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Opinion Open Access

Some Properties of Lie Algebras

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Opinion

Traditionally, Lie algebras have been used in physics in the context of symmetry groups of dynamical systems, as a powerful tool to study the underlying conservation laws [1,2]. At present, space-time symmetries and symmetries related to degrees of freedom are considered. For instance, non-trivial Heidelberg algebra arises right in the base of the Hamiltonian mechanics. Hamiltonian mechanics describes the state of a dynamic system with 2n variables (n coordinates and n momenta), and the other interesting observable physics quantities are functions of them. Kuranishi [3] proved that for any finite dimensional semisimple Lie algebra L over a field F of characteristic zero there exist two elements X, Y ∈ L which generate L. Work on simple Lie algebras of prime characteristic began nearly 75 years ago. Much of this work has concentrated on the case of restricted Lie algebras (also called Lie p-algebras). Robert Zeier and Zoltán Zimborás [4] given a subalgebra h of a compact semisimple Lie algebra g and a finite dimensional, faithful representation θ of g, then h=g iff dim(com [(θ ⊗ θ)/h])=dim(com[θ ⊗ θ]). Bai Ruipu, Gao Yansha and Li Zhengheng [5] proved L be a Lie algebra, D be an idempotent derivation. Then the image of D on L, is denoted by I=D(L), is an abelian ideal of L, and the kernel of D, is denoted by K=KerD is a subalgebra of L . Zhang Chengcheng, Zhang Qingcheng [6] Let L be a Lie color algebra. Then adL={adx | x ∈ L} is a Lie color subalgebra of End(L), which is said to be the inner derivation algebra, where a Lie color algebra is a G-graded F-vector space $L=\bigoplus_{g}\in_G$ Lg with the bilinear product [·,·]: L × L → L satisfying some conditions. Recently, David A. Towers [7] proved there are many interesting results concerning the question of what certain intrinsic properties of the maximal subalgebras of a Lie algebra L imply about the structure of L itself. In this paper, we give some properties on subalgebras and semisimple of Lie algebras with others concepts.

Preliminaries

A finite-dimensional Lie algebra is a finite-dimensional vector space L over a field F together with a map [·,·] : L × L → L, with the following properties: (1) [·,·] is bilinear: [x+v, y]=[x, y]+[v, y] for all x, v, y ∈ L and [αx, y]=α[x, y] for all x, y ∈ L and α ∈ F, [x, y+w]=[x, y]+[x,w] for all x, y, w ∈ L and [x,βy]=β[x, y] for all x, y ∈ L and β ∈ F, (2)[·,·] is skew-symmetric: [x, x]=0 for all x ∈ L, (3) The Jacobi identity holds: [x, [y, z]]+[y, [z, x]]+[z, [x, y]]=0 for all x, y, z ∈ L. A Lie algebra L is said to be semi-simple if Rad(L)=0. If $L=F^n$ then gl(L) is denoted gl(n, F). This is the vector space of all n×n matrices with coefficients in F with Lie bracket given by commutator: [xy]=xy-yx. A subalgebra is given by a subset of gl(n, F) which is closed under this bracket and under addition and scalar multiplication. Let sl(n, F) ⊆gl(n, F) denote the set of all n×n matrices with trace equal to zero. Given a Lie algebra g, dene the following descending sequences of ideals of g; (central series) $g>[g, g]=g^1>[[g, g],g]=g^3>[g^3, g]=g^4 \ldots [g^{k-1},g]=g^k \ldots$ (1) (Derived series) $g > [g, g]=g'>[g', g']=g'' \ldots>[g^{(k-1)}, g^{(k-1)}]=g^{(k)}>\ldots$ (2). A Lie algebra is called nilpotent (resp. solvable) if $g^k=0$ (resp. $g^{(k)}=0$) for k sufficiently large. A Lie algebra L is associative if [L,[L,L]]=0. In fact, any abelian or nilpotent Lie algebra is solvable Lie algebra and every solvable Lie algebra is semi simple Lie algebra. Let us define a bilinear form K: g×g → k by the formula K(a,b)=Tr(ad(a)ad(b)) (the trace of the composition of linear transformations ad(a) and ad(b), sending x ∈ g to [a[bx]]). It is called the Killing form of g. The Killing form is non-degenerate if for all y=0,κ(x,y)=0 implies x=0.

