Journal of
Applied & Computational Mathematics

Safavi, J Applied Computat Mathemat 2013, 2:1
DOI: 10.4172/2168-9679.1000123

Research Article Open Access

Solutions of System of Volterra Integro Differential Equation of Second Kind by Using Piecewise Constant Functions

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Abstract

In this paper we may use piece wise constant functions for the special type of system of second kind integro differential equation of the first order. The main problem is reduced to linear system of algebraic equations. Some numerical examples are dedicated for showing efficiency and validity of the method.

Keywords: Second kind integro differential equation; Block pulse functions; Collocation points

Introduction

Many different basis functions are used for approximating the solution of integro differential equations like Haar wavelets, lagrange functions, Taylor polynomials, Chebyshev polynomials, sine-cosine wavelets, Tau method, Adomian decomposition method, hybrid Legendre and Block Pulse functions and so on [1-8]. In this study, we use Block Pulse functions (BPfs) for solving system of Volterra integro differential equation of the form.

$$\begin{cases} y_1(x)+\lambda\int_0^x k_1(x,t)y_1'(t)\ dt = g_1(x) \\ y_2(x)+\lambda\int_0^x k_2(x,t)y_2'(t)\ dt = g_2(x) \\ \vdots \\ y_m(x)+\lambda\int_0^x k_m(x,t)y_m'(t)\ dt = g_m(x) \end{cases} \quad (1)$$

where, $k_i\in[0,1), g_i\in[0,1)$ for $i=1,\dots m$ are known functions and $y_i(x)$ is the unknown function and $y_1(0)=y_1$, $y_2(0)=y_2$, $\dots y_m(0)=y_m$. we set

$$F_i(x)=\int_0^x k_i(x,t)y_i'(x)dt \qquad for \quad i=1,2,\dots,m \quad (2)$$

so (1) becomes

$$\begin{cases} y_1(x)+\lambda F_1(x)=g_1(x) \\ y_2(x)+\lambda F_2(x)=g_2(x) \\ \vdots \\ y_m(x)+\lambda F_m(x)=g_m(x) \end{cases} \quad (3)$$

by collocating (3) at the points x_i, 1,2,…,k we get

$$\begin{cases} y_1(x_i)+\lambda F_1(x_i)=g_1(x_i) \\ y_2(x_i)+\lambda F_2(x_i)=g_2(x_i) \\ \vdots \\ y_m(x_i)+\lambda F_m(x_i)=g_m(x_i) \end{cases} \quad (4)$$

At first, we define a k-set of BPfs for every row of (4) as:

$$B_i(t)=\begin{cases} 1 & \frac{i-1}{k}\le t\le\frac{i}{k}\ ,\ for\ all\ \ i=1,2,\dots,k \\ 0 & elsewhere \end{cases}$$

The functions $B_r(t)$ are disjoint and orthogonal. That is,

$$B_i(t)B_j(t)=\begin{cases} 0 & i\ne j \\ B_i(t) & i=j \end{cases} \quad (5)$$

$$\langle B_i(t)B_j(t)\rangle=\begin{cases} 0 & i\ne j \\ \frac{1}{k} & i=j \end{cases} \quad (6)$$

Since the BPfs is not continuous so the derivatives don't exist at these points of discontinuity therefore we can't apply the BPfs in a direct manner to solve differential equations. Therefore we may expand $y_1'(x), y_2'(x),\dots,y_m'(x)$ into the BPfs series and $y_1(x), y_2(x),\dots,y_m(x)$ will be obtain through integration:

$$\begin{aligned} y_1'(x)&=\sum_{i=1}^{\infty} y_{1\ i}B_i(x) \\ y_2'(x)&=\sum_{i=1}^{\infty} y_{2\ i}B_i(x) \\ &\vdots \\ y_m'(x)&=\sum_{i=1}^{\infty} y_{m\ i}B_i(x) \end{aligned} \quad (7)$$

where,

$$\begin{aligned} y_{1\ i}&=k\langle y_1(x),B_i(x)\rangle=k\int_0^1 y_1(x)B_i(x)dx \\ y_{2\ i}&=k\langle y_2(x),B_i(x)\rangle=k\int_0^1 y_2(x)B_i(x)dx \\ &\vdots \\ y_{m\ i}&=k\langle y_m(x),B_i(x)\rangle=k\int_0^1 y_m(x)B_i(x)dx \end{aligned}$$

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Received December 02, 2012; **Accepted** January 28, 2013; **Published** February 06, 2013

Citation: Safavi M (2013) Solutions of System of Volterra Integro Differential Equation of Second Kind by Using Piecewise Constant Functions. J Applied Computat Mathemat 2: 123. doi:10.4172/2168-9679.1000123

