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Some Characterizations of Involute-Evolute Curve Couple in the Hyperbolic Space

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Abstract

This paper aims to show that Frenet apparatus of an evolute curve can be formed according to apparatus of the involute curve in hyperbolic space. We establish relationships among Frenet frame on involute-evolute curve couple in hyperbolic 2-space, hyperbolic 3-space and de Sitter 3-space. Finally, we illustrate some numerical examples in support our main results.

Keywords: Frenet frames; Involute-evolute curves; Hyperbolic space; De Sitter space.

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Introduction

The idea of a string involute is due to, who is also known for his work in optics [1]. He discovered involutes while trying to develop a more accurate clock [2]. The involute of a given curve is a well-known concept in Euclidean 3-space $\mathbb{R}^3$ [3]. It is well-known that, if a curve is differentiable at each point of an open interval, a set of mutually orthogonal unit vectors can be constructed and called Frenet frame or Moving frame vectors [4]. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curves [5].

It is safe to report that the many important results in the theory of the curves in $\mathbb{R}^3$ were initiated by G. Monge, and G. Darboux pioneered the moving frame idea [6]. Thereafter, Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry. At the beginning of the twentieth century, Einstein's theory opened a door to use of new geometries [7]. One of them, Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold, was introduced and some of the classical differential geometry topics have been treated by the researchers [8]. In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifolds. For instance, in [9-11], the authors extended and studied spacelike involute-evolute curves in Euclidean 4-space and Minkowski space-time.

An evolute and its involute, are defined in mutual pairs. The evolute and the involute of the curve pair are well known by the mathematicians especially the differential geometry scientists. The evolute of any curve is defined as the locus of the centers of curvature of the curve. The original curves are then defined as the involute of the evolute. The simplest case is that of a circle, which has only one center of curvature (its center), which is a degenerate evolute and the circle itself is the involute of this point.

Izumiya, et al. defined the evolute curve in hyperbolic 2-space and found its equation. Following the works of them, we defined the evolute curve in hyperbolic 3-space and de Sitter 3-space and find its equations and for more details see [1,6]. In this paper, we calculate the Frenet apparatus of the evolute curve by apparatus of the involute curve in hyperbolic 2-space, hyperbolic 3-space, and de Sitter 3-space. Our results can be seen as refinement and generalization of many corresponding results exist in the literature and useful in mathematical modeling and some other applications.

Preliminaries

In this section, we use the basic notions and results in Lorentzian geometry for Frenet frame in hyperbolic 2-space, hyperbolic 3-space and de Sitter 3-space [3-8].

Hyperbolic 2-space

Let $\mathbb{R}^3=\{(x_1,x_2,x_3)\,|\,x_1,x_2,x_3\in\mathbb{R}\}$ be a 3-dimensional vector space, and $x(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ be two vectors in $\mathbb{R}^3$. The pseudo scalar product of x and y is defined by $\langle x,y\rangle=-x_1y_1+x_2y_2+x_3y_3$. We call $(\mathbb{R}^3,\langle,\rangle)$ a 3-dimensional pseudo Euclidean space, or Minkowski 3-space. We write $\mathbb{R}^3$ instead of $(\mathbb{R}^3,\langle,\rangle)$. We say that a vector x in $\mathbb{R}^3$ is spacelike, lightlike or timelike if $\langle x,x\rangle>0$, $\langle x,x\rangle=0$ or $\langle x,x\rangle<0$, respectively. We now define spheres in $\mathbb{E}_1^3$ as follows:

$$\begin{cases}\mathbb{H}_+^2=\{x\in\mathbb{E}_1^3\,|\,-x_1^2+x_2^2+x_3^2=-1,x_1\geq 1\}\\ \mathbb{H}_-^2=\{x\in\mathbb{E}_1^3\,|\,-x_1^2+x_2^2+x_3^2=-1,x_1\leq -1\}\\ \mathbb{S}_1^2=\{x\in\mathbb{E}_1^3\,|\,-x_1^2+x_2^2+x_3^2=1\}.\end{cases}$$

We call $\mathbb{H}_\pm^2$ a hyperbola and $\mathbb{S}_1^2$ a pseudo-sphere. Now, we discuss some basic facts of curves in Hyperbolic 2-space, which are needed in the sequel.

Let $\gamma:I\to\mathbb{H}_+^2\subset\mathbb{E}_1^3;\ \ \gamma(t)=(x_1(t),x_2(t),x_3(t))$ be a smooth regular curve in $\mathbb{H}_+^2$ $(i.e.,\gamma'(t)\neq 0)$ for any $t\in I$, where I is an open interval.

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It is easy to show that $\langle \gamma'(t),\gamma'(t)\rangle > 0$, for any $t \in I$. We call such a curve a spacelike curve. The norm of the vector $x \in \mathbb{E}_1^3$ is defined by $\| x \| = \sqrt{|\langle x,x\rangle|}$. The arc-length of a spacelike curve γ, measured from $\gamma(t_0), t_0 \in I$ is $s(t) = \int_{t_0}^{t} \| \gamma'(t) \| dt$. Then the parameter s is determined such that $\| \dot{\gamma}(s) \| = 1$, where $\dot{\gamma}(s) = \frac{d\gamma(s)}{ds}$. So we say that a spacelike curve γ is parameterized by arc-length, if it satisfies $\| \dot{\gamma}(s) \| = 1$. Throughout the remainder in this paper, we denote the parameter s of γ as the arc-length parameter. Let us denote $\mathbf{T}(s) = \dot{\gamma}(s)$, and we call $\boldsymbol{T}(s)$ a unit tangent vector of γ at s.

For any $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in \mathbb{E}_1^3$, the pseudo vector product of x and y is defined as follows:

$$x \wedge y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

We remark that $\langle x \wedge y, z\rangle = det(x\ y\ z)$. Hence, $x \wedge y$ is pseudo-orthogonal to x,y. We now set a vector $\mathbf{E}(s) = \gamma(s) \wedge \mathbf{T}(s)$. By definition, we can calculate that $\langle \mathbf{E}(s), \mathbf{E}(s)\rangle = 1$ and $\langle \gamma(s), \gamma(s)\rangle = -1$. We can also show that $\mathbf{T}(s) \wedge \mathbf{E}(s) = -\gamma(s)$ and $\gamma(s) \wedge \mathbf{E}(s) = -\mathbf{T}(s)$. Therefore, we have a pseudo-orthonormal frame $\{\gamma(s), \mathbf{T}(s), \mathbf{E}(s)\}$ along $\gamma(s)$. We have the following hyperbolic Frenet-Serret formula of plane curves:

$$3\begin{cases} \dot{\gamma}(s) = \mathbf{T}(s) \\ \dot{\mathbf{T}}(s) = \gamma(s) + \kappa_g(s)\mathbf{E}(s) \\ \dot{\mathbf{E}}(s) = -\kappa_g(s)\mathbf{T}(s), \end{cases} \tag{1}$$

or in the matrix form:

$$\begin{bmatrix} \dot{\gamma}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \mathbf{T}(s) \\ \mathbf{E}(s) \end{bmatrix} \tag{2}$$

where k_g is the geodesic curvature of the curve γ in $\mathbb{H}_+^2$, which is given by

$$\kappa_g(s) = det(\gamma(s)\ \mathbf{T}(s)\ \dot{\mathbf{T}}(s)).$$

Hyperbolic 3-space

Let $\mathbb{R}^4$ be a four-dimensional vector space. For any $x = (x_1,x_2,x_3,x_4), y = (y_1,y_2,y_3,y_4) \in \mathbb{R}^4$, the pseudo-scalar product of x and y is defined by $\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$.

We call $(R^4, \langle,\rangle)$ Minkowski 4-space and denoted by $\mathbb{E}_1^4$. We say that a vector $x \in \mathbb{E}_1^4$ is spacelike, lightlike or timelike if $\langle x_1,x_2\rangle > 0, \langle x_1,x_2\rangle = 0$ or $\langle x_1,x_2\rangle < 0$, respectively. The norm of the vector $x \in \mathbb{E}_1^4$ is defiend by $\| x \| = \sqrt{|\langle x,x\rangle|}$. For a non-zero vector $\upsilon \in E_1^4$ and a real number c, we define a space with pseudo normal υ by

$$\mathbb{S}(\upsilon,c) = \{x \in \mathbb{E}_1^4 \mid \langle x,\upsilon\rangle = c\}.$$

We call $\mathbb{S}(\upsilon,c)$ a spacelike space, a timelike space or a lightlike space if υ is timelike, spacelike or lightlike, respectively.

Now, we define a hyperbolic space by

$$\mathbb{H}_+^3(-1) = \{x \in \mathbb{E}_1^4 \mid \langle x,x\rangle = -1, x_1 > 0\}.$$

For any $x = (x_1,x_2,x_3,x_4)$, $y = (y_1,y_2,y_3,y_4)$ and $z = (z_1,z_2,z_3,z_4)$ $\in \mathbb{E}_1^4$, the pseudo vector product of x,y and z is defined as follows:

$$x \wedge y \wedge z = \begin{vmatrix} -i & j & k & l \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

$$= \left(-\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix}, -\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \right).$$

We now prepare some basic facts of curves in hyperbolic 3-space.

