A$ and $R=\frac{Ar}{B}$ and A=0.55, N=45.

Q ≈ 8 and R ≈ rS/2B.

When Q=8, the curve is

$$R=\frac{8u}{(8-u)(1+u^2)}$$

These are the saddle node bifurcation point that are called $a_1$(L)$a_2$

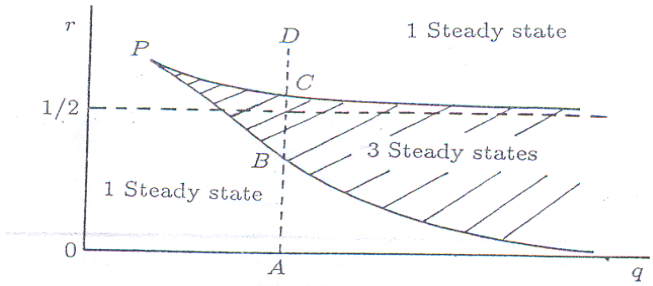


**Figure 11.** Parameter domain for the number of positive steady states for the budworm model. The boundary curves are given parametrically by

$$r(u)=\frac{2u^3}{(1+u^2)},q(u)=\frac{2u^3}{(u^2-1)}.$$

## Hysteresis effect

This model exhibit a hysterics effect suppose we have a fixed Q say, and R increases from zero along the path ABCD in the Figure 11. We see that if $u_1$=0 at R=0 the $u_1$ equilibrium simply increases monotonically with R until C as shown in Figure 11 in reaching for a large R, this steady state disappears and the equilibrium value jumps to $u_3$. If we now reduce R again the equilibrium state is the $u_3$. One and it remains so until R reaches the lower critical value, where there is again only one steady state. In other words as R increases along ABC there is a discontinuous jumps down at B.

The example of cusp catastrophe, which is illustrated schematically in Figure 12.

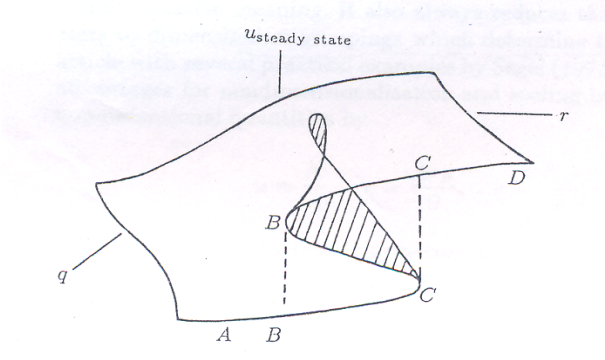


**Figure 12.** Cusp catastrophe for the equilibrium states in the (Unsteady state r, α) parameter space. As r increase from A, the path is ABCCD and as r decreases from D, the path is DCBBA. The projection of this surface onto r, α plane is given in Figure 11.

The parameters from field observation are such that there are possible steady state for the population. The smaller steady state $u_1$ is the refuge equilibrium and $u_3$ is the out brake equilibrium.

# The Spruce Budworm with Delay and without Delay and Numerical Results

## Delay Model

One of the deficiencies of single species population model like (2) is that the birth rate considers being act instant outlay whereas there may be a time delay to take into account of the time to reach maturity. The finite gestation period and so on we can incorporate such delays by considering delay differential equation model of the forms.

$$\frac{dN}{dt}=f(N(t),N(t-T)) \quad (17)$$

Where T>0, the delay parameter one such model has been used as an extension of the logistic growth model namely the delay differential equation.

$$\frac{dN}{dt}=rN(t)\left[1-\frac{N(t-T)}{K}\right] \quad (18)$$

Where r, K, T are positive constant. This say that the regulatory effect depends on the population at an earlier time t-T rather than at t. This equation is itself a model for a delay effect which should really in an intrigue differential equation. Thus more accurate model than (2) is the example of convolution type model.

$$\frac{dN}{dt}=rn(t)\left[1-K^{-1}\int_{-\infty}^{t}W(t-S)N(S)dS\right] \quad (19)$$

Where W (t) is a Wright function, which says how much emphasis, should be given to the size of the population at earlier times to determine the present effect.

