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Research Article

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Solvability of Higher Order (p,q) Laplacian Two-Point Boundary Value Problems

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Abstract

In this paper, we establish the existence of positive solutions to a coupled system of higher order (p, q)-Laplacian two-point boundary value problem,

$$(-1)^{m1^{-1}}\left[\phi_p\left(u^{(2m_1)}(t)\right)\right]^{(n_1)}=\lambda f_1(t,u(t),v(t)), t\in[0,1],\ (-1)^{n_2-1}\left[\phi_q\left(u^{(m_2)}(t)\right)\right]^{(2n_2)}=\lambda f_2(t,u(t),v(t)), t\in[0,1],\quad u^{(2i)}(0)=0=u^{(2i)}(1),\ i=0,1,2,\cdots,m_1-1,$$

$$\left[\phi_p\left(u^{(2m_1)}(\mathrm{t})\right)\right]^{(j)}_{at\,t=0}=0,\ j=0,1,2,\cdots,n_1-2,\ \left[\phi_p\left(u^{(2m_1)}(\mathrm{t})\right)\right]_{at\,t=1}=0,\ \left[\phi_q\left(v^{(m_2)}(\mathrm{t})\right)\right]^{(2i)}_{at\,t=0}=0,\left[\phi_q\left(v^{(m_2)}(\mathrm{t})\right)\right]^{(2i)}_{at\,t=1}=0,\ i=0,1,2,\cdots,n_2-1,$$

$\phi_p(y)=y\,|\,y\,|^{p-2}$ by using Guo-Krasnosel'skii fixed point theorem for operators on a cone in a Banach space.

Keywords: (p, q)-Laplacian; System of differential equations; Boundary value problem; Green's function; Eigen value; Positive solution; Cone

Introduction

Differential equations governed by nonlinear differential operators have been widely studied by many researchers. In this theory, the most investigated operator is the classical p-Laplacian, given by $\phi p(y)=y|y|p-2$ with $p>1$. In recent years, it has been generalized to other types of differential operators that preserve the monotonicity of the p-Laplacian, but are not homogeneous. This more general operator is named as Φ-Laplacian, and as involved in the modeling of non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. Much interest has been created in establishing positive solutions and multiple positive solutions for two-point, multi-point boundary value problems (BVPs) associated with ordinary differential equations. To mention the related papers along these lines, we refer to Erbe et al., Eloe et al., Wong et al., Davis et al., Henderson et al., Ntouyas et al. and Li [1-7]. Due to wide mathematical and physical background the existence of positive solutions for nonlinear BVPs with p-Laplacian operator have received wide attention. To mention a few, Agarwal, Bohner, Agarwal, Avery and Chu [8-12], and the references therein.

Recently, Prasad et al. [13] established positive solutions for (p, q)-Laplacian boundary value problem. In this paper, we consider a coupled system of higher order (p, q)-Laplacian two-point boundary value problem,

$$(-1)^{m1^{-1}}\left[\phi_p\left(u^{(2m_1)}(t)\right)\right]^{(n_1)}=\lambda f_1(t,u(t),v(t)), t\in[0,1], \tag{1.1}$$

$$(-1)^{n_2-1}\left[\phi_q\left(u^{(m_2)}(t)\right)\right]^{(2n_2)}=\lambda f_2(t,u(t),v(t)), t\in[0,1], \tag{1.2}$$

$$\left.\begin{aligned}&u^{(2i)}(0)=0=u^{(2i)}(1), i=0,1,2,\cdots,\mathrm{m}_1-1,\\&\left[\phi_p\left(u^{(2m_1)}(t)\right)\right]^{(j)}_{at\,t=0}=0,\ j=0,1,2,\cdots,n_1-2,\\&\left[\phi_p\left(u^{(2m_1)}(t)\right)\right]_{at\,t=1}=0,\end{aligned}\right\} \tag{1.3}$$

$$\left.\begin{aligned}&\left[\phi_q\left(v^{(m_2)}(t)\right)\right]^{(2\mathrm{i})}_{at\,t=0}=0\\&\left[\phi_q\left(v^{(m_2)}(t)\right)\right]^{(2\mathrm{i})}_{at\,t=1}=0, i=0,1,2,\cdots,n_2-1,\\&v^{(j)}(0)=0,\ j=0,1,2,\cdots,m_2-2,\ v(1)=0,\end{aligned}\right\} \tag{1.4}$$

Where

$m_1,\ n_1,m_2,\ n_2\in N,\ n_1\ge 2,m_2\ge 2\ and\ \frac{1}{p}+\frac{1}{q}=1$

We assume the following conditions hold throughout this paper:

(A1) f_i: $[0,1]\times R^2\to[0,\infty)$ are continuous functions, for i=1, 2,

(A2) each of

$$f_{1,0}=\lim_{u+v\to 0^+}\frac{f_1(t,u(t),v(t))}{\phi_p(u+v)},\quad f_{2,0}=\lim_{u+v\to 0^+}\frac{f_2(t,u(t),v(t))}{\phi_q(u+v)},$$

$$f_{1,\infty}=\lim_{u+v\to\infty}\frac{f_1(t,u(t),v(t))}{\phi_p(u+v)},\quad f_{2,\infty}=\lim_{u+v\to\infty}\frac{f_2(t,u(t),v(t))}{\phi_q(u+v)},$$

exist as positive real numbers.

To obtain a solution of the BVP (1.1)-(1.4), we construct the Green functions for the corresponding homogeneous BVPs. For $n_1\ge 2$ let G(t, s) be the Green's function of the BVP

$$-x^{(n_1)}(t)=0, t\in[0,1],$$

$$x^{(j)}(0)=0,\ j=0,1,2,\cdots,n_1-2,\ x(1)=0,$$

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and after simple computation it can be obtained as

$$G(t,s)=\begin{cases}\dfrac{t^{n_1-1}(1-s)^{n_1-1}}{(n_1-1)!}, & 0\le t\le s\le 1,\\[2mm] \dfrac{t^{n_1-1}(1-s)^{n_1-1}}{(n_1-1)!}-\dfrac{(t-s)^{n_1-1}}{(n_1-1)!}, & 0\le s\le t\le 1.\end{cases}$$

Let $H_{m1}(t, s)$ be the Green's function of the homogeneous BVP,

$$(-1)^{m_1}y^{(2m_1)}(t)=0,\ t\in[0,1],$$

$$y^{(2i)}(0)=0=y^{(2i)}(1),\ i=0,1,\cdots,m_1-1,$$

and is recursively defined as,

$$H_{m1}(t,s)=\int_0^1 H_{m1-1}(t,r)H_1(r,s)dr,$$

where $H_1(t, s)$ is the Green's function of the BVP,

$-y''(t)=0,\ y(0)=0=y(1),$ and is given by

$$H_1(t,s)=\begin{cases}t(1-s), & 0\le t\le s<1,\\ s(1-t), & 0<s\le t\le 1.\end{cases}$$

Then, the equivalent integral equation for (1.1) and (1.3) is

$$u(t)=\lambda\int_0^1 H_{m1}(t,s)\phi_p^{-1}\left(\int_0^1 G(s,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds$$

$$\lambda\int_0^1 H_{m1}(t,s)\phi_q\left(\int_0^1 G(s,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds.$$

Similarly for $m_2\ge 2$, let H(t, s) be the Green's function of the BVP,

$$-x^{(m_2)}(t)=0,\ t\in[0,1],$$

$$x^{(j)}(0)=0,\ j=0,1,2,\cdots,m_2-2,\ x(1)=0,$$

and after simple computation it can be obtained as

$$H(t,s)=\begin{cases}\dfrac{t^{m_2-1}(1-s)^{m_2-1}}{(m_2-1)!}, & 0\le t\le s<1,\\[2mm] \dfrac{t^{m_2-1}(1-s)^{m_2-1}}{(m_2-1)!}-\dfrac{(t-s)^{m_2-1}}{(m_2-1)!}, & 0<s\le t\le 1.\end{cases}$$

Let $G_{n2}(t, s)$ be the Green's function of the homogeneous BVP,

$$(-1)^{n_2}y^{(2n_2)}(t)=0, t\in[0,1],$$

$$y^{(2i)}(0)=0=y^{(2i)}(1), \mathrm{i}=0,1,\cdots,n_2-1,$$

and it can be recursively defined as

$$G_{n2}(t,s)=\int_0^1 G_{n2-1}(t,r)G_1(r,s)dr,$$

where $G_1(t, s)$ is the Green's function of the BVP,

$$-y''(\mathrm{t})=0,\ \mathrm{y}(0)=0=\mathrm{y}(1),$$

and is given by

$$G_1(t,s)=H_1(t,s).$$

Then, the equivalent integral equation for (1.2) and (1.4) is

$$v(t)=\lambda\int_0^1 H(t,s)\phi_q^{-1}\left(\int_0^1 G_{n2}(s,\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds$$

$$=\lambda\int_0^1 H(t,s)\phi_p\left(\int_0^1 G_{n2}(s,\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds$$

Further, it is easily seen that $H_{m1}(t, s)$, $H(t, s)$, $G_{n2}(t, s)$ and $G(t, s)$ are all non-negative on [0, 1] × [0, 1].