Let $\gamma : I \to \mathbb{H}_+^3 \subset \mathbb{E}_1^4;\ \gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a smooth regular curve in $\mathbb{H}_+^3$ $(i.e., \gamma'(t) \neq 0)$ for any $t \in I$ where I is an open interval. So that, $\langle \gamma'(t), \gamma'(t)\rangle > 0$ for any $t \in I$. The arc-length of γ, measured from $\gamma(t_\circ), t_\circ \in I$ is $s(t) = \int_{t_\circ}^{t} \| \gamma'(t) \| dt$. Then the parameter s is determined such that $\| \dot{\gamma}(s) \| = 1$, where $\dot{\gamma}(s) = \frac{d\gamma(s)}{ds}$. So we say that, a spacelike curve γ is parameterized by arc-length if it satisfies that $\| \dot{\gamma}(s) \| = 1$. Let us denote $\mathbf{T}(s) = \dot{\gamma}(s)$, and we call $\mathbf{T}(s)$ a unit tangent vector of γ at s.

Here, we construct the explicit differential geometry on curves in $\mathbb{H}_+^3(-1)$. Let $\gamma : I \to \mathbb{H}_+^3(-1)$ be a regular curve. Since $\mathbb{H}_+^3(-1)$ is a Riemannian manifold, we can reparameterize γ by the arc-length. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So we have the tangent vector $\mathbf{T}(s) = \dot{\gamma}(s)$ with $\| \mathbf{T} \| = 1$. In case, when $\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq -1$, then we have a unit vector

$$\mathbf{N}(s) = \frac{\dot{\mathbf{T}}(s) - \gamma(s)}{\| \dot{\mathbf{T}}(s) - \gamma(s) \|}.$$

Moreover, define $\mathbf{E}(s) = \gamma(s) \wedge \mathbf{T}(s) \wedge \mathbf{N}(s)$, then we have a pseudo orthonormal frame $\{\gamma(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{E}(s)\}$ of $\mathbb{E}_1^4$ along γ. By standard arguments, under the assumption that $\left\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \right\rangle \neq -1$, we have the following Frenet formulae:

$$\begin{cases} \dot{\gamma}(s) = \mathbf{T}(s), \\ \dot{\mathbf{T}}(s) = \gamma(s) + \kappa_g \mathbf{N}(s), \\ \dot{\mathbf{N}}(s) = -\kappa_g \mathbf{T}(s) + \tau_g \mathbf{E}(s), \\ \dot{\mathbf{E}}(s) = -\tau_g \mathbf{N}(s). \end{cases} \tag{3}$$

Or in the matrix form:

$$\begin{bmatrix} \dot{\gamma}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{N}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \kappa_g & 0 \\ 0 & -\kappa_g & 0 & \tau_g \\ 0 & 0 & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{E}(s) \end{bmatrix}$$

where

$$\kappa_g = \| \dot{\mathbf{T}}(s) - \gamma(s) \|, \tag{4}$$

$$\tau_g = -\frac{det\left(\gamma(s), \dot{\gamma}(s), \ddot{\gamma}(s), \dddot{\gamma}(s)\right)}{(\kappa_g(s))^2},$$

are the geodesic curvature and geodesic torsion respectively, of the curve γ in $\mathbb{H}_+^3(-1)$.

Since $\langle \dot{\mathbf{T}}(s)-\gamma(s), \dot{\mathbf{T}}(s)-\gamma(s)\rangle = \langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s)\rangle + 1$, the condition

$\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s)\rangle \neq -1$,

is equivalent to the condition $\kappa_g(s)\neq 0$. Moreover, we can show that the curve $\gamma(s)$ satisfies the condition $\kappa_g(s)\equiv 0$ if and only if there exists a lightlike vector c such that $\gamma(s)-c$ is a geodesic. Such a curve is called an equidistant curve (see [5,8]).

De Sitter 3-space

Moreover, in this subsection we introduce some basic facts which we will use in this paper. Let $\mathbb{R}^4$ be a four-dimensional vector space. For any $x=(x_1,x_2,x_3,x_4), y=(y_1,y_2,y_3,y_4)\in\mathbb{R}^4$, the pseudo-scalar product of x and y is defined by $\langle x,y\rangle = -x_1y_1+x_2y_2+x_3y_3+x_4y_4$. We call $(\mathbb{R}^4,\langle,\rangle)$ Minkowski 4-space. We write $\mathbb{E}_1^4$ instead of $(\mathbb{R}^4,\langle,\rangle)$. We say that a vector $x\in\mathbb{E}_1^4$ is spacelike, lightlike or timelike if $\langle x_1,x_2\rangle>0, \langle x_1,x_2\rangle=0$ or $\langle x_1,x_2\rangle<0$, respectively. The norm of the vector $x\in\mathbb{E}_1^4$ is defined by $\|x\|=\sqrt{|\langle x,x\rangle|}$.

We now define de Sitter 3-space by

$\mathbb{S}_1^3=\{x\in\mathbb{E}_1^4 \mid \langle x,x\rangle = 1\}$.

For any $x=(x_1,x_2,x_3,x_4), y=(y_1,y_2,y_3,y_4), z=(z_1,z_2,z_3,z_4)\in\mathbb{E}_1^4$, the pseudo vector product of x,y and z is defined as follows:

$$x\wedge y\wedge z=\begin{vmatrix} -e_1 & e_2 & e_3 & e_3\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix}.$$

Let $\gamma: I\to\mathbb{S}_1^3$ be a smooth and regular spacelike curve in $\mathbb{S}_1^3$. We can parameterize it by arc-length s, since $\mathbb{S}_1^3$ is a Riemannian manifold, we can reparameterize γ by the arc-length. Hence, we may assume that $\gamma(s)$ is a unit speed curve. So, we have the tangent vector $\mathbf{T}(s)=\dot{\gamma}(s)$ with $\|\mathbf{T}\|=1$. In this case, we call γ a unit speed spacelike curve. If $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s)\rangle\neq 1$, then $\|\dot{\mathbf{T}}(s)+\gamma(s)\|\neq 0$, and we define the unit vector $\mathbf{N}(s)=\dfrac{\dot{\mathbf{T}}(s)+\gamma(s)}{\|\dot{\mathbf{T}}(s)+\gamma(s)\|}$. Moreover, define $\mathbf{E}(s)=\gamma(s)\wedge\mathbf{T}(s)\wedge\mathbf{N}(s)$, then we have a pseudo orthonormal frame $\{\gamma(s),\mathbf{T}(s),\mathbf{N}(s),\mathbf{E}(s)\}$ of $\mathbb{E}_1^4$ along γ. By standard arguments, under the assumption that $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s)\rangle\neq 1$, we have the following Frenet-Serret type formula:

$$\begin{cases} \dot{\gamma}(s)=\mathbf{T}(s)\\ \dot{\mathbf{T}}(s)=-\gamma(s)+\kappa_g\mathbf{N}(s)\\ \dot{\mathbf{N}}(s)=-\delta(\gamma)\kappa_g\mathbf{T}(s)+\tau_g\mathbf{E}(s)\\ \dot{\mathbf{E}}(s)=\tau_g\mathbf{N}(s).\end{cases} \tag{5}$$

Or in the matrix form:

$$\begin{bmatrix}\dot{\gamma}(s)\\ \dot{\mathbf{T}}(s)\\ \dot{\mathbf{N}}(s)\\ \dot{\mathbf{E}}(s)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & \kappa_g & 0\\ 0 & -\delta(\gamma)\kappa_g & 0 & \tau_g\\ 0 & 0 & \tau_g & 0\end{bmatrix}\begin{bmatrix}\gamma(s)\\ \mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{E}(s)\end{bmatrix}$$

where $\delta(\gamma)=sign\,(\mathbf{N}(s))$ (which we shall write as simply δ) and

$$\begin{cases} \kappa_g=\|\dot{\mathbf{T}}(s)+\gamma(s)\|,\\ \tau_g=\dfrac{\delta\det(\gamma(s),\dot{\gamma}(s),\ddot{\gamma}(s),\dddot{\gamma}(s))}{(\kappa_g(s))^2},\end{cases} \tag{6}$$

are the geodesic curvature and geodesic torsion respectively, of the curve γ in $\mathbb{S}_1^3$.

Since $\langle \dot{\mathbf{T}}(s)+\gamma(s), \dot{\mathbf{T}}(s)+\gamma(s)\rangle = \langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s)\rangle - 1$, the condition $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s)\rangle \neq 1$ is equivalent to the condition $\kappa_g(s)\neq 0$ (see [7]).

The Frenet Apparatus of an Evolute Curve in Hyperbolic 2-Space

In this section, we introduce the Frenet apparatus of an evolute curve according to the Frenet apparatus of the involute curve in $\mathbb{H}_+^2$.