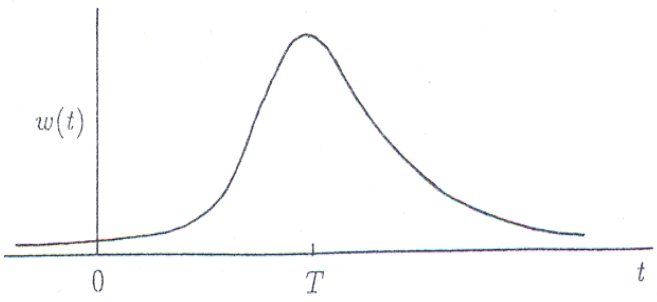


**Figure 13.** Typical weight function w(t) for an integrated delay effect or growth limation for delay model.

On resource availability partially W(t) will tend to zero for large negative and positive t and will probably have a maximum at same reprehensive time T-typically W(t) is an illustrated in Figure 13. If W(t) in sharper in sense that the region around T is narrower are larger than in the limit or can think of W(t) an approximately the direct function S(T-t).

Where

$$\int_{-\infty}^{\infty}\delta(t-f)f(t)dt=f(T)$$

Now equation (9) in this case reduces to

$$\int_{-\infty}^{t}\delta(t-T-S)N(S)dS$$

=N (t-T)

The character of the solution of equation (18) and the type of boundary

conditions required are quite similar to those of logistic equation. Even with seemingly innocuous equation (18) the solutions in general have to be found numerically N(t) for $-T \leq T\ 0$. We can however get some qualitative impression of the kind of the solution of equation (18) which are possible by hearistic reasoning.

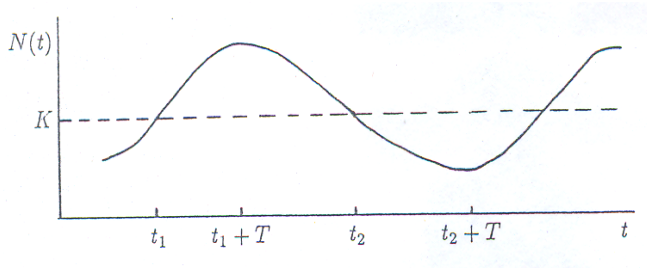


**Figure 14.** Schematize periodic solution of the delay equation of population model 2.

Rater now to the above Figure 14 and support that for some

$t=t_1$, $N(t_1)=K$ and for some $t<t_1$

$N(t-T)<K$ (20)

Thus from the equation 18 we have

$N(t-T)<K$

$$\Rightarrow \frac{N(t-T)}{K}<1$$

$$\Rightarrow 1-\frac{N(t-T)}{K}>0$$

$$\Rightarrow \frac{dN}{dt}<0 \quad (21)$$

So N(t) at $t_1$ is still increasing.

When $t= t_1+T$, $N(f-T)=N(f_1)=K$ and so

$$\Rightarrow \frac{dN}{dt}<0 \quad (22)$$

For $t_1+T<t<t_2$, $N(t-T)>K$

$$\Rightarrow \frac{dN}{dt}<0 \quad (23)$$

and N(t) decreases until $t=t_2+T$ since $\frac{dN}{dt}=0$ again because $N(t_2+T-T)=N(t_2)=K$ (24)

Therefore this is a probability of oscillatory behaviors for example, with linear delay equation

$$\frac{dN}{dt}=\frac{1T}{2T}N(t-T)$$

$$\Rightarrow \mathrm{N(t)}=\mathrm{Acos}\frac{1Tt}{2T} \quad (25)$$

Which is periodic in time.

## The effect of delay of the spruce budworm

The spruce budworm model by delay effects become

$$\frac{dN}{dt}=rN\left(1-\frac{N(f-\tau)}{K}\right)-\frac{BN^2}{A^2+N^2}$$

Which is in the form of

$$\frac{dN}{dt}=f(N)$$

Where $f(N)=rN\left(1-\frac{N(\mathrm{t}-\tau)}{K}\right)-\frac{BN^2}{A^2+N^2}$ (26)

Now we simulate this model numerically. We solve this numerically by using mat lab programming.

