Solution of Voltera-Fredholm Integro-Differential Equations using Chebyshev Collocation Method

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Abstract

In this paper, we use chebyshev polynomial basis functions to solve the Fredholm and Volterra integro-differential equations. We directly calculate integrals and other terms are calculated by approximating the functions with the Chebyshev polynomials. This method affects the computational aspect of the numerical calculations. Application of the method on some examples show its accuracy and efficiency.

Keywords: Integro-differential equation; Chebyshev polynomial; Collocation method

Introduction

We consider the integro-differential equations of Fredholm, Volterra and Fredholm-Volterra types in the forms

\[ D_y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, \quad y(x) \approx \sum_{l=0}^N a_l T_l(x), \]

where \( \lambda \), \( a_l \), and \( T_l(x) \) are real parameters. The functions \( f(x) \), \( k(x,t) \), \( k(x,t) \) and \( k(x,t) \) are known, \( y(x) \) is the unknown function to be determined and \( D \) is a linear differential operator. We suppose, without loss of generality, that the interval of integration is \([-1,1]\). Many problems in engineering and mechanics can be transformed into integral equations. For example it is usually required to solve Fredholm integral equations(FIE) in the calculations of plasma physics [1].

The numerical solution of these equations is a well-studied problem and a large variety of numerical methods have been developed to rapidly and accurately obtain approximations to \( y(x) \). Overviews and references to the literature for many existing methods are available in [2,3].

The paper is organized as follows: In Section Approximations we describe numerical approximations for differential operator and functions of integro-differential equation. The numerical results are presented in Section Numerical examples.

Methods and Approximations

Let \( D \) be a linear differential operator of order \( \nu \) with polynomial coefficients defined by

\[ D := \sum_{r=0}^\nu p_r(x) \frac{d^r}{dx^r}, \quad p_r(x) = \sum_{j=0}^N p_{rj}x^j, \]

where \( p_r(x) \) are known, \( p_{rj} \) are given constants.

Let \( y(x) \) be the exact solution of the integro-differential equation

\[ D_y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, \quad x \in [a,b], \]

with

\[ \sum_{k=1}^N \sum_{r=0}^\nu C^{(r)}_{jk} y^{(r-1)}(a) + C^{(2)}_{jk} y^{(r-1)}(b) = d_j, \quad j = 1, ..., v, \]

where \( f(x) \) and \( k(x,t) \) are continuous functions and \( \lambda, a, b, \) and \( d_j \) some given constants.

Matrix representation for \( D_y(x) \)

Let \( V := \{v_0(x), v_1(x), ..., \} \) be a polynomial basis given by

\[ V = V \chi \quad \text{where} \quad \chi = \{1, x, x^2, ..., \} \]

and \( V \) is a non-singular lower triangular matrix with degree \( \nu \) for \( i = 0,1,2, ..., \). According to [17] the effect of differentiation, shifting and integration on the

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and \( V \) is a non-singular lower triangular matrix with degree \( \nu \) for \( i = 0,1,2, ..., \). According to [17] the effect of differentiation, shifting and integration on the
coefficient vector \( \tilde{\alpha} = (\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_n, 0, 0, ...) \) of a polynomial \( u_n(x) = \tilde{\alpha}X \) is the same as that of post-multiplication of \( \tilde{\alpha} \) by the matrices \( \eta, \eta^T \) and \( l \) respectively,

\[
\eta = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}, \quad \mu = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}, \quad l = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}.
\]

We have
\[
\eta = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
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\end{bmatrix}, \quad \mu = \begin{bmatrix}
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\end{bmatrix}, \quad l = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}.
\]

We recall now the following theorem given by Oritez and Samara [15].

**Theorem 4.1.1** For any linear differential operator \( D \) defined by Eq. (6) and any series
\[
y(x) := AT(X, A) = [a_0, a_1, a_2, ...], \quad A = [\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, ...],
\]
we have
\[
Dy(x) = \tilde{A}QX = \tilde{A}T,
\]
where
\[
Q = \sum_{j=0}^{N} p_j(\mu), \quad Z = QT^{-1}, \quad T = \begin{bmatrix}
1 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}.
\]

**Function approximation**

The solution of Eqs. (1), (2) and (3) can be expressed as a truncated Chebyshev series. Therefore, the approximate solution (4) can be written in the matrix form
\[
y(x) = T^T(X)A,
\]
where
\[
T(x) = [T_0(x), T_1(x), ..., T_N(x)]^T, \quad A = [a_0, a_1, a_2, ..., a_N]^T.
\]

Consequently, using Theorem 2.1 and substituting Eq. (10) in Eq. (1), we get
\[
A^T Z T(X) = f(x) + \lambda \int_0^1 K(x, t)T^T(t)A dt.
\]

Now using the Chebyshev collocation point (5) in Eq. (11) we obtain the following new system of algebraic equations
\[
A^T Z T(x_i) = f(x_i) + \lambda \int_0^1 K(x_i, t)T^T(t)A dt,
\]
where
\[
x_i = \cos\left(\frac{i\pi}{N}\right), \quad i = 0, 1, 2, ..., N,
\]
and so, unknown coefficients \( a_j \) are found.