Definition 3.1 :We define the hyperbolic evolute curve of $\gamma(s)$ under the assumption that $\kappa_g^2(s)\neq\pm 1$ in $\mathbb{H}_+^2$ as;

$$\alpha(s)=\frac{1}{\sqrt{|\kappa_g^2(s)-1|}}\left(\kappa_g(s)\gamma(s)+\mathbf{E}(s)\right).$$

We remark that $\alpha(s)$ is located in $\mathbb{H}_+^2\cup\mathbb{H}_-^2$ if and only if $\kappa_g^2>1$. If $\alpha(s)$ is located in $\mathbb{H}_-^2$, we may consider $-\alpha(s)$ instead of $\alpha(s)$ and we call $\gamma(s)$ an involute curve of $\alpha(s)$ (for more details see [6]).

The Frenet apparatus of an evolute curve $\alpha(s)$ denoted by

$\{\alpha(s),\mathbf{T}_\alpha(s),\mathbf{E}_\alpha(s),\mathcal{K}_\alpha(s)\}$ can be formed according to the Frenet apparatus of the involute curve $\gamma(s)$.

Theorem 1: If $\alpha(s)$ is a unit speed space-like curve and $\alpha(s)$ is an evolute curve of $\gamma(s)$. Then the Frenet apparatus of the evolute curve $\alpha(s)$ is as follows:

$$\begin{cases}\alpha(s)=\dfrac{1}{\sqrt{|\kappa_g^2-1|}}\left(\kappa_g\gamma(s)+\mathbf{E}(s)\right),\\ \mathbf{T}_\alpha(s)=\dfrac{-\dot{\kappa}_g}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\left(\gamma(s)+\kappa_g\mathbf{E}(s)\right),\end{cases}\quad \begin{cases}\mathbf{E}_\alpha(s)=\left(\dfrac{\dot{\kappa}_g}{\kappa_g^2-1}\right)\mathbf{T}(s),\\ \mathcal{K}_\alpha(s)=\dfrac{\dot{\kappa}_g{}^2}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}.\end{cases} \tag{7}$$

Proof. It follows from the definition of the evolute curve in hyperbolic 2-space that

$$\alpha(s)=\frac{1}{\sqrt{|\kappa_g^2-1|}}\left(\kappa_g\gamma(s)+\mathbf{E}(s)\right). \tag{8}$$

Differentiating both sides of the previous equation with respect to s and substitute from Eqs.(1), we obtain

$$\mathbf{T}_\alpha(s)=\frac{-\kappa_g\dot{\kappa}_g}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\left(\kappa_g\gamma+\mathbf{E}\right)+\frac{1}{\sqrt{|\kappa_g^2-1|}}\left(\dot{\kappa}_g\gamma+\kappa_g\mathbf{T}-\kappa_g\mathbf{T}\right),$$

which, can be written as

$$\mathbf{T}_\alpha(s)=\frac{-\kappa_g\dot{\kappa}_g}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\left(\kappa_g\gamma+\mathbf{E}\right)+\frac{\kappa_g^2-1}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\left(\dot{\kappa}_g\gamma+\kappa_g\mathbf{T}-\kappa_g\mathbf{T}\right)$$

$$=\frac{-\kappa_g^2\dot{\kappa}_g}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\gamma-\frac{\kappa_g\dot{\kappa}_g}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\mathbf{E}+\frac{\kappa_g^2\dot{\kappa}_g}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\gamma-\frac{\dot{\kappa}_g}{\left(|\kappa_g^2-1|\right)^{\frac{3}{2}}}\gamma,$$

then, we get

$$\mathbf{T}_\alpha(s) = \frac{-\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}\gamma(s) - \frac{\kappa_g\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}\mathbf{E}(s). \tag{9}$$

Moreover, from definition of the vector $E_\alpha(s)$, where

$$\mathbf{E}_\alpha(s) = \alpha(s)\wedge\mathbf{T}_\alpha(s),$$

we have

$$\mathbf{E}_\alpha(s) = \begin{vmatrix} -\gamma(s) & \mathbf{T}(s) & \mathbf{E}(s) \\ \frac{\kappa_g}{\sqrt{\left|\kappa_g^2-1\right|}} & 0 & \frac{1}{\sqrt{\left|\kappa_g^2-1\right|}} \\ \frac{-\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}} & 0 & \frac{-\kappa_g\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}} \end{vmatrix},$$

then, we get

$$\mathbf{E}_\alpha(s) = \left(\frac{\dot{\kappa}_g}{\kappa_g^2-1}\right)\mathbf{T}(s).$$

Also, by differentiating the eqn.(9) and substitute from eqn.(1), we have

$$2\dot{\mathbf{T}}_\alpha(s) = \begin{cases} \left(\frac{\ddot{\kappa}_g - \kappa_g^2\ddot{\kappa}_g + 3\kappa_g\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right)\gamma(s) + \frac{\dot{\kappa}_g}{\sqrt{\left|\kappa_g^2-1\right|}}\mathbf{T}(s) \\ + \left(\frac{\kappa_g\ddot{\kappa}_g - \kappa_g^3\ddot{\kappa}_g + 3\kappa_g^2\dot{\kappa}_g - \kappa_g^2\dot{\kappa}_g^{\,2} + \dot{\kappa}_g^{\,2}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right)\mathbf{E}(s), \end{cases} \tag{10}$$

where

$$\mathcal{K}_\alpha = \det\left(\alpha(s)\ \mathbf{T}_\alpha\ \dot{\mathbf{T}}_\alpha\right).$$

In eqn. (8), (9) and (10), we have

$$\mathcal{K}_\alpha(s) = \begin{vmatrix} \frac{\kappa_g}{\sqrt{\left|\kappa_g^2-1\right|}} & 0 & \frac{1}{\sqrt{\left|\kappa_g^2-1\right|}} \\ \frac{-\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}} & 0 & \frac{-\kappa_g\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}} \\ \frac{\ddot{\kappa}_g - \kappa_g^2\ddot{\kappa}_g + 3\kappa_g\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}} & \frac{\dot{\kappa}_g}{\sqrt{\left|\kappa_g^2-1\right|}} & \frac{\kappa_g\ddot{\kappa}_g - \kappa_g^3\ddot{\kappa}_g + 3\kappa_g^2\dot{\kappa}_g - \kappa_g^2\dot{\kappa}_g^2 + \dot{\kappa}_g^2}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}} \end{vmatrix},$$

from the previous determinant, we have

$$\mathcal{K}_\alpha = \left(\frac{\kappa_g}{\sqrt{\left|\kappa_g^2-1\right|}}\right)\left(\frac{\dot{\kappa}_g}{\sqrt{\left|\kappa_g^2-1\right|}}\right)\left(\frac{\kappa_g\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}\right)$$

$$-\left(\frac{1}{\sqrt{\left|\kappa_g^2-1\right|}}\right)\left(\frac{\dot{\kappa}_g}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}\right)\left(\frac{\dot{\kappa}_g}{\sqrt{\left|\kappa_g^2-1\right|}}\right)$$

$$= \left(\frac{\kappa_g^2\dot{\kappa}_g^{\,2}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right) - \left(\frac{\dot{\kappa}_g^{\,2}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right)$$

$$= \left(\frac{\dot{\kappa}_g^{\,2}\left(\kappa_g^2-1\right)}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right),$$

then, we get

$$\mathcal{K}_\alpha = \frac{\dot{\kappa}_g^{\,2}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}.$$

This completes the prove.

The Frenet Apparatus of an Evolute Curve in Hyperbolic 3-Space

In this section, we study the Frenet apparatus of an evolute curve according to the Frenet apparatus of an involute curve in hyperbolic 3-space and define the equation of the evolute curve in $\mathbb{H}_+^3(-1)$.

Definition 4.1

We define the hyperbolic evolute curve $\beta : I \to \mathbb{H}_+^3(-1)$ of $\gamma(s)$ by

$$\beta(s) = \frac{1}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - 1\right|}}\left(\kappa_g\gamma(s) + \mathbf{N}(s) - \frac{\dot{\kappa}_g}{\kappa_g\tau_g}\mathbf{E}(s)\right),$$

under the assumption that $\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 > 1$.

We remark that $\beta(s)$ is located in $\mathbb{H}_+^3(-1)$ if and only if $\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 > 1$, where $\gamma(s)$ is an involute curve of $\beta(s)$ (see [1]).

The Frenet apparatus of an evolute curve in $\mathbb{H}_+^3(-1)$ denoted by

$$\left\{\beta(s), \mathbf{T}_\beta(s), \mathbf{N}_\beta(s), \mathbf{E}_\beta(s), \mathcal{K}_\beta(s), \mathcal{T}_\beta(s)\right\},$$

can be formed according to the Frenet apparatus of an involute curve $\gamma(s)$.

