Solvability of Higher Order (p, q) Laplacian Two-Point Boundary Value Problems

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Abstract

In this paper, we establish the existence of positive solutions to a coupled system of higher order (p, q)-Laplacian two-point boundary value problem,

\[ (-1)^{m-1} \left[ \phi_{q} \left( \phi_{p} (u^{(m)}(t)) \right) \right]^{(m-1)} = \lambda f_{i}(t, u(t), v(t)), \quad t \in [0,1], \quad (-1)^{m-1} \left[ \phi_{q} \left( \phi_{p} (v^{(m)}(t)) \right) \right]^{(m-1)} = \lambda f_{i}(t, v(t), u(t)), \quad t \in [0,1], \]

where

\[ u^{(i)}(0) = 0 = u^{(i)}(1), \quad i = 0, 1, 2, \ldots, m-1, \quad \phi_{q}(0) = \phi_{q}(1) = 0, \quad j = 0, 1, 2, \ldots, n_{j} - 2, \]

and

\[ \phi_{q}(y) = y^{q} - 1 \]

by using Guo-Krasnosel’skii fixed point theorem for operators on a cone in a Banach space.

Keywords: (p, q)-Laplacian; System of differential equations; Boundary value problem; Green’s function; Eigen value; Positive solution; Cone

Introduction

Differential equations governed by nonlinear differential operators have been widely studied by many researchers. In this theory, the most investigated operator is the classical p-Laplacian, given by \( \phi_{p}(y) = |y|^{p-2}y \) with \( p > 1 \). In recent years, it has been generalized to other types of differential operators that preserve the monotonicity of the p-Laplacian, but are not homogeneous. This more general operator is named as \( \Phi \)-Laplacian, and as involved in the modeling of non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. Much interest has been created in establishing positive solutions and multiple positive solutions for two-point, multi-point boundary value problems (BVPs) associated with ordinary differential equations. To mention the related papers along these lines, we refer to Erbe et al., Eloe et al., Wong et al., and Li [1-7]. Due to the wide mathematical and physical background the existence of positive solutions for nonlinear BVPs with \( \Phi \)-Laplacian operator have received wide attention. To mention a few, Agarwal, Bohner, Agarwal, Avery and Chu [8-12], and the references therein.

Recently, Prasad et al. [13] established positive solutions for (p, q)-Laplacian boundary value problem, where

\[ (-1)^{m-1} \left[ \phi_{q} \left( \phi_{p} (u^{(m)}(t)) \right) \right]^{(m-1)} = \lambda f_{i}(t, u(t), v(t)), \quad t \in [0,1], \]

\[ u^{(i)}(0) = 0 = u^{(i)}(1), \quad i = 0, 1, 2, \ldots, m-1, \]

and

\[ \phi_{q}(y) = y^{q} - 1 \]

exist as positive real numbers.

We assume the following conditions throughout this paper:

\[ f_{i}: [0,1] \times R^{2} \rightarrow [0, \infty) \]

are continuous functions, for \( i = 1, 2, \ldots, \) each of

\[ f_{i,0} = \lim_{u \rightarrow -0^{+}} \frac{f_{i}(t, u(t), v(t))}{\phi_{p}(u + v)}, \quad f_{i,0} = \lim_{u \rightarrow -0^{+}} \frac{f_{i}(t, u(t), v(t))}{\phi_{p}(u + v)} \]

exist as positive real numbers.

To obtain a solution of the BVP (1.1)-(1.4), we construct the Green functions for the corresponding homogeneous BVPs. For \( n_{i} \geq 2 \) let \( G(t, s) \) be the Green’s function of the BVP

\[ x^{(n)}(t) = 0, \quad t \in [0,1], \]

\[ x^{(i)}(0) = 0, \quad j = 0, 1, 2, \ldots, n_{j} - 2, \]

where

\[ m_{1}, m_{2}, n_{1} \in N, \quad n_{i} \geq 2, m_{i} \geq 2 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1 \]