A solution of the BVP (1.1)-(1.4), is a function $(u,v)\in c^{2m_1}[0,1]\times c^{m_2}[0,1]$ such that $\left(\phi_p\circ u^{(2m_1)},\phi_q\circ v^{(m_2)}\right)\in c^{n_1}[0,1]\times c^{2n_2}[0,1]$ and (u, v) satisfies the BVP (1.1)-(1.4).

A positive solution (u, v) of the BVP (1.1)-(1.4) is a solution of the BVP (1.1)-(1.4) such that u and v are non-negative on [0, 1]. The rest of the paper is organized as follows. In Part 4, we estimate the bounds of the Green functions which will be used in defining the positive operator. In Part 5, we provide an appropriate Banach space and a cone in order to apply the Guo-Krasnosel'skii fixed point theorem [14] to characterize the values of λ so that the BVP (1.1)-(1.4) has a positive solution. Finally as an application, we give an example to demonstrate our result.

Bounds of the Green's Functions

In this section, we state some lemmas to estimate the bounds of the Green functions which are needed later. Let $I=\left[\frac{1}{4},\frac{3}{4}\right]$. For the preceding Lemmas we refer the reader to [13].

Lemma 2.1 [13] For (t, s) ∈ I×[0, 1],

$$G_j(t,s)\ge\left(\frac{11}{6}\right)^{j-1}\frac{1}{4^{3j-2}}(1-s)s \tag{2.1}$$

Lemma 2.2 For t, s ∈ [0, 1],

$$G_j(t,s)\ge\frac{1}{6^{j-1}}(1-s)s \tag{2.2}$$

Lemma 2.3 For (t, s) ∈ I×[0, 1],

$$H_j(t,s)\ge\left(\frac{11}{6}\right)^{j-1}\frac{1}{4^{3j-2}}(1-s)s \tag{2.3}$$

Lemma 2.4 For t, s ∈ [0, 1],

$$H_j(t,s)\le\frac{1}{6^{j-1}}(1-s)s \tag{2.4}$$

Lemma 2.5 For (t, s) ∈ I×[0, 1],

$$H(t,s)\ge\frac{1}{4^{m_2-1}}\mathrm{H(s,s)}. \tag{2.5}$$

Lemma 2.6 For t, s ∈ [0, 1],

$$H(t,s)\le\mathrm{H(s,s)}. \tag{2.6}$$

Lemma 2.7 For (t, s) ∈ I×[0, 1],

$$G(t,s)\ge\frac{1}{4^{n_1-1}}G(s,s). \tag{2.7}$$

Lemma 2.8 For t, s ∈ [0, 1],

$$G(t,s)\le G(s,s). \tag{2.8}$$

Denote

$$M=\min\left\{\frac{11^{m_1-1}}{4^{3m_1-2}}.\phi_p^{-1}\left(\frac{1}{4^{n_1-1}}\right),\frac{1}{4^{m_2-1}}.\phi_q^{-1}\left(\frac{11^{n_2-1}}{4^{3n_2-2}}\right)\right\}. \tag{2.9}$$

The values of the parameter λ will be determined for which there exist positive solutions of the BVP (1.1)-(1.4), using the following Guo-Krasnosel'skii fixed point theorem [14].

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Theorem 2.9 [14] Let X be a Banach Space, $\kappa \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : k \cap (\overline{\Omega}_2 \setminus \Omega_1) \to k$ is completely continuous operator such that either

i) $\| Tu \| \le \| u \|, u \in k \cap \partial\Omega_1$ *and* $\| Tu \| \ge \| u \|, u \in k \cap \partial\Omega_2$ or

ii) $\| Tu \| \ge \| u \|, u \in k \cap \partial\Omega_1$ *and* $\| Tu \| \le \| u \|, u \in k \cap \partial\Omega_2$ holds

Then T has a fixed point in $k \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Positive Solution in a Cone

In this section, we establish the criteria to determine eigen values for which the BVP (1.1)-(1.4) has at least one positive solution. Consider the Banach space E={y : y ∈ C[0, 1]} with the norm //•// and the Banach space B=E×E, with the norm $//(u, v)//=//u//_0 + //v//_0$, where $\| u \|_0 = \max_{t \in [0,1]} | u(t) |$.