Theorem 2 : If $\beta(s)$ is a unit speed spacelike curve and an evolute curve of $\gamma(s)$. Then the Frenet apparatus of the evolute curve $\beta(s)$ is as follows:

Proof: It follows from the definition (4.1) of the evolute curve in hyperbolic 3-space that

$$3\beta(s) = \frac{\kappa_g}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - 1\right|}}\gamma + \frac{1}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - 1\right|}}\mathbf{N} - \frac{\dot{\kappa}_g}{\kappa_g\tau_g\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - 1\right|}}\mathbf{E}.$$

If we denote

$$\mu_1 = \frac{\kappa_g}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - 1\right|}},\ \mu_2 = \frac{1}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - 1\right|}},$$

$$\mu_3 = -\frac{\dot{\kappa}_g}{\kappa_g\tau_g\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - 1\right|}}. \tag{11}$$

Then, the evolute curve $\beta(s)$ can be written in the formula

$$\beta(s) = \mu_1\gamma(s) + \mu_2\mathbf{N}(s) + \mu_3\mathbf{E}(s). \tag{12}$$

Differentiating both sides of the previous equation with respect to s and substituted from Eqs.(3), we obtain

$$\begin{aligned}\mathbf{T}_\beta(s) &= \dot{\mu}_1\gamma(s) + \mu_1\dot{\gamma}(s) + \dot{\mu}_2\mathbf{N}(s) + \mu_2\dot{\mathbf{N}}(s) + \dot{\mu}_3\mathbf{E}(s) + \mu_3\dot{\mathbf{E}}(s)\\ &= \dot{\mu}_1\gamma(s) + \mu_1\mathbf{T}(s) + \dot{\mu}_2\mathbf{N}(s) + \mu_2(-\kappa_g\mathbf{T}(s) + \tau_g\mathbf{E}(s))\\ &+\dot{\mu}_3\mathbf{E}(s) - \mu_3(\tau_g\mathbf{N}(s))\\ &= \dot{\mu}_1\gamma(s) + (\mu_1 - \kappa_g\mu_2)\mathbf{T}(s) + (\dot{\mu}_2 - \tau_g\mu_3)\mathbf{N}(s)\\ &+(\dot{\mu}_3 + \tau_g\mu_2)\mathbf{E}(s).\end{aligned} \tag{13}$$

Now, we need to find $\dot{\mathbf{T}}_\beta(s)$, by differentiating Eq.(??), we have

$$3\dot{\mathbf{T}}_\beta(s) = \begin{cases}(\ddot{\mu}_1 + \mu_1 - \kappa_g\mu_2)\gamma(s) + (2\dot{\mu}_1 - \dot{\kappa}_g\mu_2 - 2\kappa_g\dot{\mu}_2 + \kappa_g\tau_g\mu_3)\mathbf{T}(s)\\ +(\kappa_g\mu_1 - \kappa_g^2\mu_2 + \ddot{\mu}_2 - \dot{\tau}_g\mu_3 - 2\tau_g\dot{\mu}_3 - \tau_g^2\mu_2)\mathbf{N}(s)\\ +(\tau_g\dot{\mu}_2 - \tau_g^2\mu_3 + \ddot{\mu}_3 + \dot{\tau}_g\mu_2 + \tau_g\dot{\mu}_2)\mathbf{E}(s),\end{cases} \tag{14}$$

or in another form

$$\dot{\mathbf{T}}_\beta(s) = \eta_1\gamma(s) + \eta_2\mathbf{T}(s) + \eta_3\mathbf{N}(s) + \eta_4\mathbf{E}(s), \tag{15}$$

where

$$3\begin{cases}\eta_1 = (\ddot{\mu}_1 + \mu_1 - \kappa_g\mu_2)\\ \eta_2 = (2\dot{\mu}_1 - \dot{\kappa}_g\mu_2 - 2\kappa_g\dot{\mu}_2 + \kappa_g\tau_g\mu_3)\\ \eta_3 = (\kappa_g\mu_1 - \kappa_g^2\mu_2 + \ddot{\mu}_2 - \dot{\tau}_g\mu_3 - 2\tau_g\dot{\mu}_3 - \tau_g^2\mu_2)\\ \eta_4 = (\tau_g\dot{\mu}_2 - \tau_g^2\mu_3 + \ddot{\mu}_3 + \dot{\tau}_g\mu_2 + \tau_g\dot{\mu}_2).\end{cases} \tag{16}$$

Thus, in eqn.(12) and (14), we can compute the value of the vector $N_\beta(s)$, where

$$\mathbf{N}_\beta(s) = \frac{\dot{\mathbf{T}}_\beta(s) - \beta(s)}{\|\dot{\mathbf{T}}_\beta(s) - \beta(s)\|},$$

It follows that

$$\begin{aligned}\mathbf{N}_\beta(s) &= \left(\left|-(\eta_1 - \mu_1)^2 + \eta_2^2 + (\eta_3 - \mu_2)^2 + (\eta_4 - \mu_3)^2\right|\right)^{\frac{-1}{2}}(((\eta_1 - \mu_1)\gamma(s)\\ &+\eta_2\mathbf{T}(s) + (\eta_3 - \mu_2)\mathbf{N}(s) + (\eta_4 - \mu_3)\mathbf{E}(s)))\end{aligned} \tag{17}$$

Also, in eqn.(16) and eqn.(17), we obtain

$$\mathcal{K}_\beta = \sqrt{\left|-(\eta_1 - \mu_1)^2 + \eta_2^2 + (\eta_3 - \mu_2)^2 + (\eta_4 - \mu_3)^2\right|}.$$

Therefore, by differentiating in eqn. (14) and from Frenet formulae, we can obtain

$$\begin{aligned}\ddot{\beta}(s) &= (\dddot{\mu}_1 + 3\dot{\mu}_1 - 2\dot{\kappa}_g\mu_2 - 3\kappa_g\dot{\mu}_2 + \kappa_g\tau_g\mu_3)\gamma(s) + (3\ddot{\mu}_1 + \mu_1 - \kappa_g\mu_2 - \ddot{\kappa}_g\mu_2\\ &-3\dot{\kappa}_g\dot{\mu}_2 - 3\kappa_g\ddot{\mu}_2 + \dot{\kappa}_g\tau_g\mu_3 + 2\kappa_g\dot{\tau}_g\mu_3 + 3\kappa_g\tau_g\dot{\mu}_3)\mathbf{T}(s) + (3\kappa_g\dot{\mu}_1 - 3\kappa_g\dot{\kappa}_g\mu_2\\ &-3\kappa_g^2\dot{\mu}_2 + \dot{\kappa}_g\mu_1 + \kappa_g^2\tau_g\mu_3 - 3\tau_g\dot{\tau}_g\mu_2 - 2\tau_g^2\dot{\mu}_2 + \dddot{\mu}_2 - \ddot{\tau}_g\mu_3 - 3\dot{\tau}_g\dot{\mu}_3 - 3\tau_g\ddot{\mu}_3\\ &-\tau_g^2\dot{\mu}_2 - \tau_g^2\mu_3)\mathbf{N}(s) + (\kappa_g\tau_g\mu_1 - \kappa_g^2\tau_g\mu_2 - \tau_g^3\mu_2 + 3\tau_g\ddot{\mu}_2 - 3\tau_g\dot{\tau}_g\mu_3 + \dddot{\mu}_3\\ &-3\tau_g^2\dot{\mu}_3 + 3\dot{\tau}_g\dot{\mu}_2 + \ddot{\tau}_g\mu_2)\mathbf{E}(s),\end{aligned} \tag{18}$$

which, can be written as

$$\ddot{\beta}(s) = \zeta_1\gamma(s) + \zeta_2\mathbf{T}(s) + \zeta_3\mathbf{N}(s) + \zeta_4\mathbf{E}(s), \tag{19}$$

i.e.,

$$2\begin{cases}\zeta_1 = (\dddot{\mu}_1 + 3\dot{\mu}_1 - 2\dot{\kappa}_g\mu_2 - 3\kappa_g\dot{\mu}_2 + \kappa_g\tau_g\mu_3)\\ \zeta_2 = (3\ddot{\mu}_1 + \mu_1 - \kappa_g\mu_2 - \ddot{\kappa}_g\mu_2 - 3\dot{\kappa}_g\dot{\mu}_2 - 3\kappa_g\ddot{\mu}_2\\ \quad +\dot{\kappa}_g\tau_g\mu_3 + 2\kappa_g\dot{\tau}_g\mu_3 + 3\kappa_g\tau_g\dot{\mu}_3)\\ \zeta_3 = (3\kappa_g\dot{\mu}_1 - 3\kappa_g\dot{\kappa}_g\mu_2 - 3\kappa_g^2\dot{\mu}_2 + \dot{\kappa}_g\mu_1 + \kappa_g^2\tau_g\mu_3\\ \quad -3\tau_g\dot{\tau}_g\mu_2 - 2\tau_g^2\dot{\mu}_2 + \dddot{\mu}_2 - \ddot{\tau}_g\mu_3 - 3\dot{\tau}_g\dot{\mu}_3 - 3\tau_g\ddot{\mu}_3\\ \quad -\tau_g^2\dot{\mu}_2 - \tau_g^2\mu_3)\\ \zeta_4 = (\kappa_g\tau_g\mu_1 - \kappa_g^2\tau_g\mu_2 - \tau_g^3\mu_2 + 3\tau_g\ddot{\mu}_2 - 3\tau_g\dot{\tau}_g\mu_3\\ \quad +\dddot{\mu}_3 - 3\tau_g^2\dot{\mu}_3 + 3\dot{\tau}_g\dot{\mu}_2 + \ddot{\tau}_g\mu_2).\end{cases} \tag{20}$$