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and after simple computation it can be obtained as

$$G(t, s) = \begin{cases} \frac{t^{n_1-1}(1-s)^{n_1-1}}{(n_1-1)!}, & 0 \leq t \leq s \leq 1, \\ \frac{(n_1-1)!}{t^{n_1-1}(1-s)^{n_1-1}} - \frac{(t-s)^{n_1-1}}{(n_1-1)!}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $H_m(t, s)$ be the Green’s function of the homogeneous BVP,

$$(-1)^m y^{(2m)}(t) = 0, \quad t \in [0, 1],$$

$$y(0) = 0 = y^{(2j-1)}(1), \quad j = 0, 1, \ldots, m_1 - 1,$$

and is recursively defined as,

$$H_m(t, s) = \int_0^t H_{m-1}(t, r)H_1(r, s)dr,$$

where $H_1(t, s)$ is the Green’s function of the BVP,

$$y''(t) = 0 \quad y(0) = 0 = y(1),$$

and is given by

$$H_1(t, s) = \begin{cases} \frac{t^{n_1-1}(1-s)^{n_1-1}}{(n_1-1)!}, & 0 \leq t \leq s \leq 1, \\ \frac{(n_1-1)!}{t^{n_1-1}(1-s)^{n_1-1}} - \frac{(t-s)^{n_1-1}}{(n_1-1)!}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Then, the equivalent integral equation for (1.1) and (1.3) is

$$u(t) = \int_0^1 H_m(t, s)f(s, u(s), v(s))ds + \lambda \int_0^1 G_{m_1}(s, r)f_2(r, u(r), v(r))dr,$$

where $G_{m_1}(t, s)$ is the Green’s function of the BVP,

$$y''(t) = 0 \quad y(0) = 0 = y(1).$$

Similarly for $m_1 \geq 2$, let $H(t, s)$ be the Green’s function of the BVP,

$$y''(t) = 0 \quad y(0) = 0 = y(1),$$

and is recursively defined as

$$H(t, s) = \int_0^1 H_{m_1-1}(t, r)H_1(r, s)dr,$$

where $H_1(t, s)$ is the Green’s function of the BVP,

$$y''(t) = 0 \quad y(0) = 0 = y(1),$$

and is given by

$$H_1(t, s) = \begin{cases} \frac{t^{n_1-1}(1-s)^{n_1-1}}{(n_1-1)!}, & 0 \leq t \leq s \leq 1, \\ \frac{(n_1-1)!}{t^{n_1-1}(1-s)^{n_1-1}} - \frac{(t-s)^{n_1-1}}{(n_1-1)!}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Further, it is easily seen that $H_1(t, s), H(t, s), G_{m_1}(t, s)$ and $G(t, s)$ are all non-negative on $[0, 1] \times [0, 1]$.

A solution of the BVP (1.1)-(1.4) is a solution of the BVP (1.1)-(1.4) such that $u$ and $v$ are non-negative on $[0, 1]$. The rest of the paper is organized as follows. In Part 4, we estimate the bounds of the Green functions which will be used in defining the positive operator. In Part 5, we provide an appropriate Banach space and a cone in order to apply the Guo-Krasnosel’skii fixed point theorem [14] to characterize the values of $\lambda$ so that the BVP (1.1)-(1.4) has a positive solution. Finally as an application, we give an example to demonstrate our result.

**Bounds of the Green’s Functions**

In this section, we state some lemmas to estimate the bounds of the Green functions which are needed later. Let $I = \left[ \frac{1}{4}, \frac{3}{4} \right]$. For the preceding Lemmas we refer the reader to [13].

**Lemma 2.1 [13]** For $(t, s) \in I \times [0, 1],$

$$G_{m_1}(t, s) \leq \frac{11}{6} \left( \frac{1}{4} \right)^{m_1-1} (1-s)s$$

**Lemma 2.2** For $t, s \in [0, 1],$

$$G_{m_1}(t, s) \geq \frac{1}{6^{m_1-1}} (1-s)s$$

**Lemma 2.3** For $(t, s) \in I \times [0, 1],$

$$H(t, s) \leq \frac{11}{6} \left( \frac{1}{4} \right)^{m_1-1} (1-s)s$$

**Lemma 2.4** For $t, s \in [0, 1],$

$$H(t, s) \leq \frac{1}{6^{m_1-1}} (1-s)s$$

**Lemma 2.5** For $(t, s) \in I \times [0, 1],$

$$H(t, s) \geq \frac{1}{4^{m_1-1}} H(s, s)$$

**Lemma 2.6** For $t, s \in [0, 1],$

$$H(t, s) \leq H(s, s)$$

**Lemma 2.7** For $(t, s) \in I \times [0, 1],$

$$G(t, s) \geq \frac{1}{4^{m_1-1}} G(s, s)$$

**Lemma 2.8** For $t, s \in [0, 1],$

$$G(t, s) \leq G(s, s)$$

Denote

$$M = \min \left\{ \frac{11^{n_1-1}}{4^{m_1-1}}, \frac{1}{4^{n_1-1}}, \frac{1}{4^{m_1-1}}, \frac{11^{n_1-1}}{4^{m_1-1}} \right\}.$$
Consider the Banach space $E = \{y : y \in C[0, 1]\}$ with the norm $\| \cdot \|$. The integral operator $T : P \to B$ is defined by