Define the cone $P \subset B$ by,

$$P = \{(u,v) \in B \mid u(t) \ge 0, v(t) \ge 0 \; on \, [0,1] \; and \; \min_{t \in I}\{u(t) + v(t)\} \ge M \, \| (u,v) \| \}.$$

For our first result, we define the positive numbers R_1, R_2 by

$$R_1 = \max\left\{ \left[\frac{2M^2}{6^{m1}} \phi_p^{-1}\left(\int_0^1 G(\tau,\tau) f_{1,\infty} d\tau \right) \right]^{-1}, \left[2M^2 \int_0^1 H(s,s) ds \phi_q^{-1}\left(\frac{1}{6^{n_2}} f_{2,\infty} \right) \right]^{-1} \right\}$$

And

$$R_2 = \min\left\{ \left[\frac{2}{6^{m1}} \phi_p^{-1}\left(\int_0^1 G(\tau,\tau) f_{1,0} d\tau \right) \right]^{-1}, \left[2\int_0^1 H(s,s) ds \phi_q^{-1}\left(\frac{1}{6^{n_2}} f_{2,0} \right) \right]^{-1} \right\}.$$

Theorem 3.1 Assume that the conditions (A_1), (A_2) are satisfied. Then, for each λ satisfying

$$R_1 < \lambda < R_2, \tag{3.1}$$

There exists at least one positive solution (u, v) of the BVP (1.1)-(1.4).

Proof: Let λ be as in (3.1). And let ϵ>0 be chosen such that

$$\max\left\{ \left[\frac{2M^2}{6^{m_1}} \phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty} - \epsilon) d\tau \right) \right]^{-1}, \left[2M^2 \int_0^1 H(s,s) ds \, \phi_q^{-1}\left(\frac{1}{6^{n_2}} (f_{2,\infty} - \epsilon) \right) \right]^{-1} \right\} \le \lambda$$

And

$$\lambda \le \min\left\{ \left[\frac{2}{6^{m_1}} \phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0} + \epsilon) d\tau \right) \right]^{-1}, \left[2\int_0^1 H(s,s) ds \, \phi_q^{-1}\left(\frac{1}{6^{n_2}} (f_{2,0} + \epsilon) \right) \right]^{-1} \right\}.$$

The integral operator T : P → B is defined by

$$T(u,v) = \left(\lambda \int_0^1 H_{m_1}(t,s) \, \phi_p^{-1}\left(\int_0^1 G(s,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds, \; \lambda \int_0^1 H(t,s) \, \phi_q^{-1}\left(\int_0^1 G_{n_2}(s,\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right). \tag{3.2}$$

And also we denote

$$T_1(u,v) = \lambda \int_0^1 H_{m_1}(t,s) \, \phi_p^{-1}\left(\int_0^1 G(s,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds,$$

$$T_2(u,v) = \lambda \int_0^1 H(t,s) \, \phi_q^{-1}\left(\int_0^1 G_{n_2}(s,\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \Big).$$

From (A1) and the positivity of the Green functions that, for (u, v) ∈ P, T1(u, v)(t) ≥ 0, T2(u, v)(t) ≥ 0, for t ∈ [0, 1]. Now for (u, v)∈ P and Lemma 2.4, Lemma 2.8, we have

$$\begin{aligned} T_1(u,v)(t) &= \lambda \int_0^1 H_{m_1}(t,s) \phi_p^{-1}\left(\int_0^1 G(s,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\le \lambda \frac{1}{6^{m1-1}} \int_0^1 (1-s) s \phi_p^{-1}\left(\int_0^1 G(\tau,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\le \lambda \frac{1}{6^{m_1}} \phi_p^{-1}\left(\int_0^1 G(\tau,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) \end{aligned}$$

so that

$$\| T_1(u,v) \| 0 \le \lambda \frac{1}{6^{m_1}} \phi_p^{-1}\left(\int_0^1 G(\tau,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right).$$