In practice, only the first k-term of (7) are considered, that is,

$$y_1'(x)\simeq\sum_{i=1}^{k} y_{1\ i}B_i(x)$$
$$y_2'(x)\simeq\sum_{i=1}^{k} y_{2\ i}B_i(x)$$
$$\vdots$$
$$y_m'(x)\simeq\sum_{i=1}^{k} y_{m\ i}B_i(x)$$

with matrix form

$$\begin{aligned} y_1'(x)&\simeq B^t(x)Y_1\\ y_2'(x)&\simeq B^t(x)Y_2\\ &\vdots\\ y_m'(x)&\simeq B^t(x)Y_m \end{aligned} \tag{8}$$

With $Y_1=[y_{1\,1},y_{1\,2},\dots y_{1\,k}]^t, Y_2=[y_{2\,1},y_{2\,2},\dots y_{2\,k}]^t,\dots,Y_m=[y_{m\,1},y_{m\,2},\dots y_{m\,k}]^t$ and $B(x)=[B_1(x),B_2(x),\dots B_k(x)]^t$.

Also the integration of BPfs is expandable into BPfs series:

$$\int_0^x B(x)=PB(x)\ , \tag{9}$$

the k-square matrix P is called the operational matrix of integration of the transform and is defined as follows:

$$P_{k\times k}=\frac{1}{2}\begin{pmatrix}1&2&2&\dots&2\\0&1&2&\dots&2\\0&0&1&\dots&2\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\end{pmatrix} \tag{10}$$

Integrating (8) over the interval $[0, x]$ and using (9) yields

$$\begin{aligned} y_1(x)&\simeq y_1+(\int_0^x B^t(x)\ dx)\ Y_1\\ &=y_1+B^t(x)P^tY_1\\ y_2(x)&\simeq y_2+(\int_0^x B^t(x)\ dx)\ Y_2\\ &=y_2+B^t(x)P^tY_2\\ &\vdots\\ y_m(x)&\simeq y_m+(\int_0^x B^t(x)\ dx)\ Y_2\\ &=y_m+B^t(x)P^tY_m \end{aligned} \tag{11}$$

Similarly, $k_i(x,t)\in L^2[0,1)^2\ ,\forall i=1,2,\dots,m$ may be approximated as:

$$\begin{aligned} k_1(x,t)&\simeq B^t(x)K_1B(x)\\ k_2(x,t)&\simeq B^t(x)K_2B(x)\\ &\vdots\\ k_m(x,t)&\simeq B^t(x)K_mB(x) \end{aligned} \tag{12}$$

Where $K_1=[K_{ij}]_{1\le i,j\le k}\ ,K_2=[K_{ij}]_{1\le i,j\le k}\ ,\dots,K_m=[K_{ij}]_{1\le i,j\le k}$ and $K_{i,j}=k^2\langle B_i(x),\langle k_s(x,t),B_j(x)\rangle\rangle\ \ for\ s=1,2,\dots,m$. Now using (5) leads to

$$\begin{aligned} B(x)B^t(x)&=\begin{pmatrix}B_1(x)&&&0\\&B_2(x)&&\\&&\ddots&\\0&&&B_k(x)\end{pmatrix}\\ &=diag[B_1(x),B_2(x),\dots,B_k(x)] \end{aligned} \tag{13}$$

Now we choose

$$x_i=\frac{i}{k},i=1,2,\dots,k$$

so for evaluating $F_j(x_i)\ ,\forall j=1,2,\dots,m$ by substituting the matrix form of functions $k_j(x,t)\ ,\forall j=1,2,\dots,m$ and $y_1'(t)\ ,y_2'(t)\ ,\dots y_m'(t)$. and using the fact that $B(x_i)$ where e_i is the i-th column of the identity matrix of order k, we may proceed as follows:

$$\begin{aligned} F_1(x_i)&=B^t(x_i)K_1\int_0^{x_i}B(t)\ B^t(t)\ dt\ Y_1\\ &=e_i^t\ K_1\sum_{j=1}^{i}\int_{\frac{j-1}{k}}^{\frac{j}{k}} diag[B_1(x),B_2(x),\dots,B_k(x)]\ dt\ Y_1\\ &=e_i^tK_1\left[\int_0^{\frac{1}{k}}diag[1,0,\dots,0]dt+\dots+\int_{\frac{j-1}{k}}^{\frac{j}{k}}diag[0,\dots,0,1,\dots,0,\dots,0]\ dt\right]Y_1\\ &=e_i^tK_1\left[diag[\frac{1}{k},0,\dots,0]+\dots+diag[0,\dots,0,\frac{1}{k},\dots,0,\dots,0]\right]Y_1\\ &=\frac{1}{k}e_i^tK_1\ d^i\ Y_1\\ F_2(x_i)&=\frac{1}{k}e_i^tK_2\ d^i\ Y_2\\ &\vdots\\ F_m(x_i)&=\frac{1}{k}e_i^tK_m\ d^i\ Y_m \end{aligned} \tag{14}$$

where d^i is the $k\times k$ diagonal matrix defined as follows:

$$d_{pq}^i=\begin{cases}1 & p=q=1,2,\dots,i\\ 0 & p=q=i+1,\dots,k\end{cases}\qquad for\ i=1,2,\dots,k.$$