$$T(u, v) = \lambda \int_0^1 H_n(t, s) \phi_n^{-1} \left( \int_0^t G(s, \tau) f_1(t, \tau, u(t), v(t)) d\tau \right) ds,$$

$$\lambda \int_0^1 H_n(t, s) \phi_n^{-1} \left( \int_0^t G(s, \tau) f_2(t, \tau, u(t), v(t)) d\tau \right) ds.$$  

(3.2)

And also we denote

$$T_1(u, v) = \lambda \int_0^1 H_n(t, s) \phi_n^{-1} \left( \int_0^t G(s, \tau) f_1(t, \tau, u(t), v(t)) d\tau \right) ds,$$

$$T_2(u, v) = \lambda \int_0^1 H_n(t, s) \phi_n^{-1} \left( \int_0^t G(s, \tau) f_2(t, \tau, u(t), v(t)) d\tau \right) ds.$$

From (A1) and the positivity of the Green functions that, for $(u, v) \in P$, $T_1(u, v) \geq 0$, $T_2(u, v) \geq 0$, for $t \in [0, 1]$. Now for $(u, v) \in P$ and Lemma 2.4, Lemma 2.8, we have

$$T_1(u, v)(t) = \lambda \int_0^1 H_n(t, s) \phi_n^{-1} \left( \int_0^t G(s, \tau) f_1(t, \tau, u(t), v(t)) d\tau \right) ds,$$

$$\leq \lambda \int_0^1 \left( \int_0^1 (1-s) \phi_n^{-1} \left( \int_0^t G(s, \tau) f_1(t, \tau, u(t), v(t)) d\tau \right) ds \right) ds.$$

so that

$$\| T_1(u, v) \| \leq \lambda \int_0^1 \left( \int_0^1 (1-s) \phi_n^{-1} \left( \int_0^t G(s, \tau) f_1(t, \tau, u(t), v(t)) d\tau \right) ds \right) ds.$$

Now for $(u, v) \in P$ from (2.9), Lemma 2.3 and Lemma 2.7, we have

$$\min_{t \in [0, 1]} \| T_1(u, v) \| \geq M \| (u, v) \|,$$

where

$$M = \max_{t \in [0, 1]} \| u(t) \|.$$

Define the cone $P \subset B$ by,

$$P = \{ (u, v) \in B : u(t) \geq 0, v(t) \geq 0 \text{ on } [0, 1] \}.$$
\[ f_1(t, u(t), v(t)) \leq (f_{1,0} + \epsilon)\phi_1(u + v), \ 0 < \phi_1(u + v) \leq K_1, \]
\[ f_2(t, u(t), v(t)) \leq (f_{2,0} + \epsilon)\phi_2(u + v), \ 0 < \phi_2(u + v) \leq K_2. \]

Let \((u, v) \in P\) with \(||(u, v)|| = K_1\). Then, from Lemma 2.4, Lemma 2.8 and choice of \(\epsilon\),
\[ T(u, v)(t) = \lambda \int_0^t H(t, s)\phi_1^2 \left( \int_0^s G(s, r)f_1(r, u(r), v(r))dr \right) ds \]
\[ \leq \lambda \frac{1}{6} \int_0^t (1-s) \phi_1^2 \left( \int_0^s G(s, r)f_1(r, u(r), v(r))dr \right) ds \]
\[ \leq \lambda \frac{1}{6} \phi_1^2 \int_0^t (G(t, r) ||f_1(r, u(r), v(r))||) dr \leq \frac{1}{2} ||(u, v)|| \frac{K_2}{2}. \]

In a similar manner, we conclude that
\[ T(u, v)(t) = \lambda \int_0^t H(t, s)\phi_2^2 \left( \int_0^s G(s, r)f_2(r, u(r), v(r))dr \right) ds \]
\[ \leq \lambda \frac{1}{6} \phi_2^2 \int_0^t (G(t, r) ||f_2(r, u(r), v(r))||) dr \leq \frac{1}{2} ||(u, v)|| \frac{K_2}{2}. \]

Hence, \(||T(u, v)|| \leq K_1 = ||(u, v)||\). If we set
\[ \Omega_1 = \{(u, v) \in B ||(u, v)|| < K_1\}, \]
then
\[ ||T(u, v)|| \leq ||(u, v)||, \quad (u, v) \in P \cap \partial\Omega_1. \] (3.3)