In eqn. (12), (16), (15), (19), and (20), we can compute the torsion T_β of the evolute curve $\beta(s)$ as

$$\mathcal{T}_\beta(s) = -\frac{1}{\mathcal{K}_\beta^2}\begin{vmatrix}\mu_1 & 0 & \mu_2 & \mu_3\\ \dot{\mu}_1 & (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2)\\ \eta_1 & \eta_2 & \eta_3 & \eta_4\\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4\end{vmatrix},$$

or in the form

$$\mathcal{T}_\beta(s) = \frac{1}{\mathcal{K}_\beta^2}\begin{cases}(\mu_1((\mu_1 - \kappa_g\mu_2)(\eta_3\zeta_4 - \eta_4\zeta_3) - (\dot{\mu}_2 - \tau_g\mu_3)(\eta_2\zeta_4 - \eta_4\zeta_2)\\ \quad +(\dot{\mu}_3 - \tau_g\mu_2)(\eta_2\zeta_3 - \eta_3\zeta_2))\\ -\mu_2(\dot{\mu}_1(\eta_2\zeta_4 - \eta_4\zeta_2) - (\mu_1 - \kappa_g\mu_2)(\eta_1\zeta_4 - \eta_4\zeta_1)\\ \quad +(\dot{\mu}_3 - \tau_g\mu_2)(\eta_1\zeta_2 - \eta_2\zeta_1))\\ +\mu_3(\dot{\mu}_1(\eta_2\zeta_3 - \eta_3\zeta_2) - (\mu_1 - \kappa_g\mu_2)(\eta_1\zeta_3 - \eta_3\zeta_1)\\ \quad +(\dot{\mu}_2 - \tau_g\mu_3)(\eta_1\zeta_2 - \eta_2\zeta_1))).\end{cases} \tag{21}$$

Also, in eqn. (12-14), we can compute the equation of the vector $\mathbf{E}_\beta(s)$, where

$$\mathbf{E}_\beta(s) = \beta(s) \wedge \mathbf{T}_\beta(s) \wedge \mathbf{N}_\beta(s),$$

leads us to

$$\mathbf{E}_\beta(s) = \frac{1}{\mathcal{K}_\beta}\begin{vmatrix}-\gamma(s) & \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{E}(s)\\ \mu_1 & 0 & \mu_2 & \mu_3\\ \dot{\mu}_1 & (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2)\\ (\eta_1 - \mu_1) & \eta_2 & (\eta_3 - \mu_2) & (\eta_4 - \mu_3)\end{vmatrix},$$

or in the format

$$\begin{aligned}\mathbf{E}_\beta(s) &= -\frac{1}{\mathcal{K}_\beta}\begin{vmatrix}0 & \mu_2 & \mu_3\\ (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2)\\ \eta_2 & (\eta_3 - \mu_2) & (\eta_4 - \mu_3)\end{vmatrix}\gamma(s)\\ &-\frac{1}{\mathcal{K}_\beta}\begin{vmatrix}\mu_1 & \mu_2 & \mu_3\\ \dot{\mu}_1 & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2)\\ (\eta_1 - \mu_1) & (\eta_3 - \mu_2) & (\eta_4 - \mu_3)\end{vmatrix}\mathbf{T}(s)\\ &+\frac{1}{\mathcal{K}_\beta}\begin{vmatrix}\mu_1 & 0 & \mu_3\\ \dot{\mu}_1 & (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_3 + \tau_g\mu_2)\\ (\eta_1 - \mu_1) & \eta_2 & (\eta_4 - \mu_3)\end{vmatrix}\mathbf{N}(s)\\ &-\frac{1}{\mathcal{K}_\beta}\begin{vmatrix}\mu_1 & 0 & \mu_2\\ \dot{\mu}_1 & (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_2 - \tau_g\mu_3)\\ (\eta_1 - \mu_1) & \eta_2 & (\eta_3 - \mu_2)\end{vmatrix}\mathbf{E}(s),\end{aligned}$$

which completes the proof.

The Frenet Apparatus of an Evolute Curve in De Sitter 3-Space

Moreover, we introduce the Frenet apparatus of an evolute curve according to the Frenet apparatus of an involute curve in de Sitter 3-space. Otherwise, we define the hyperbolic evolute curve in de Sitter 3-space as follows:

Definition 5.1 We define the hyperbolic evolute curve $\psi : I \to \mathbb{S}_1^3$ of $\gamma(s)$ by

$$\psi(s) = \frac{1}{\sqrt{\left|\left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - \kappa_g^2 + 1\right|}}\left(\kappa_g\gamma(s) + \mathbf{N}(s) + \frac{\dot{\kappa}_g}{\kappa_g\tau_g}\mathbf{E}(s)\right)$$

under the assumption that $\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 < 1$.

We remark that $\psi_\gamma(s)$ is located in $\mathbb{S}_1^3$ if and only if $\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 < 1$, where $\gamma(s)$ is an involute curve of $\psi(s)$ (see [1]).

The Frenet apparatus of the evolute curve in $\mathbb{S}_1^3$ denoted by

$$\{\psi(s), \mathbf{T}_\psi(s), \mathbf{N}_\psi(s), \mathbf{E}_\psi(s), \mathcal{K}_\psi(s), \mathcal{T}_\psi(s)\},$$

can be formed according to the Frenet apparatus of the involute curve $\gamma(s)$.

Theorem 3: If $\psi(s)$ is a unit speed spacelike curve and is an evolute curve of $\gamma(s)$. Then, the Frenet apparatus of the evolute curve $\psi(s)$ is as follows:

Proof: It follows from the definition of an evolute curve in de Sitter 3-space that

$$\psi(s) = \frac{\kappa_g}{\sqrt{\left|\left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - \kappa_g^2 + 1\right|}}\gamma + \frac{1}{\sqrt{\left|\left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - \kappa_g^2 + 1\right|}}\mathbf{N} - \frac{\dot{\kappa}_g}{\kappa_g\tau_g\sqrt{\left|\left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - \kappa_g^2 + 1\right|}}\mathbf{E}. \tag{22}$$

If, we refer to

$$\lambda_1 = \frac{\kappa_g}{\sqrt{\left|\left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - \kappa_g^2 + 1\right|}}, \quad \lambda_2 = \frac{1}{\sqrt{\left|\left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - \kappa_g^2 + 1\right|}},$$

$$\lambda_3 = -\frac{\dot{\kappa}_g}{\kappa_g\tau_g\sqrt{\left|\left(\frac{\dot{\kappa}_g}{\kappa_g\tau_g}\right)^2 - \kappa_g^2 + 1\right|}}. \tag{23}$$

Then, we have

$$\psi(s) = \lambda_1\gamma(s) + \lambda_2\mathbf{N}(s) + \lambda_3\mathbf{E}(s). \tag{24}$$

Differentiating both sides of Eq.(24), and from Eqs (5), we have

$$\begin{aligned}\mathbf{T}_\psi(s) &= \dot{\lambda}_1\gamma(s) + \lambda_1\dot{\gamma}(s) + \dot{\lambda}_2\mathbf{N}(s) + \lambda_2\dot{\mathbf{N}}(s) + \dot{\lambda}_3\mathbf{E}(s) + \lambda_3\dot{\mathbf{E}}(s)\\ &= \dot{\lambda}_1\gamma(s) + \lambda_1\mathbf{T}(s) + \dot{\lambda}_2\mathbf{N}(s) + \lambda_2(-\delta\kappa_g\mathbf{T}(s) + \tau_g\mathbf{E}(s))\\ &\quad + \dot{\lambda}_3\mathbf{E}(s) + \lambda_3(\tau_g\mathbf{N}(s))\\ &= \dot{\lambda}_1\gamma(s) + (\lambda_1 - \delta\kappa_g\lambda_2)\mathbf{T}(s) + (\dot{\lambda}_2 + \tau_g\lambda_3)\mathbf{N}(s)\\ &\quad + (\dot{\lambda}_3 + \tau_g\lambda_2)\mathbf{E}(s).\end{aligned} \tag{25}$$