**Mat lab programming (Figure 15):** Function y dot=DIPAK (t, y)

global r, k, A, B log

y dot= $\left(r\times y(1)\times\left(1-y\,lag(\tau)/k\right)-\left(B\times y(1)\times y(1)\right)/\left(A\times A+y(1)\times y(1)\right)\right)$

global r, k, A, B log τ

r=0.2;
k=0.8;
A=0.1;
B=0.01;
lag τ=;
$t_s$=[0, 1000 ];
$Z_0$ = [0.9];
[ t, z ] = ode 40 ('DIPAK', $t_s$, $Z_0$);
plot (t, y);

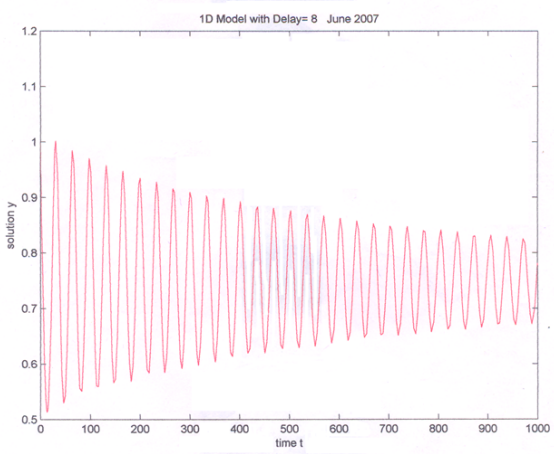


**Figure 15.** 1D model with delay=8.

## Simulation of spruce budworm

Simulate the Spruce budworm model by using Mat lab (Figures 16-18).

Programming-

Function y dot=DIPAK (f, y);

Global r, k, A, B

'y dot= $r\,\&\,y(1)-\left(1-y(1)/K\right)-B\,\&\,y(1)\,\&\,y(1)/\left(A\,\&\,A+y(1)+y(1)\right)$

We write the main code as follows

'r=0.5;
K=0.8;
A=0.1;
B=0.5;
$f_s$=[0, 40]
$y_0$= [10]
[t, z]=ode 45 ['DIPAK', $t_s$=0]
plot (f , z);