Next, from the definitions of \(f_{1,0}\) and \(f_{2,0}\), there exists \(K_2 > 0\) such that
\[ f_1(t, u(t), v(t)) \geq (f_{1,0} - \epsilon)\phi_1(u + v), \quad \phi_1(u + v) \geq K_2, \]
\[ f_2(t, u(t), v(t)) \geq (f_{2,0} - \epsilon)\phi_2(u + v), \quad \phi_2(u + v) \geq K_2. \]

Let \(K_2 = \max \left\{ 2K_1, K_2 \right\} \).
Choose \((u, v) \in P\) with \(||(u, v)|| = K_2\). Then, \(\min||u(t) + v(t)|| \geq M ||(u, v)|| \geq K_2\). Consequently, from (2.9), Lemma 2.3, Lemma 2.7 and choice \(\epsilon\),
\[ T(u, v)(t) = \lambda \int_0^t H(t, s)\phi_1^2 \left( \int_0^s G(s, r)f_1(r, u(r), v(r))dr \right) ds \]
\[ \geq \lambda \frac{1}{6} \phi_1^2 \int_0^t (1-s) \phi_1^2 \left( \int_0^s G(s, r)f_1(r, u(r), v(r))dr \right) ds \]
\[ \geq \lambda \frac{1}{6} \phi_1^2 \int_0^t (1-s) \phi_1^2 \left( \frac{1}{4} \int_0^s f_1(r, u(r), v(r)) dr \right) ds \]
\[ \geq \lambda \frac{1}{6} \phi_1^2 \int_0^t (1-s) \phi_1^2 \left( \frac{1}{4} \int_0^s f_1(r, u(r), v(r)) dr \right) ds \]
\[ = \lambda \frac{1}{6} \phi_1^2 \int_0^t (G(t, r)f_1(r, u(r), v(r)) dr \]
\[ \geq \lambda \frac{M^2}{6} \phi_1^2 \int_0^t (G(t, r)f_1(r, u(r), v(r)) dr \]
\[ \geq \frac{1}{2} ||(u, v)|| \frac{K_2}{2}. \]

In a similar manner, we conclude that
\[ T(u, v)(t) = \lambda \int_0^t H(t, s)\phi_2^2 \left( \int_0^s G(s, r)f_2(r, u(r), v(r))dr \right) ds \]
\[ \geq \frac{1}{2} ||(u, v)|| \frac{K_2}{2}. \]

Therefore, \(T(u, v)(t) \geq K_2\).
Hence, \(||T(u, v)|| \geq K_2 ||(u, v)||\). If we set
\[ \Omega_2 = \{(u, v) \in B ||(u, v)|| < K_2\}, \]
then \(||T(u, v)|| \geq ||(u, v)||\) for \((u, v) \in P \cap \partial\Omega_2\) (3.4)
Applying Theorem 2.9 to (3.3) and (3.4) we obtain that \(T\) has a fixed point \((u, v)\) in \(P \cap (\Omega_2 \setminus \Omega_1)\) and hence the BVP (1.1)-(1.4) has a positive solution such that \(K_2 \leq ||(u, v)|| \leq K_2\). The proof is complete.

For our next result, we define the positive numbers \(R_1, R_2, R_3\) by
\[ R_1 = \max \left[ \left( \frac{2M^2}{6} \phi_1^2 \left( \int \int G(r, s)ds \right) ds \right)^{-1} \right], \]
\[ R_2 = \min \left[ \left( \frac{2M^2}{6} \phi_1^2 \left( \int \int G(r, s)ds \right) ds \right)^{-1} \right], \]
and
\[ R_3 = \min \left[ \left( \frac{2M^2}{6} \phi_1^2 \left( \int \int G(r, s)ds \right) ds \right)^{-1} \right]. \]

**Theorem 3.2** Assume that the conditions (A1), (A2) are satisfied. Then, for each \(\lambda\) satisfying
\[ R_3 < \lambda < R_4 \] (3.5)
there exists at least one positive solution \((u, v)\) of the BVP (1.1)-(1.4).