Therefore, by differentiating Eq.(??), we get

$$3\dot{\mathbf{T}}_\psi(s) = \begin{cases}(\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2)\gamma(s) + (2\dot{\lambda}_1 - \delta\dot{\kappa}_g\lambda_2 - 2\delta\kappa_g\dot{\lambda}_2 - \delta\kappa_g\tau_g\lambda_3)\mathbf{T}(s)\\ +(\kappa_g\lambda_1 - \delta\kappa_g^2\lambda_2 + \ddot{\lambda}_2 + \dot{\tau}_g\lambda_3 + 2\tau_g\dot{\lambda}_3 + \tau_g^2\lambda_2)\mathbf{N}(s)\\ +(\tau_g\dot{\lambda}_2 + \tau_g^2\lambda_3 + \ddot{\lambda}_3 + \dot{\tau}_g\lambda_2 + \tau_g\dot{\lambda}_2)\mathbf{E}(s),\end{cases} \tag{26}$$

or in the formula

$$\dot{\mathbf{T}}_\psi(s) = \xi_1\gamma(s) + \xi_2\mathbf{T}(s) + \xi_3\mathbf{N}(s) + \xi_4\mathbf{E}(s), \tag{27}$$

where

$$3\begin{cases}\xi_1 = (\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2)\\ \xi_2 = (2\dot{\lambda}_1 - \delta\dot{\kappa}_g\lambda_2 - 2\delta\kappa_g\dot{\lambda}_2 - \delta\kappa_g\tau_g\lambda_3)\\ \xi_3 = (\kappa_g\lambda_1 - \delta\kappa_g^2\lambda_2 + \ddot{\lambda}_2 + \dot{\tau}_g\lambda_3 + 2\tau_g\dot{\lambda}_3 + \tau_g^2\lambda_2)\\ \xi_4 = (\tau_g\dot{\lambda}_2 + \tau_g^2\lambda_3 + \ddot{\lambda}_3 + \dot{\tau}_g\lambda_2 + \tau_g\dot{\lambda}_2).\end{cases}$$

Thus, in eqn. (24) and (27) we can compute the value of the vector $\mathbf{N}_\psi(s)$, where

$$\mathbf{N}_\psi(s) = \frac{\dot{\mathbf{T}}_\psi(s) + \psi(s)}{\|\dot{\mathbf{T}}_\psi(s) + \psi(s)\|}.$$

Then, we have

$$\begin{aligned}\mathbf{N}_\psi(s) &= \left(\left|-(\xi_1 + \lambda_1)^2 + \xi_2^2 + (\xi_3 + \lambda_2)^2 + (\xi_4 + \lambda_3)^2\right|\right)^{\frac{-1}{2}}(((\xi_1 + \lambda_1)\gamma(s)\\ &\quad + \xi_2\mathbf{T}(s) + (\xi_3 + \lambda_2)\mathbf{N}(s) + (\xi_4 + \lambda_3)\mathbf{E}(s))).\end{aligned} \tag{28}$$

Also, in eqn.(27) and (28), we have

$$\mathcal{K}_\psi = \sqrt{\left|-(\xi_1 + \lambda_1)^2 + \xi_2^2 + (\xi_3 + \lambda_2)^2 + (\xi_4 + \lambda_3)^2\right|}.$$

Therefore, by differentiating in eqn.(26) with respect to s, one can obtain

$$\begin{aligned}\dddot{\psi}(s) &= (\dddot{\lambda}_1 - 3\dot{\lambda}_1 + 2\delta\dot{\kappa}_g\lambda_2 + 3\delta\kappa_g\dot{\lambda}_2 + \delta\kappa_g\tau_g\lambda_3)\gamma(s) + (3\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2\\ &\quad - \delta\ddot{\kappa}_g\lambda_2 - 3\delta\dot{\kappa}_g\dot{\lambda}_2 - 3\delta\kappa_g\ddot{\lambda}_2 - \delta\dot{\kappa}_g\tau_g\lambda_3 - \delta\kappa_g\dot{\tau}_g\lambda_3 - 3\kappa_g\tau_g\dot{\lambda}_3 - \delta\kappa_g^2\lambda_1\\ &\quad + \delta\kappa_g^3\lambda_2 - \delta\kappa_g\tau_g^2\lambda_2)\mathbf{T}(s) + (3\kappa_g\dot{\lambda}_1 - 3\delta\kappa_g\dot{\kappa}_g\lambda_2 - 3\delta\kappa_g^2\dot{\lambda}_2 + \dot{\kappa}_g\lambda_1\\ &\quad - \delta\kappa_g^2\tau_g\lambda_3 + 3\tau_g\dot{\tau}_g\lambda_2 + 3\tau_g^2\dot{\lambda}_2 + \dddot{\lambda}_2 + \ddot{\tau}_g\lambda_3 + 3\dot{\tau}_g\dot{\lambda}_3 + \tau_g\ddot{\lambda}_3 + \tau_g^3\lambda_3\\ &\quad + 2\dot{\tau}_g\ddot{\lambda}_3)\mathbf{N}(s) + (\kappa_g\tau_g\lambda_1 - \delta\kappa_g^2\tau_g\lambda_2 + \tau_g^3\lambda_2 + 3\tau_g\ddot{\lambda}_2 + 3\tau_g\dot{\tau}_g\lambda_3 + \dddot{\lambda}_3\\ &\quad + 3\tau_g^2\dot{\lambda}_3 + 3\dot{\tau}_g\dot{\lambda}_2 + \ddot{\tau}_g\lambda_2)\mathbf{E}(s),\end{aligned} \tag{29}$$

then, we have

$$\dddot{\psi}(s) = B_1\gamma(s) + B_2\mathbf{T}(s) + B_3\mathbf{N}(s) + B_4\mathbf{E}(s), \tag{30}$$

where

$$2\begin{cases}B_1 = (\dddot{\lambda}_1 - 3\dot{\lambda}_1 + 2\delta\dot{\kappa}_g\lambda_2 + 3\delta\kappa_g\dot{\lambda}_2 + \delta\kappa_g\tau_g\lambda_3)\\ B_2 = (3\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2 - \delta\ddot{\kappa}_g\lambda_2 - 3\delta\dot{\kappa}_g\dot{\lambda}_2 - 3\delta\kappa_g\ddot{\lambda}_2 - \delta\dot{\kappa}_g\tau_g\lambda_3\\ \quad -\delta\kappa_g\dot{\tau}_g\lambda_3 - 3\kappa_g\tau_g\dot{\lambda}_3 - \delta\kappa_g^2\lambda_1 + \delta\kappa_g^3\lambda_2 - \delta\kappa_g\tau_g^2\lambda_2)\\ B_3 = (3\kappa_g\dot{\lambda}_1 - 3\delta\kappa_g\dot{\kappa}_g\lambda_2 - 3\delta\kappa_g^2\dot{\lambda}_2 + \dot{\kappa}_g\lambda_1 - \delta\kappa_g^2\tau_g\lambda_3 + 3\tau_g\dot{\tau}_g\lambda_2\\ \quad +3\tau_g^2\dot{\lambda}_2 + \dddot{\lambda}_2 + \ddot{\tau}_g\lambda_3 + 3\dot{\tau}_g\dot{\lambda}_3 + \tau_g\ddot{\lambda}_3 + \tau_g^3\lambda_3 + 2\dot{\tau}_g\ddot{\lambda}_3)\\ B_4 = (\kappa_g\tau_g\lambda_1 - \delta\kappa_g^2\tau_g\lambda_2 + \tau_g^3\lambda_2 + 3\tau_g\ddot{\lambda}_2 + 3\tau_g\dot{\tau}_g\lambda_3 + \dddot{\lambda}_3 + 3\tau_g^2\dot{\lambda}_3\\ \quad +3\dot{\tau}_g\dot{\lambda}_2 + \ddot{\tau}_g\lambda_2).\end{cases} \tag{31}$$

In eqn.(24), (25), (27), and (30), we can compute the torsion $\mathcal{T}_\psi$ of the evolute curve $\psi(s)$ as

$$\mathcal{T}_\psi(s) = -\frac{1}{\mathcal{K}_\psi^2}\begin{vmatrix}\lambda_1 & 0 & \lambda_2 & \lambda_3\\ \dot{\lambda}_1 & (\lambda_1 - \kappa_g\lambda_2) & (\dot{\lambda}_2 + \tau_g\lambda_3) & (\dot{\lambda}_3 + \tau_g\lambda_2)\\ \xi_1 & \xi_2 & \xi_3 & \xi_4\\ B_1 & B_2 & B_3 & B_4\end{vmatrix},$$

which implies

$$\mathcal{T}_\beta(s)=\frac{1}{\mathcal{K}_\beta^2}\begin{cases} (\lambda_1((\lambda_1-\kappa_g\lambda_2)(\xi_3B_4-\xi_4B_3)-(\dot{\lambda}_2+\tau_g\lambda_3)(\xi_2B_4-\xi_4B_2) \\ \quad +(\dot{\lambda}_3+\tau_g\lambda_2)(\xi_2B_3-\xi_3B_2)) \\ -\lambda_2(\dot{\lambda}_1(\xi_2B_4-\xi_4B_2)-(\lambda_1-\kappa_g\lambda_2)(\xi_1B_4-\xi_4B_1) \\ \quad +(\dot{\lambda}_3+\tau_g\lambda_2)(\xi_1B_2-\xi_2B_1)) \\ +\lambda_3(\dot{\lambda}_1(\xi_2B_3-\xi_3B_2)-(\lambda_1-\kappa_g\lambda_2)(\xi_1B_3-\xi_3B_1) \\ \quad +(\dot{\lambda}_2+\tau_g\lambda_3)(\xi_1B_2-\xi_2B_1))). \end{cases} \qquad (32)$$