Now for (u, v) ∈ P from (2.9), Lemma 2.3 and Lemma 2.7, we have

$$\begin{aligned} \min_{t \in I} T_1(u,v)(t) &= \min_{t \in I} \lambda \int_0^1 H_{m_1}(t,s) \phi_p^{-1}\left(\int_0^1 G(s,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\ge \lambda \left(\frac{11}{6} \right)^{m_1 - 1} \frac{1}{4^{3m_1 - 2}} \int_0^1 (1-s) S \\ &\quad \times \phi_p^{-1}\left(\frac{1}{4^{n_1 - 1}} \int_0^1 G(\tau,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\ge \lambda \frac{11^{m_1 - 1}}{4^{3m_1 - 2}} \frac{1}{6^{m_1 - 1}} \int_0^1 (1-s) s \, \phi_p^{-1}\left(\frac{1}{4^{n_1 - 1}} \right) \\ &\quad \times \phi_p^{-1}\left(\int_0^1 G(\tau,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\ge \lambda \frac{11^{m_1 - 1}}{4^{3m_1 - 2}} . \phi_p^{-1}\left(\frac{1}{4^{n_1 - 1}} \right) \\ &\quad \times \frac{1}{6^{m_1}} \phi_p^{-1}\left(\int_0^1 G(\tau,\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) \\ &\ge M \, \| T_1(u,v) \|_0 . \end{aligned}$$

In a similar manner, we conclude that

$$\min_{t \in I} T_2(u,v)(t) \ge M \, \| T_2(u,v) \|_0 .$$

Therefore,

$$\begin{aligned} \min_{t \in I}\{T_1(u,v)(t) + T_2(u,v)(t)\} &\ge M \, \| T_1(u,v) \|_0 + M \, \| T_2(u,v) \|_0 \\ &= M \, \| T(u,v) \| . \end{aligned}$$

Thus, T : P → P. Further, the operator T is completely continuous by an application of the Arzela–Ascoli theorem.

Now, from the definitions of $f_{1,0}$ and $f_{2,0}$, there exists an $K_1>0$ such that

$$f_1(t,u(t),v(t)) \le (f_{1,0}+\epsilon)\phi_p(u+v),\ 0<\phi_p(u+v)\le K_1,$$

$$f_2(t,u(t),v(t)) \le (f_{2,0}+\epsilon)\phi_q(u+v),\ 0<\phi_q(u+v)\le K_1.$$

Let (u, v) ∈ P with ||(u, v)||=K1. Then, from Lemma 2.4, Lemma 2.8 and choice of ϵ,

$$\begin{aligned}T_1(u,v)(\mathrm{t}) &= \lambda\int_0^1 H_{m_1}(t,s)\phi_p^{-1}\left(\int_0^1 G(s,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&\le \lambda\frac{1}{6^{m1-1}}\int_0^1(1-s)s\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&\le \lambda\frac{1}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0}+\epsilon)d\tau\right)(u+v)\\
&\le \lambda\frac{1}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0}+\epsilon)d\tau\right)\|(u,v)\|\\
&\le \frac{1}{2}\|(u,v)\| = \frac{K_1}{2}.\end{aligned}$$

In a similar manner, we conclude that

$$\begin{aligned}T_2(u,v)(\mathrm{t}) &= \lambda\int_0^1 H(t,s)\phi_q^{-1}\left(\int_0^1 G_{n_2}(s,\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&\le \frac{1}{2}\|(u,v)\| = \frac{K_1}{2}.\end{aligned}$$

Hence, ||T(u, v) || ≤ K1 = || (u, v)||. If we set

$$\Omega_1 = \{(\mathrm{u},\mathrm{v})\in \mathrm{B}\,|\,\|(\mathrm{u},\mathrm{v})\|<\mathrm{K}_1\},$$

Then

$$\|T(u,v)\| \le \|(u,v)\|,\ (u,v)\in P\cap\partial\Omega_1. \tag{3.3}$$

Next, from the definitions of $f_{1,\infty}$ and $f_{2,\infty}$, there exists $\overline{K}_2>0$ such that

$$f_1(t,u(t),v(t)) \ge (f_{1,\infty}-\epsilon)\phi_p(u+v),\quad \phi_p(u+v)\ge \overline{K}_2,$$

$$f_2(t,u(t),v(t)) \ge (f_{2,\infty}-\epsilon)\phi_q(u+v),\quad \phi_q(u+v)\ge \overline{K}_2.$$