**Proof:** Let \(\lambda\) be as in (3.5). And let \(\epsilon > 0\) be chosen such that
\[ m \left[ \left( \frac{2M^2}{6} \phi_1^2 \left( \int \int G(r, s)ds \right) ds \right)^{-1} \right], \]
\[ \left( \frac{2M^2}{6} \phi_1^2 \left( \int \int G(r, s)ds \right) ds \right)^{-1} \leq \lambda \]
and
\[ \lambda \leq \min \left[ \left( \frac{2}{6} \phi_1^2 \left( \int \int G(r, s)ds \right) ds \right)^{-1} \right]. \]

Let \(T\) be the cone preserving completely continuous operator that was defined by (3.2). From the definitions of \(f_{1,0}\) and \(f_{2,0}\), there exists \(J > 0\) such that
\[ f_1(t, u(t), v(t)) \geq (f_{1,0} - \epsilon)\phi_1(u + v), \quad \phi_1(u + v) \leq J, \]
\[ f_2(t, u(t), v(t)) \geq (f_{2,0} - \epsilon)\phi_2(u + v), \quad \phi_2(u + v) \leq J. \]
Choose \((u, v) \in P\) with \(||(u, v)|| = J\). Then, from (2.9), Lemma 2.3 and Lemma 2.7 we have
\begin{align*}
T_z(u, v)(t) &= \lambda \int_0^1 H_n(t, s) s^\alpha \left( \int_0^1 G(s, \tau) f(s, u(\tau), v(\tau)) d\tau \right) ds \\
&\geq \lambda \left( \frac{11}{6} \right)^{1-n} \int_0^1 (1-s) s^\alpha \phi^\alpha_p \left( \int_0^1 G(s, \tau) f(s, u(\tau), v(\tau)) d\tau \right) ds \\
&\geq \lambda \left( \frac{11}{6} \right)^{1-n} \frac{1}{6^{n-1}} \int_0^1 (1-s) s^\alpha \phi^\alpha_p \left( \int_0^1 G(s, \tau) (f_{s, w} - e) \phi_p(u + v) d\tau \right) ds \\
&\geq \frac{1}{6^n} \phi^\alpha_p \left( \int_0^1 G(s, \tau) (f_{s, w} - e) d\tau \right) (u + v) \\
&\geq \frac{1}{2} \| (u, v) \| = \frac{J_z}{2}.
\end{align*}

In a similar manner, we conclude that
\begin{align*}
T_z(u, v)(t) &= \lambda \int_0^1 H_n(t, s) s^\alpha \left( \int_0^1 G(s, \tau) f_z(s, u(\tau), v(\tau)) d\tau \right) ds \\
&\geq \frac{1}{2} \| (u, v) \| = \frac{J_z}{2}.
\end{align*}

Therefore, \( T(u, v)(t) \geq J_z \).

Hence, \( \| T(u, v) \| \geq J_z = \| (u, v) \| \). If we set
\begin{align*}
\Omega_z = \{(u, v) \in B \| \| (u, v) \| < J_z \},
\end{align*}

Then
\begin{align*}
\| T(u, v) \| &\geq \| (u, v) \| \quad (u, v) \in P \cap \mathcal{C} \Omega_z 
\end{align*}

Let \( f(t, u(t), v(t)) \) be the Green's function. Now, we define the functions \( f^1_1, f^1_2 : \mathbb{R}^2 \rightarrow \mathbb{R} \) by
\begin{align*}
f^1_1(x) &= \max_{u, v \in [0, 1]} f_j(t, u(t), v(t)), \\
f^1_2(x) &= \max_{u, v \in [0, 1]} f_j(t, u(t), v(t)).
\end{align*}

Then \( f^1_1(x) \leq f^1_2(x) \) and \( f^1_2(x) \leq f^2(x) \).

It follows that the functions \( f^1_1, f^1_2 \) are non-decreasing and satisfy the conditions
\begin{align*}
\lim_{x \to -\infty} f^1_1(x) \leq f_{1_{\infty}} \quad \text{and} \quad \lim_{x \to +\infty} f^1_2(x) = f_{2_{\infty}}.
\end{align*}

Next, by the definitions of \( f_{1_{\infty}} \) and \( f_{2_{\infty}} \), there exist \( J_z > 0 \) such that
\begin{align*}
f^1_1(x) &\leq (f_{1_{\infty}} + e) \phi^\alpha_p(x) \quad \text{and} \quad f^1_2(x) \leq (f_{2_{\infty}} + e) \phi^\alpha_p(x), \quad x \leq J_z.
\end{align*}

Then, for \( J_z = \max \{ 2J^1_1, J^1_2 \} \) we have
\begin{align*}
f^1_1(x) &\leq f^1_2(J_z) \quad \text{and} \quad f^1_2(x) \leq f^2(J_z), \quad 0 < x \leq J_z.
\end{align*}