Also, from eqn. (24), (28) and (29), we can compute the equation of the vector $\mathbf{E}_\psi(s)$, where

$$\mathbf{E}_\psi(s)=\psi(s)\wedge\mathbf{T}_\psi(s)\wedge\mathbf{N}_\psi(s),$$

then, we get

$$\mathbf{E}_\psi(s)=\frac{1}{\mathcal{K}_\psi}\begin{vmatrix} -\gamma(s) & \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{E}(s) \\ \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ \dot{\lambda}_1 & (\lambda_1-\kappa_g\lambda_2) & (\dot{\lambda}_2+\tau_g\lambda_3) & (\dot{\lambda}_3+\tau_g\lambda_2) \\ (\xi_1+\lambda_1) & \xi_2 & (\xi_3+\lambda_2) & (\xi_4+\lambda_3) \end{vmatrix},$$

i.e.,

$$\mathbf{E}_\psi(s)=-\frac{1}{\mathcal{K}_\psi}\begin{vmatrix} 0 & \lambda_2 & \lambda_3 \\ (\lambda_1-\kappa_g\lambda_2) & (\dot{\lambda}_2+\tau_g\lambda_3) & (\dot{\lambda}_3+\tau_g\lambda_2) \\ \xi_2 & (\xi_3+\lambda_2) & (\xi_4+\lambda_3) \end{vmatrix}\gamma(s)$$

$$-\frac{1}{\mathcal{K}_\psi}\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \dot{\lambda}_1 & (\dot{\lambda}_2+\tau_g\lambda_3) & (\dot{\lambda}_3+\tau_g\lambda_2) \\ (\xi_1+\lambda_1) & (\xi_3+\lambda_2) & (\xi_4+\lambda_3) \end{vmatrix}\mathbf{T}(s)$$

$$+\frac{1}{\mathcal{K}_\psi}\begin{vmatrix} \lambda_1 & 0 & \lambda_3 \\ \dot{\lambda}_1 & (\lambda_1-\kappa_g\lambda_2) & (\dot{\lambda}_3+\tau_g\lambda_2) \\ (\xi_1+\lambda_1) & \xi_2 & (\xi_4+\lambda_3) \end{vmatrix}\mathbf{N}(s)$$

$$-\frac{1}{\mathcal{K}_\psi}\begin{vmatrix} \lambda_1 & 0 & \lambda_2 \\ \dot{\lambda}_1 & (\lambda_1-\kappa_g\lambda_2) & (\dot{\lambda}_2+\tau_g\lambda_3) \\ (\xi_1+\lambda_1) & \xi_2 & (\xi_3+\lambda_2) \end{vmatrix}\mathbf{E}(s),$$

which completes the proof.

Examples

In this section, we construct two examples of an evolute curves in hyperbolic 2-space and hyperbolic 3-space and show that the Frenet apparatus of an envolute curves in hyperbolic space can be obtained from Frenet apparatus of an involute curves.

Example 6.1 Consider the general helix curve γ in $\mathbb{H}_+^2(-1)$, where (Figure 1)

$$\gamma(s)=\left(\frac{s}{\sqrt{2}},\sin\left(\sqrt{\frac{3}{2}}s\right),\cos\left(\sqrt{\frac{3}{2}}s\right)\right). \qquad (33)$$

Now, we need to find the Frenet frame on the curve γ. From equation (33), the tangent vector of the curve γ is given by

$$\mathbf{T}(s)=\dot{\gamma}(s)=\left(\frac{1}{\sqrt{2}},\sqrt{\frac{3}{2}}\cos\left(\sqrt{\frac{3}{2}}s\right),-\sqrt{\frac{3}{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\right). \qquad (34)$$

Thus, from Eqs. (33) and (34), we get

$$\gamma(s)\wedge\mathbf{T}(s)=\begin{vmatrix} -i & j & k \\ \frac{s}{\sqrt{2}} & \sin\left(\sqrt{\frac{3}{2}}s\right) & \cos\left(\sqrt{\frac{3}{2}}s\right) \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}}\cos\left(\sqrt{\frac{3}{2}}s\right) & -\sqrt{\frac{3}{2}}\sin\left(\sqrt{\frac{3}{2}}s\right) \end{vmatrix}, \qquad (35)$$

since

$$\mathbf{E}(s)=\gamma(s)\wedge\mathbf{T}(s).$$

Then, we get

$$\mathbf{E}(s)=\left(\sqrt{\frac{3}{2}}\sin^2\left(\sqrt{\frac{3}{2}}s\right)+\sqrt{\frac{3}{2}}\cos^2\left(\sqrt{\frac{3}{2}}s\right)\right)i$$

$$+\left(\frac{\sqrt{3}}{2}s\sin\left(\sqrt{\frac{3}{2}}s\right)+\frac{1}{\sqrt{2}}\cos\left(\sqrt{\frac{3}{2}}s\right)\right)j$$

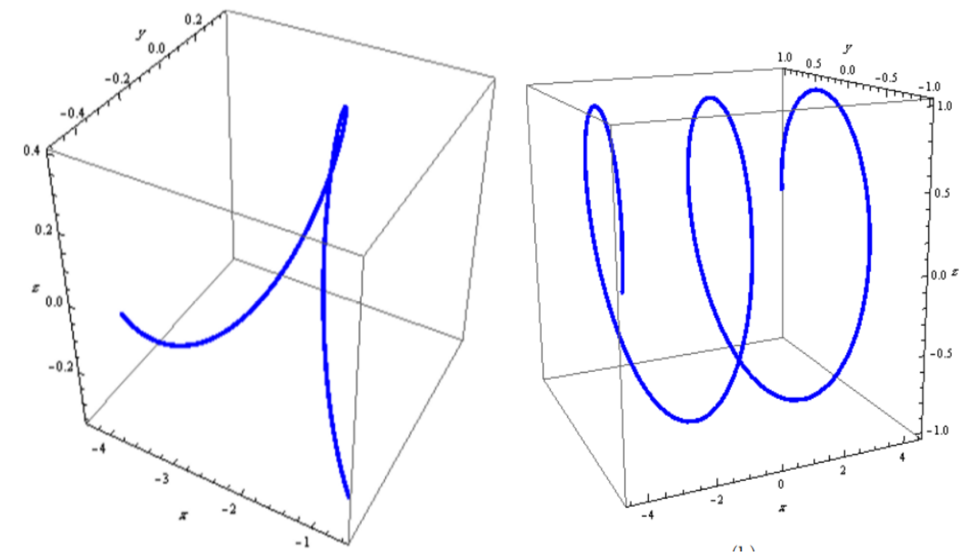

Figure 1: (a) Evolute curve α(s), (b) Involute curve γ(s).

$$+\left(\frac{\sqrt{3}}{2}s\cos\left(\sqrt{\frac{3}{2}}s\right)-\frac{1}{\sqrt{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\right)k, \tag{36}$$

or in simplified format

$$\mathbf{E}(s)=\left(\sqrt{\frac{3}{2}}\right)i+\left(\frac{\sqrt{3}}{2}s\sin\left(\sqrt{\frac{3}{2}}s\right)+\frac{1}{\sqrt{2}}\cos\left(\sqrt{\frac{3}{2}}s\right)\right)j$$

$$+\left(\frac{\sqrt{3}}{2}s\cos\left(\sqrt{\frac{3}{2}}s\right)-\frac{1}{\sqrt{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\right)k. \tag{37}$$

Also, from in eqn. (34), we have

$$\dot{\mathbf{T}}(s)=\left(0,-\frac{3}{2}\sin\left(\sqrt{\frac{3}{2}}s\right),-\frac{3}{2}\cos\left(\sqrt{\frac{3}{2}}s\right)\right),$$

we can compute the curvature of the curve γ as follows:

$$\kappa_g(s)=\begin{vmatrix}\frac{s}{\sqrt{2}} & \sin\left(\sqrt{\frac{3}{2}}s\right) & \cos\left(\sqrt{\frac{3}{2}}s\right)\\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}}\cos\left(\sqrt{\frac{3}{2}}s\right) & -\sqrt{\frac{3}{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\\ 0 & -\frac{3}{2}\sin\left(\sqrt{\frac{3}{2}}s\right) & -\frac{3}{2}\cos(\sqrt{\frac{3}{2}}s)\end{vmatrix},$$

or

$$\kappa_g(s)=-\frac{s}{\sqrt{2}}\left(\frac{3}{2}\sqrt{\frac{3}{2}}\cos^2\left(\sqrt{\frac{3}{2}}s\right)+\frac{3}{2}\sqrt{\frac{3}{2}}\sin^2\left(\sqrt{\frac{3}{2}}s\right)\right)$$

$$+\frac{3}{2\sqrt{2}}\left(\sin\left(\sqrt{\frac{3}{2}}s\right)\cos\left(\sqrt{\frac{3}{2}}s\right)-\sin\left(\sqrt{\frac{3}{2}}s\right)\cos\left(\sqrt{\frac{3}{2}}s\right)\right),$$

then, we have

$$\kappa_g(s)=-\frac{\sqrt{27}}{4}s.$$

Then, we can find the equation of an evolute curve from definition (3.1) as

$$2\alpha(s)=\frac{1}{\sqrt{|27s^2-16|}}\begin{cases}(\sqrt{\frac{3}{2}}(4-3s^2),\frac{4}{\sqrt{2}}\cos\left(\sqrt{\frac{3}{2}}s\right)-\sqrt{3}s\sin\left(\sqrt{\frac{3}{2}}s\right)\\ ,-\frac{4}{\sqrt{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)-\sqrt{3}s\cos(\sqrt{\frac{3}{2}}s)).\end{cases} \tag{38}$$