Let $K_2 = \max\left\{2K_1, \frac{\overline{K}_2}{M}\right\}.$

Choose (u, v) ∈ P with ||(u, v)||=K_2. Then,

$$\min_{t\in I}\{u(t)+v(t)\} \ge M\,\|(u,v)\| \ge \overline{K}_2.$$

Consequently, from (2.9), Lemma 2.3, Lemma 2.7 and choice ϵ,

$$\begin{aligned}T_1(u,v)(\mathrm{t}) &= \lambda\int_0^1 H_{m_1}(t,s)\phi_p^{-1}\left(\int_0^1 G(s,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&\ge \lambda\left(\frac{11}{6}\right)^{m_1-1}\frac{1}{4^{3m_1-2}}\int_0^1(1-s)\mathrm{s}\\
&\quad\times\phi_p^{-1}\left(\frac{1}{4^{n_1-1}}\int_0^1 G(\tau,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&\ge \lambda\frac{11^{m_1-1}}{4^{3m_1-2}}\frac{1}{6^{m_1-1}}\int_0^1(1-s)s\phi_p^{-1}\left(\frac{1}{4^{n_1-1}}\right)\\
&\quad\times\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}-\epsilon)\phi_p(u+v)d\tau\right)ds\\
&\ge \lambda\frac{1}{6^{m_1}}\frac{11^{m_1-1}}{4^{3m_1-2}}\cdot\phi_p^{-1}\left(\frac{1}{4^{n_1-1}}\right)\\
&\quad\times\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}-\epsilon)\phi_p(u+v)d\tau\right)\\
&= \lambda\frac{1}{6^{m_1}}M\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}-\epsilon)d\tau\right)(u+v)\\
&\ge \lambda\frac{M^2}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}-\epsilon)d\tau\right)\|(u,v)\|\\
&\ge \frac{1}{2}\|(u+v)\| = \frac{K_2}{2}.\end{aligned}$$

In a similar manner, we conclude that

$$\begin{aligned}T_2(u,v)(\mathrm{t}) &= \lambda\int_0^1 H(t,s)\phi_q^{-1}\left(\int_0^1 G_{n_2}(s,\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&\ge \frac{1}{2}\|(u+v)\| = \frac{K_2}{2}.\end{aligned}$$

Therefore, $T_2(u,v)(\mathrm{t}) \ge K_2$.

Hence, $\|T(u,v)\| \ge K_2 = \|(u,v)\|$. If we set,

$$\Omega_2 = \{(u,v)\in B\,|\,\|(u,v)\|<K_2\},$$

Then $\|T(u,v)\| \ge \|(u,v)\|$ for $(u,v)\in P\cap\partial\Omega_2$ (3.4)

Applying Theorem 2.9 to (3.3) and (3.4) we obtain that T has a fixed point (u, v) in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$ and hence the BVP (1.1)-(1.4) has a positive solution such that K_1 ≤ ||(u, v)|| ≤ K_2. The proof is complete.

For our next result, we define the positive numbers R_3, R_4 by

$$R_3 = \max\left\{\left[\frac{2M^2}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)d\tau f_{1,0}\right)\right]^{-1},\right.$$
$$\left.\left[2M^2\int_0^1 H(s,s)\phi_p^{-1}\left(\frac{1}{6^{n_2}}f_{2,0}\right)ds\right]^{-1}\right\}$$

and

$$R_4 = \min\left\{\left[\frac{2}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)d\tau f_{1,\infty}\right)\right]^{-1},\right.$$
$$\left.\left[2\int_0^1 H(s,s)\phi_q^{-1}\left(\frac{1}{6^{n_2}}f_{2,\infty}\right)ds\right]^{-1}\right\}.$$

Theorem 3.2 Assume that the conditions (A1), (A2) are satisfied. Then, for each λ satisfying

$$R_3 < \quad < R_4 \tag{3.5}$$

there exists at least one positive solution (u, v) of the BVP (1.1)-(1.4).

Proof: Let λ be as in (3.5). And let ϵ>0 be chosen such that

$$\max\left\{\left[\frac{2M^2}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0}-\epsilon)d\tau\right)\right]^{-1},\right.$$
$$\left.\left[2M^2\int_0^1 H(s,s)\,\mathrm{ds}\,\phi_q^{-1}\left(\frac{1}{6^{n_2}}(f_{2,0}-\epsilon)\right)\right]^{-1}\right\}\le\lambda$$

and

$$\lambda\le\min\left\{\left[\frac{2}{6^{m_1}}\phi_q^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}+\epsilon)d\tau\right)\right]^{-1},\right.$$
$$\left.\left[2\int_0^1 H(s,s)ds\,\phi_p^{-1}\left(\frac{1}{6^{n_2}}(f_{2,\infty}+\epsilon)\right)\right]^{-1}\right\}.$$

Let T be the cone preserving completely continuous operator that was defined by (3.2). From the definitions of $f_{1,0}$ and $f_{2,0}$, there exists J_1>0 such that

$$f_1(t,u(t),v(t)) \ge (f_{1,0}-\epsilon)\phi_p(u+v),\quad \phi_p(u+v)\le J_1,$$

$$f_2(t,u(t),v(t)) \ge (f_{2,0}-\epsilon)\phi_q(u+v),\quad \phi_q(u+v)\le J_1.$$

Choose (u, v) ∈ P with ||(u, v)||=J_1. Then, from (2.9), Lemma 2.3 and Lemma 2.7 we have