Choose \( (u, v) \in P \) with \( \| (u, v) \| = J_z \). Then we have
\begin{align*}
T_z(u, v)(t) &= \lambda \int_0^1 H_n(t, s) s^\alpha \left( \int_0^1 G(s, \tau) f_z(s, u(\tau), v(\tau)) d\tau \right) ds \\
&\leq \lambda \left( \frac{11}{6} \right)^{1-n} \frac{1}{6^{n-1}} \int_0^1 (1-s) s^\alpha \phi^\alpha_p \left( \int_0^1 G(s, \tau) (f_{s, w} + e) \phi_p(u + v) d\tau \right) ds \\
&\leq \frac{1}{2} \| (u, v) \| = \frac{J_z}{2}.
\end{align*}

Then, for \( \| T(u, v) \| \leq \| (u, v) \| \) \( (u, v) \in P \cap \mathcal{C} \Omega_z \)

Applying Theorem 2.9 to (3.6) and (3.7), we obtain that \( \| T(u, v) \| \). If we set
\begin{align*}
\Omega_z = \{(u, v) \in B \| \| (u, v) \| < J_z \},
\end{align*}

Then
\begin{align*}
\| T(u, v) \| &\leq \| (u, v) \| \quad (u, v) \in P \cap \mathcal{C} \Omega_z 
\end{align*}

Applying Theorem 2.9 to (3.6) and (3.7), we obtain that \( T \) has a fixed point \( (u, v) \) in \( P \cap \mathcal{C} \Omega_z \) and hence the BVP (1.1)-(1.4) has a positive solution such that \( J_z \leq \| (u, v) \| \leq J_z \). The proof is complete.

**Example**

Let us consider an example to illustrate the above result. Now, we consider two-point \((p, q)\) Laplacian boundary value problems
\begin{align*}
-(\phi_p(u^0(t)))^{\omega} &= \lambda_1 f_1(t, u(t), v(t)), \quad t \in [0, 1], \\
-(\phi_q(v^0(t)))^{\omega} &= \lambda_2 f_2(t, u(t), v(t)), \quad \frac{1}{p} + \frac{1}{q} = 1,
\end{align*}

with \( u^{(2)}(0) = 0 = u^{(2)}(1), \ i = 0, 1 \),
\begin{align*}
[\phi_p(u^{(i)}(t))]_{t=0}^{t=1} &= 0, \ j = 0, 1, 2, \\
[\phi_q(v^{(i)}(t))]_{t=0}^{t=1} &= 0, \ i = 0, 1, 2,
\end{align*}

where
\begin{align*}
f_1(t, u(t), v(t)) &= (u + v)[9995000000 - 9994999845 e^{-5(t+e)}], \\
f_2(t, u(t), v(t)) &= (u + v)[2400000000 - 239999970 e^{-7(t+e)}].
\end{align*}

The Green's function \( G(t, s) \) for the homogeneous BVP,
\[-x''(t) = 0,\]
\[x(0) = x'(0) = 0, x(1) = 0,\]
is given by
\[G(t, s) = \begin{cases} 
\frac{t^2(1-s)^2}{2}, & 0 \leq t \leq s < 1, \\
\frac{t^2(1-s)^2 - (t-s)^2}{2}, & 0 < s \leq t \leq 1.
\end{cases}
\]
The Green's function \(H(t, s)\) for the homogeneous BVP,
\[-x''(t) = 0,\]
\[x(0) = 0, x(1) = 0,\]
is given by
\[H(t, s) = \begin{cases} 
t(1-s), & 0 \leq t \leq s < 1, \\
(1-t), & 0 < s \leq t \leq 1.
\end{cases}
\]
By direct calculations, we have
\[M = 0.0001678466 \quad (p = 2), \quad \int_0^1 G(t, t) dt = \frac{1}{60}, \quad \int_0^1 H(s, s) ds = \frac{1}{6},
\]
\[f_{1,0} = 155, \quad f_{2,0} = 90,
\]
\[f_{1,\infty} = 9995000000, \quad f_{2,\infty} = 2400000000,
\]
\[R_1 = \max\{3.835473314, 3.108280305\} = 3.835473314,
\]
\[R_2 = \min\{6.967742005, 7.2\} = 6.967742005.
\]
Applying Theorem 3.1, we get an optimal eigenvalue interval 3.835473314 < \lambda < 6.967742005 for which the boundary value problem (4.1)-(4.4) has at least one positive solution [14-19].

References