Therefore, in eqn. (7) we obtain the Frenet apparatus of the evolute curve $\alpha(s)$ as follows:

$$2\mathbf{T}_\alpha(s)=\frac{6\sqrt{3}}{|27s^2-16|^{\frac{3}{2}}}\begin{cases}(-5\sqrt{2}s,(9s^2-2)\sin\left(\sqrt{\frac{3}{2}}s\right)-3\sqrt{6}s\cos(\sqrt{\frac{3}{2}}s)\\ ,(9s^2-2)\cos\left(\sqrt{\frac{3}{2}}s\right)+3\sqrt{6}s\sin\left(\sqrt{\frac{3}{2}}s\right)),\end{cases}$$

and

$$\mathbf{E}_\alpha(s)=\frac{-12\sqrt{3}}{|27s^2-16|}\left(\frac{1}{\sqrt{2}},\sqrt{\frac{3}{2}}\cos\left(\sqrt{\frac{3}{2}}s\right),-\sqrt{\frac{3}{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\right).$$

Also, we get

$$\mathcal{K}_\alpha(s)=\frac{27}{|27s^2-16|^{\frac{3}{2}}}.$$

Example 6.2 Let γ be a general helix in $\mathbb{H}^3_+(-1)$, where

$$\gamma(s)=\left(\sqrt{2}\cosh(s),\sqrt{2}\sinh(s),\sin(s),\cos(s)\right). \tag{39}$$

Now, we need to find the Frenet frame on the curve γ. From in eqn. (39), the tangent vector of the curve γ given from

$$\mathbf{T}(s)=\left(\sqrt{2}\sinh(s),\sqrt{2}\cosh(s),\cos(s),-\sin(s)\right), \tag{40}$$

and we get

$$\dot{\mathbf{T}}(s)=\left(\sqrt{2}\cosh(s),\sqrt{2}\sinh(s),-\sin(s),-\cos(s)\right). \tag{41}$$

In eqn. (39) and (41), we get

$$\mathbf{N}(s)=\frac{\dot{\mathbf{T}}(s)-\gamma(s)}{\|\dot{\mathbf{T}}(s)-\gamma(s)\|}=\left(0,0,-\sin(s),-\cos(s)\right), \tag{42}$$

we can compute the curvature of the curve γ as follows:

$$\kappa_g(s)=\|\dot{\mathbf{T}}(s)-\gamma(s)\|=2.$$

Also, we get

$$\mathbf{E}(s)=\gamma(s)\wedge\mathbf{T}(s)\wedge\mathbf{N}(s)$$

$$=\begin{vmatrix}-i & j & k & l\\ \sqrt{2}\cosh(s) & \sqrt{2}\sinh(s) & \sin(s) & \cos(s)\\ \sqrt{2}\sinh(s) & \sqrt{2}\cosh(s) & \cos(s) & -\sin(s)\\ 0 & 0 & -\sin(s) & -\cos(s)\end{vmatrix},$$

or in the form

$$\mathbf{E}(s)=-\begin{vmatrix}\sqrt{2}\sinh(s) & \sin(s) & \cos(s)\\ \sqrt{2}\cosh(s) & \cos(s) & -\sin(s)\\ 0 & -\sin(s) & -\cos(s)\end{vmatrix}i$$

$$-\begin{vmatrix}\sqrt{2}\cosh(s) & \sin(s) & \cos(s)\\ \sqrt{2}\sinh(s) & \cos(s) & -\sin(s)\\ 0 & -\sin(s) & -\cos(s)\end{vmatrix}j$$

$$+\begin{vmatrix}\sqrt{2}\cosh(s) & \sqrt{2}\sinh(s) & \cos(s)\\ \sqrt{2}\sinh(s) & \sqrt{2}\cosh(s) & -\sin(s)\\ 0 & 0 & -\cos(s)\end{vmatrix}k$$

$$-\begin{vmatrix}\sqrt{2}\cosh(s) & \sqrt{2}\sinh(s) & \sin(s)\\ \sqrt{2}\sinh(s) & \sqrt{2}\cosh(s) & \cos(s)\\ 0 & 0 & -\sin(s)\end{vmatrix}l,$$

then, we have

$$\mathbf{E}(s)=\left(\sqrt{2}\sinh(s),\sqrt{2}\cosh(s),-2\cos(s),2\sin(s)\right).$$

Therefore, we obtain

$$\det(\gamma,\dot{\gamma},\ddot{\gamma},\dddot{\gamma})=\begin{vmatrix}\sqrt{2}\cosh(s) & \sqrt{2}\sinh(s) & \sin(s) & \cos(s)\\ \sqrt{2}\sinh(s) & \sqrt{2}\cosh(s) & \cos(s) & -\sin(s)\\ \sqrt{2}\cosh(s) & \sqrt{2}\sinh(s) & -\sin(s) & -\cos(s)\\ \sqrt{2}\sinh(s) & \sqrt{2}\cosh(s) & -\cos(s) & \sin(s)\end{vmatrix},$$

where

$$\det(\gamma,\dot{\gamma},\ddot{\gamma},\dddot{\gamma})=\sqrt{2}\cosh(s)[-\sqrt{2}\cosh(s)-\cos(s)(\sqrt{2}\sin(s)\sinh(s)$$

$$+\sqrt{2}\cos(s)\cosh(s))-\sin(s)(-\sqrt{2}\cos(s)\sinh(s)$$
$$+\sqrt{2}\sin(s)\cosh(s))]-\sqrt{2}\sinh(s)[-\sqrt{2}\sinh(s)$$
$$-\cos(s)(\sqrt{2}\sin(s)\cosh(s)+\sqrt{2}\cos(s)\sinh(s))$$
$$-\sin(s)(-\sqrt{2}\cos(s)\cosh(s)+\sqrt{2}\sin(s)\sinh(s)]$$
$$+\sin(s)[\sqrt{2}\sinh(s)(\sqrt{2}\sin(s)\sinh(s)+\sqrt{2}\cos(s)\cosh(s))$$
$$-\sqrt{2}\cosh(s)(\sqrt{2}\sin(s)\cosh(s)+\sqrt{2}\cos(s)\sinh(s))$$
$$-2\sin(s))]-\cos(s)[\sqrt{2}\sinh(s)(-\sqrt{2}\cos(s)\sinh(s)$$
$$+\sqrt{2}\sin(s)\cosh(s))-\sqrt{2}\cosh(s)(-\sqrt{2}\cos(s)\cosh(s)$$
$$+\sqrt{2}\sin(s)\sinh(s))+2\cos(s))],$$

then, we get

$$\det(\gamma,\dot{\gamma},\ddot{\gamma},\dddot{\gamma})=-8.$$

Therefore, we have

$$\tau_g(s)=-\frac{\det(\gamma,\dot{\gamma},\ddot{\gamma},\dddot{\gamma})}{\kappa_g^2}=2.$$

Thus, we can find the equation of the evolute curve as follows:

$$\beta(s)=\frac{1}{\sqrt{3}}\left(2\sqrt{2}\cosh(s),2\sqrt{2}\sinh(s),\sin(s),\cos(s)\right).$$

In eqn. (15), (16) and (20), we get

$$3\begin{cases}\mu_1=\frac{2}{\sqrt{3}},\mu_2=\frac{1}{\sqrt{3}},\mu_3=0,\\ \eta_1=\eta_2=\eta_4=0,\quad \eta_3=\frac{-4}{3},\\ \zeta_1=\zeta_2=\zeta_3=0,\quad \zeta_4=\frac{-8}{\sqrt{3}},\end{cases}$$

and then, we obtain the Frenet apparatus of the evolute curve $\beta(s)$ as

$$\mathbf{T}_\beta(s)=\frac{2}{\sqrt{3}}\left(\sqrt{2}\sinh(s),\sqrt{2}\cosh(s),-2\cos(s),2\sin(s)\right),$$

$$\mathbf{N}_\beta(s)=\frac{-1}{\sqrt{5}}\left(-2\sinh(s),-2\cosh(s),(-2\sqrt{2}+5)\sin(s),(-2\sqrt{2}+5)\cos(s)\right),$$

$$\mathbf{E}_\beta(s)=\frac{-16}{3\sqrt{5}}\left(\sqrt{2}\sinh(s),\sqrt{2}\cosh(s),\cos(s),-\sin(s)\right),$$

Also, we get $\mathcal{K}_\beta(s)=\sqrt{\frac{5}{3}}$, and $\mathcal{T}_\beta(s)=0$.

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