Proposition 1

Let sl(n, F) be the subset of gl(n, F) consisting of matrices with trace 0. This is an subalgebra of gl(n; F).

Proof: We have the relation sl(n,F) ⊆gl(n, F), so we not need to prove sl(n,F) is a linear subspace. Let $x=(x_{ij})$, $y=(y_{ij})$ ∈ sl(n, F).Then trace $xy=\sum_{k=1}^{n}\sum_{l=1}^{n} x_{kl}y_{lk}=\sum_{l=1}^{n}\sum_{k=1}^{n} y_{lk}x_{kl}$. Thus, we get the trace (xy-yx)=0 for all x, y ∈ gl(n, F), so in particular [x, y] ∈ sl(n, F) for all x,y ∈ sl(n, F),as required.

Proposition 2

Let L be Lie algebra and L^n is nilpotent Lie algebra, then

(i) if the Killing form κ(L^n, RadL) is non-degenerate then L is semi simple Lie algebra.

(ii) if the Killing form is κ(x, L^n), then κ is non-degenerate.

Proof: (i) According to our hypothesis, we have L^n is a nilpotent Lie algebra with the Killing form κ(L^n,RadL)=0. Suppose κ is non-degenerate which is implies Rad L=0. Therefore, L is semi simple Lie algebra as required.

(ii) We have we have L^n is a nilpotent Lie algebra with the Killing form defined as κ(x, L^n)=0.Since L^n=0,we get x=0. Therefore, κ is non-degenerate as required.

Proposition 3

If L be associative Lie algebra then it is semi simple.

Proof: According to our hypothesis, L is associative which means satisfying the following relation [L,[L,L]]=0. Now discuss the behavior of the bracket [L,[L,L]]. Firstly, if [L,L]=0. Then

L=[L,L]=0, which means $L^{(1)}$=[L,L].

$L^{(2)}=[L^{(1)},L^{(1)}]=[[L,L],[L,L]]=0,\ldots\ldots, L^{(n)}=[L^{(n-1)},L^{(n-1)}]=0$, which leads to $L^{(n)}$=0. Thus, L is solvable Lie algebra. In other words, L is semisimple Lie algebra. Secondly, if L=0. Then

L^1=[L,L]=0.

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$L^2=[L.L^1]=[L,[L,L]]=0.$

$L^3=[L,L^2]=[L,[L,[L,L]]]=0,\ldots, L^{n=}0$. Then L is a nilpotent Lie algebra, where any abelian or nilpotent Lie algebra is solvable Lie algebra, which means L is semi simple Lie algebra as required.

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References

1. Gilmore R (2005) Lie Groups, Lie Algebras, and Some of Their Applications. Dover Publications, Mineola, NY, USA.
2. Sattinger DH, Weaver OL (1986) Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics. Springer-Verlag, New York, USA.
3. Kuranishi M (1951) On everywhere dense imbeddings of free groups in Lie groups. Nagoya Math J 2: 63-71.
4. Zeier R, Zimborás Z (2015) On squares of representations of compact Lie algebras. Journal of Mathematical Physics.
5. Ruipu B, Yansha G, Zhengheng L (2015) Lie algebras with idempotent derivations. Mathematica Aeterna 5: 159-162.
6. Chengcheng Z, Qingcheng Z (2012) Some Results of Lie Color Algebras. Advances in Mathematics.
7. David AT (2015) On n-Maximal Subalgebras of Lie Algebras. Proceedings of the American Mathematical Society 144: 1457-1466.