Now by substituting (11) and (14) into (4) we obtain:

$$\begin{cases} y_1+\dfrac{\lambda}{k}e_i^tK_1\ d^i\ Y_1=g_1(x_i)\\ y_2+\dfrac{\lambda}{k}e_i^tK_2\ d^i\ Y_2=g_2(x_i) & for\ \ i=1,2,\dots,k\\ \quad\vdots\\ y_m+\dfrac{\lambda}{k}e_i^tK_m\ d^i\ Y_m=g_m(x_i) \end{cases} \tag{15}$$

$$\begin{cases} A^1Y_1=g_1(x_i)-y_1\\ A^2Y_2=g_2(x_i)-y_2\\ \quad\vdots\\ A^mY_m=g_m(x_i)-y_m \end{cases}\ ,\quad i=1,2,\dots,k, \tag{16}$$

where,

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$$\begin{cases} A^i{}_1 = e_i^t P^t + \dfrac{\lambda}{k} e_i^t K_1 \; d^i \\ A^i{}_2 = e_i^t P^t + \dfrac{\lambda}{k} e_i^t K_2 \; d^i \\ \vdots \\ A^i{}_m = e_i^t P^t + \dfrac{\lambda}{k} e_i^t K_m \; d^i \end{cases} \quad , \quad i = 1, 2, \ldots, k, \tag{17}$$

Solving the linear system of algebraic equations (16) gives column vectors $Y_1, Y_2, \ldots, Ym$ and we can approximate $y_1(x), y_2(x), \ldots, y_m(x)$ by (11) at every points $x_i \in [1, 0]$.

Illustrative Example

Consider system of second kind volterra integro differential equation with initial conditions

$$\begin{cases} y_1(x) + \int_0^x \cos(x-t) y_1'(t) \; dt = x^2 - 2\cos(x) + 2, \\ y_2(x) + \lambda \int_0^x x t y_2'(t) \; dt = x + \dfrac{1}{2} x^3, \end{cases} , \begin{cases} y_1(0) = 0 \\ y_1(0) = 0 \end{cases}$$

the exact solution of system is

$$\begin{cases} y_1(x) = x^2 \\ y_1(x) = x \end{cases} .$$

Table 1 show the computed error for the example with k=16 and table 2 show the computed error for the example with k=32.

t	*Exact*		$y_1(x)$	$y_2(x)$
	$y_1(x)$	$y_2(x)$		
0.1	0.01	0.1	0.0135	0.1257
0.2	0.04	0.2	0.0463	0.2498
0.3	0.09	0.3	0.0994	0.3452
0.4	0.16	0.4	0.1727	0.4285
0.5	0.25	0.5	0.2648	0.5262
0.6	0.36	0.6	0.3786	0.6365
0.7	0.49	0.7	0.5111	0.7386
0.8	0.64	0.8	0.6638	0.8389
0.9	0.81	0.9	0.8367	0.9400

Table 1: Approximate and exact solution of system for k=16.

t	*Exact*		$y_1(x)$	$y_2(x)$
	$y_1(x)$	$y_2(x)$		
0.1	0.01	0.1	0.0115	0.1249
0.2	0.04	0.2	0.0432	0.2128
0.3	0.09	0.3	0.0947	0.3153
0.4	0.16	0.4	0.166	0.4189
0.5	0.25	0.5	0.2575	0.5160
0.6	0.36	0.6	0.3691	0.6182
0.7	0.49	0.7	0.5006	0.7186
0.8	0.64	0.8	0.6519	0.8193
0.9	0.81	0.9	0.8231	0.9199

Table 2: Approximate and exact solution of system for k=32.

Conclusions

In this paper, Block Pulse functions were used to solve special type of system of the second kind volterra integro differential equation which convert the main problem to solve linear system of algebraic equations. For showing validity and efficiency, the method is applied for test problem with two different values of the parameter k which states approximations may be more accurate using larger k. The benefit of the method is simplicity for execution and using sparse matrices which make the method cheap as computational costs.

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