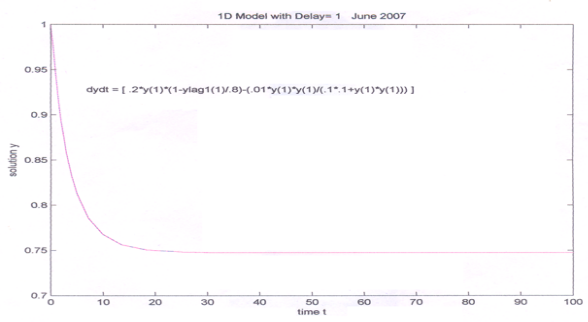


**Figure 16.** 1D model with delay=1.

$t_s$=[0, 100], r=0.2, k=0.8

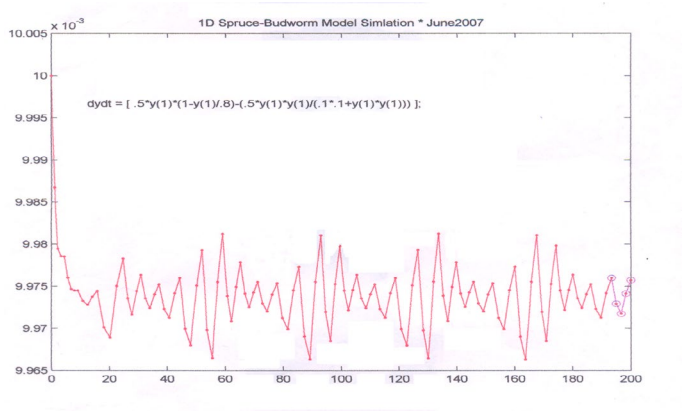


**Figure 17.** 1D Spruce-budworm model simulation.

$t_s$=[0, 200], r=0.5, k=0.8

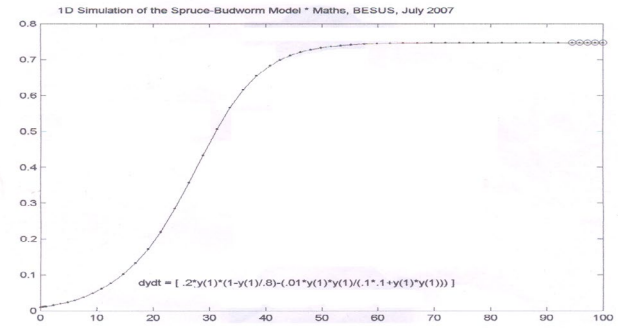


**Figure 18.** 1D Simulation of the spruce-budworm model.

## The numerical solution of spruce budworm problems

|*Solution of spruce budworm model*|

# include<stdio .h>

# include<math .h>

# include<conio.h>

float r, k, $a_1$, $b_1$

Float function (float x)

answer=1/ (($r \times x \times \left(1-\frac{x}{k}\right)$) – ($b1 \times x \times x$)/($a1 \times a1 + x \times x$)

return (answer);

}

void main ( )

{

int n;

float b, a, h, x, s1, s2, s4, intregal;

printf (" Enter the value of r\n");

scanf ("%f " , and r);

printf ("Enter the value of k\n");

scanf ("%f", and k );

printf ("Enter the value of a1\n");

scanf ("%f', and a1);

printf ("Enter the value of b1\n");

scanf ("%f", and b1);

printf ("Enter the value of upper limit\n");

scanf ("%f", and b);

printf ("Enter the value of lower limit\n");

scanf ("%f", and a);

h=(b-a)/n;

s1=function (a)+function (b);

s2=0;

x=a;

While (x<b)

{

h=x+2 × h;

s4=s4+function(x);

}

integral = (s1+4 × s4+s2) × ($\frac{h}{3.0}$);

printf ("our integral value is %f", integral);

printf ("our s1 value is %f", s1);

printf ("our s1 value is %f", s4);

printf ("our s1 value is %f", s2);

printf ("number of subinterval %d", n);

}

Output

'r=0.5

k=0.8

a1=0 .5

b1=0.1

upper limit=140

lower limit=20

subinterval=n=10

our integral value is -0.040718

our s1 value is -0.00424

our s4 value is -0.0010874

our s2 value is -0.001584

## Discussion and Conclusion

1. When the forest is young, so s is small, and R<$a_1$ , then there is only one small positive equilibrium point which sink, so the budworm population is controlled by birds, the equilibrium is kept at a low level, which we call refuge

2. When the forest grows, R passes $a_1$, then there are three positive equilibrium points, two of them are sinks, the refuge and a much larger one, which we call out break level. Outbreak level of budworm is dangerous for the forest. But since when the forest grows, the budworm is kept at refuge level, then it can jump to the outbreak level. Thus the forest is still in good shape since budworm is stalled at low level

3. But when the forest grows such that R passes $a_2$, there is only one equilibrium point left, which is the outbreak level. So the budworm population has a sudden increase in a short time we say an outbreak occurs

4. When an outbreak occurs the budworm population is in a height level, then the forest growth cannot keep up with the budworm, so R in fact

decreases but when R decreases, the budworm population cannot drop back to refuge level, which is called hysteresis effect

5. The fir tree forest are defoliated by budworm, and the forest is taken over by birch trees but they are less efficient at using nutrients and eventually the fir trees come back, but this recovery will take about 50-100 years

6. In general view, our results suggest that predator maturation delay and functional response are potentially the key causative factors generating population cycles. They appear to determine the cycle period and amplitude. Maturation delay has the strongest effect on the period whereas the functional response has the largest effect on amplitude and existence of the cycle. In particular, maturation periods that are too long or too short inhibit population cycles, as does a Holling Type I functional response

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**How to cite this article:** Maji, Dipak. "Spruce Budworm Model with and without Delay." J Generalized Lie Theory Appl 14 (2020): 304. doi: 10.37421/GLTA.2020.14.304.