**Definition 4.2.1** The polynomial \( y_n(x) = A_nT^T = T(x) \), will be called an approximate solution of Eqs. (8) and (9), if the vector \( A_n = [a_0, a_1, ..., a_n] \) is the solution of the system of linear algebraic equations (12).

Similarly we can develop the method for the problem defined in the domain \([0, 1]\)
\[
Dy(x) = f(x) + \lambda \int_0^1 K(x, t)y(t) dt
\]
In this case we obtain the solution in terms of shifted Chebyshev polynomials \( T'_n(x) \) in the form
\[
y(x) = \sum_{j=0}^{N} a_jT'_n(x), \quad 0 \leq x \leq 1,
\]
where \( T'_n(x) = T_n(2x - 1) \) Similar to the previous procedure and using the collocation points defined by
\[
x_i = \frac{1}{2} (1 + \cos(\frac{i\pi}{N})), \quad i = 0, 1, 2, ..., N,
\]
one can get the following system of algebraic equations
\[
A^T Z T'(x) = f(x) + \lambda \int_0^1 K(x, t)A^T T'(t) dt,
\]
where
\[
T'(x) = [T'_0(x), T'_1(x), ..., T'_N(x)] \text{ and } (Z' = T'_Q T'^T) \text{ and } (Z' = T' A) .
\]

Solving the nonlinear system, unknown coefficients \( a_j \) are found. Similarly, we obtain the fundamental equation for Volterra and Fredholm-Volterra integral equation. In this study, instead of approximating integral terms, we directly calculate integrals. Examples show that this method affects the computational aspect of the numerical calculations.

**Results and Numerical Examples**

The results obtained in previous sections are used to introduce a direct efficient and simple method to solve integro-differential equations of Volterra and Fredholm type.

**Example 5.1** We consider the following Fredholm integro-differential equation of the second kind
\[
y'(x) = y(x) + 1 - \frac{4}{3} x + \int_0^1 xy(t) dt, \quad y(0) = 0
\]
and the exact solution is \( y(x) = cx \). We assume the solution of \( y(x) \) as a truncated Chebyshev series
\[
y(x) = a_0T_0(x) + a_1T_1(x), \quad 0 \leq x \leq 1.
\]

Here, we have
\[
f(x) = 1 - \frac{4}{3} x, \quad k(x, t) = xt, \quad \lambda = 1, \quad N = 1
\]
and the fundamental equation of the problem is defined by
\[
A^T Z T'(x_i) = f(x_i) + \lambda \int_0^1 x_i A T'(t) dt, \quad x_0 = 0, \quad x_i = 0,
\]
where
\[
T'(x_i) = [T'_0(x_i), T'_1(x_i)] = [1, 2x_i - 1], \quad A = [a_0, a_1]^T
\]
and
\[
T'(t) = [T'_0(t), T'_1(t)] = [1, 2t - 1],
\]

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\[ \eta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ T' = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad T'^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \]

Therefore, using Theorem 4.1.1 we obtain
\[ Q = \sum_{i=0}^{N} \eta^i p_i(\mu) = \sum_{i=0}^{N} p_i(\mu) = -1 + \eta = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \]

The system yields the solution
\[ a_0 = \frac{1}{2}, \quad a_i = \frac{1}{2} \]

Substituting these values in (15), we get the exact solution of the problem
\[ y(x) = \frac{1}{2} + \frac{1}{2}(2x - 1) = x \]

**Example 5.2** We consider the following Fredholm-Volterra integro-differential equation
\[ y'(x) = -2\sin(x) - x^2 \sin(2x) - 2x \cos(2x) + 2\sin(2x) - 2e^x + 5e^{-x} + 2x + \int_0^x \cos(x + t)y(x)dt + \int_0^x e^{x-t}y(x)dt, \]
\[ y(0) = 0 \]

The exact solution is \( y(x) = x^2 \). Let us suppose that \( y(x) \) is approximated by Chebyshev series
\[ y(x) = \sum_{j=0}^{N} a_j T_j(x), \quad 0 \leq x \leq 1 \]

Using the procedure in section Approximations, we obtain the approximate solution of the problem.

In Table 1, we compare the numerical results of the problem by the proposed method of N=3 with the method discussed in an earlier study [16].

**Example 5.3** We consider the following Fredholm integro-differential equation of the second kind
\[ y''(x) = y(x) - \frac{1}{20} \int_0^x y(t)dt - x^2 - 2x + 2521/68800, \]
\[ \begin{align*}
  y(0) &= 0 \\
  (1) - y' (1) &= 9
\end{align*} \]

The exact solution is \( y(x) = x^2 + 2x + 2 \). Talking \( N = 2, 4 \), the approximate solutions are obtained by this method. Results are compared with those of the methods in literature [17] as shown in Table 2.

**Example 5.4** We consider the following Volterra integro-differential equation of the second kind
\[ y''(x) + \int_0^x y'(x - t) + x(t) + e^{-x} - 2\sin(x) + \int_0^x \sin(x)e^{-x}y(t)dt, \quad -1 \leq x \leq 1 \]
\[ y(0) = 1, \quad y'(0) = 1 \]

The exact solution is \( y(x) = e^x \). See Table 3 for the numerical results.

### References