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$$T_1(u,v)(t)=\lambda\int_0^1 H_{m_1}(t,s)\phi_p^{-1}\left(\int_0^1 G(s,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds$$
$$\geq\lambda\left(\frac{11}{6}\right)^{m_1-1}\frac{1}{4^{3m_1-2}}\int_0^1(1-s)s$$
$$\times\phi_p^{-1}\left(\frac{1}{4^{n_1-1}}\int_0^1 G(\tau,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds$$
$$\geq\lambda\frac{11^{m_1-1}}{4^{3m_1-2}}\frac{1}{6^{m_1-1}}\int_0^1(1-s)s\phi_p^{-1}\left(\frac{1}{4^{n_1-1}}\right)$$
$$\times\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0}-\in)\phi_p(u+v)d\tau\right)ds$$
$$\geq\lambda\frac{1}{6^{m_1}}\frac{11^{m_1-1}}{4^{3m_1-2}}.\phi_p^{-1}\left(\frac{1}{4^{n_1-1}}\right)$$
$$\times\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0}-\in)\phi_p(u+v)d\tau\right)$$
$$\geq\lambda\frac{M}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0}-\in)d\tau\right)(u+v)$$
$$\geq\lambda\frac{M^2}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,0}-\in)d\tau\right)\|(u,v)\|$$
$$\geq\frac{1}{2}\|(u,v)\|=\frac{J_1}{2}.$$

In a similar manner, we conclude that

$$T_2(u,v)(t)=\lambda\int_0^1 H(t,s)\phi_q^{-1}\left(\int_0^1 G_{n_2}(s,\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds$$
$$\geq\frac{1}{2}\|(u,v)\|=\frac{J_1}{2}.$$

Therefore, $T(u,v)(t)\geq J_1$

Hence $\|T(u,v)\|\geq J_1=\|(u,v)\|$ If we set,

$$\Omega_1=\{(u,v)\in B\,|\,\|(u,v)\|<J_1\},$$

Then

$$\|T(u,v)\|\geq\|(u,v)\|\quad (u,v)\in P\cap\partial\Omega_1 \tag{3.6}$$

Let $\overline{f}\ (t)\quad f(t,u(t),v(t))\ \overline{f}_2(t)=f_2(t,u(t),v(t))$. Now, we define the functions $f_1^*,f_2^*:R^+\to R^+$ by $f_1^*(x)=\max_{0\leq t\leq\phi_p(x)}\overline{f}_1(t)$ and $f_2^*(x)=\max_{0\leq t\leq\phi_q(x)}\overline{f}_2(t)$, for all $x\in R^+$.

Then $\overline{f}_1(t)\leq f_1^*(x)$ and $\overline{f}_2(t)\leq f_2^*(x)$

It follows that the functions f_1^*, f_2^* are non-decreasing and satisfy the conditions

$$\lim_{x\to\infty}\frac{f_1^*(x)}{\phi_p(x)}\leq f_{1,\infty}\ \text{and}\ \lim_{x\to\infty}\frac{f_2^*(x)}{\phi_q(x)}\leq f_{2,\infty}.$$

Next, by the definitions of $f_{1,\infty}$ and $f_{2,\infty}$ there exist $J_2>0$ such that

$$f_1^*(x)\leq(f_{1,\infty}+\in)\phi_p(x)\ \text{ and }\ f_2^*(x)\leq(f_{2,\infty}+\in)\phi_q(x)\ ,\ x\leq\overline{J}_2$$

Then, for $J_2=\max\{2J_1,\overline{J}_2\}$ we have

$$f_1^*(x)\leq f_1^*(J_2)\ \text{and}\ f_2^*(x)\leq f_2^*(J_2),\ 0<x\leq J_2$$

Choose $(u, v)\in P$ with $\|(u, v)\| = J_2$. Then we have

$$T_1(u,v)(t)=\lambda\int_0^1 H_{m_1}(t,s)\phi_p^{-1}\left(\int_0^1 G(s,\tau)f_1(\tau,u(\tau),v(\tau))d\tau\right)ds$$
$$\leq\lambda\frac{1}{6^{m_1-1}}\int_0^1(1-s)s\,\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)f_1^*(x)d\tau\right)ds$$
$$\leq\lambda\frac{1}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)f_1^*(J_2)d\tau\right)$$
$$\leq\lambda\frac{1}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}+\in)\phi_p(J_2)d\tau\right)$$
$$\leq\lambda\frac{1}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}+\in)\phi_p(J_2)d\tau\right)$$
$$\leq\lambda\frac{1}{6^{m_1}}\phi_p^{-1}\left(\int_0^1 G(\tau,\tau)(f_{1,\infty}+\in)d\tau\right)J_2$$
$$\leq\frac{J_2}{2}$$

In a similar manner, we conclude that

$$T_2(u,v)(t)=\lambda\int_0^1 H(t,s)\phi_q^{-1}\left(\int_0^1 G_{n_2}(s,\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds$$
$$\leq\frac{J_2}{2}.$$

Therefore, $T(u,v)(t)\geq J_2$

Hence, $\|T(u, v)\|\leq J2 = \|(u, v)\|$. If we set

$$\Omega_2=\{(u,v)\in B\,|\,\|(u,v)\|<J_2\},$$

And $\|T(u,v)\|\leq\|(u,v)\|\quad (u,v)\in P\cap\partial\Omega_2$ (3.7)

Applying Theorem 2.9 to (3.6) and (3.7), we obtain that T has a fixed point (u, v) in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$ and hence the BVP (1.1)-(1.4) has a positive solution such that $J_1\leq\|(u, v)\|\leq J_2$. The proof is complete.

Example

Let us consider an example to illustrate the above result. Now, we consider two-point (p, q) Laplacian boundary value problems

$$(-1)^1\left[\phi_p\left(u^{(4)}(t)\right)\right]'''=\lambda f_1(t,u(t),v(t)),\ t\in[0,1], \tag{4.1}$$

$$(-1)^2\left[\phi_q\left(v''(t)\right)\right]^{(6)}=\lambda f_2(t,u(t),v(t)),\ \frac{1}{p}+\frac{1}{q}=1, \tag{4.2}$$

$$\left.\begin{array}{l} u^{(2i)}(0)=0=u^{(2i)}(1),\ i=0,1,\\ [\phi_p(u^{(4)}(t))]^{(j)}_{at\ t=0}=0,\ j=0,1,\\ [\phi_p(u^{(4)}(t))]_{at\ t=1}=0,\end{array}\right\} \tag{4.3}$$

$$\left.\begin{array}{l}[\phi_q(v''(t))]^{(2i)}_{at\ t=0}=0=[\phi_q(v''(t))]^{(2i)}_{at\ t=1},\ i=0,1,2,\\ v(0)=0,\ v(1)=0\end{array}\right\} \tag{4.4}$$

Where

$$f_1(t,u(t),v(t))=(u+v)[9995000000-9994999845e^{-5(u+v)^3}],$$

$$f_2(t,u(t),v(t))=(u+v)[2400000000-2399999970e^{-7(u+v)^5}].$$

The Green's function $G(t, s)$ for the homogeneous BVP,

$$-x'''(t) = 0,$$

$$x(0) = x'(0) = 0, \; x(1) = 0,$$

is given by

$$G(t,s) = \begin{cases} \dfrac{t^2(1-s)^2}{2}, & 0 \le t \le s < 1, \\ \dfrac{t^2(1-s)^2}{2} - \dfrac{(t-s)^2}{2}, & 0 < s \le t \le 1. \end{cases}$$

The Green's function $H(t, s)$ for the homogeneous BVP,

$$-x'''(t) = 0,$$

$$x(0) = 0, \; x(1) = 0,$$

is given by

$$H(t,s) = \begin{cases} t(1-s), & 0 \le t \le s < 1, \\ s(1-t), & 0 < s \le t \le 1. \end{cases}$$

By direct calculations, we have

$$M = 0.0001678466 \, (p = 2), \quad \int_0^1 G(\tau,\tau)d\tau = \frac{1}{60}, \quad \int_0^1 H(s,s)ds = \frac{1}{6},$$

$$f_{1,0} = 155, \; f_{2,0} = 90,$$

$$f_{1,\infty} = 9995000000, \quad f_{2,\infty} = 2400000000,$$

$$R_1 = \max\{3.835473314, 3.108280305\} = 3.835473314,$$

And $R_2 = \min\{6.967742005, 7.2\} = 6.967742005.$

Applying Theorem 3.1, we get an optimal eigenvalue interval $3.835473314<\lambda<6.967742005$ for which the boundary value problem (4.1)-(4.4) has at least one positive solution [14-19].

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