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Simplicial Hochschild cochains as an Amitsur complex [1]

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Abstract

It is shown how the cochain complex of the relative Hochschild A-valued cochains of a depth two extension $A\,|\,B$ under cup product is isomorphic as a differential graded algebra with the Amitsur complex of the coring $S = \mathrm{End}\,_B A_B$ over the centralizer $R = A^B$ with grouplike element 1_S. This specializes to finite dimensional algebras, Hopf-Galois extensions and H-separable extensions.

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1 Introduction

Relative Hochschild cohomology of a subring $B \subseteq A$ or ring homomorphism $B \to A$ is set forth in [4]. The coefficients of the general form of the cohomology theory are taken in a bimodule M over A. If $M = A^*$ is the k-dual of the k-algebra A, this gives rise to a cyclic symmetry exploited in cyclic cohomology. If $M = A$, this has been shown to be related to the simplicial cohomology of a finitely triangulated space via barycentric subdivision, the poset algebra of incidence relations and the separable subalgebra of simplices by Gerstenhaber and Schack in a series of papers beginning with [3]. The A-valued relative cohomology groups of (A, B) are also of interest in deformation theory. Thus we refer to the relative Hochchild cochains and cohomology groups $H^n(A, B; A)$ as simplicial Hochschild cohomology.

In this note we will extend the following algebraic result in [6]: given a depth two ring extension $A\,|\,B$ with centralizer $R = A^B$ and endomorphism ring $S = \mathrm{End}\,_B A_B$, the simplicial Hochschild cochains under cup product are isomorphic as a graded algebra to the tensor algebra of the (R, R)-bimodule S. Since S is a left bialgebroid over R, it is in particular an R-coring with grouplike element $1_S = \mathrm{id}_A$. The Amitsur complex of such a coring is a differential graded algebra explained in [2, 29.2]. We note below that the algebra isomorphism in [6] extends to an isomorphism of differential graded algebras. We remark on the consequences for cohomology of various types of Galois extensions with bialgebroid action or coaction.

2 Preliminaries on depth two extensions

All rings and algebras have a unit and are associative; homomorphisms between them preserve the unit and modules are unital. Let R be a ring, and M_R, N_R be two right R-modules. The notation M/N denotes that M is R-module isomorphic to a direct summand of an n-fold direct sum power of N: $M \oplus * \cong N^n$. Recall that M and N are similar [1, p. 268] if M/N and N/M. A ring homomorphism $B \to A$ is sometimes called a ring extension $A\,|\,B$ (proper ring extension if $B \hookrightarrow A$).

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Definition 2.1. A ring homomorphism $B \to A$ is said to be a right depth two (rD2) extension if the natural (A, B)-bimodules $A \otimes_B A$ and A are similar.

Left D2 extension is defined similarly using the natural (B, A)-bimodule structures: a D2 extension is both rD2 and ℓD2. Note that in either case any ring extension satisfies $A / A \otimes_B A$.

Note some obvious cases of depth two: 1) A a finite dimensional algebra, B the ground field. 2) A a finite dimensional algebra, B a separable algebra, since the canonical epi $A \otimes A \to A \otimes_B A$ splits. 3) $A \,|\, B$ an H-separable extension. 4) $A \,|\, B$ a finite Hopf-Galois extension, since the Galois isomorphism $A \otimes_B A \xrightarrow{\cong} A \otimes H$ is an (A, B)-bimodule arrow (and its twist by the antipode shows $A \,|\, B$ to be ℓD2 as well).

Fix the notation $S := \operatorname{End}{}_B A_B$ and $R = A^B$. Equip S with (R, R)-bimodule structure

$$r \cdot \alpha \cdot s = r\alpha(-)s = \lambda_r \circ \rho_s \circ \alpha$$

where $\lambda, \rho : R \to S$ denote left and right multiplication of $r, s \in R$ on A.

Lemma 2.2 ([5])**.** *If $A \,|\, B$ is rD2, then the module S_R is a projective generator and*

$$f_2 : \; S \otimes_R S \xrightarrow{\cong} \operatorname{Hom}({}_B A \otimes_B A_B, {}_B A_B)$$

via $f_2(\alpha \otimes_R \beta)(x \otimes_B y) = \alpha(x)\beta(y)$ for $x, y \in A$.

For example, if A is a finite dimensional algebra over ground field B, then $S = \operatorname{End} A$, the linear endomorphism algebra. If $A \,|\, B$ is H-separable, then $S \cong R \otimes_Z R^{\mathrm{op}}$, where Z is the center of A [5, 4.8]. If $A \,|\, B$ is an H^*-Hopf-Galois extension, then $S \cong R \# H$, the smash product where H has dual action on A restricted to R [5, 4.9].

Recall that a left R-bialgebroid H is a type of bialgebra over a possibly noncommutative base ring R. More specifically, H and R are rings with "target" and "source" ring anti-homomorphism and homomorphism $R \to H$, commuting at all values in H, which induce an (R, R)-bimodule structure on H from the left. W.r.t. this structure, there is an R-coring structure $(H, R, \Delta, \varepsilon)$ such that 1_H is a grouplike element (see the next section) and the left H-modules becomes a tensor category w.r.t. this coring structure. One of the main theorems in depth two theory is

Theorem 2.3 ([5])**.** *Suppose $A \,|\, B$ is a left or right D2 ring extension. Then the endomorphism ring $S := \operatorname{End}{}_B A_B$ is a left bialgebroid over the centralizer $A^B := R$ via the source map $\lambda : R \hookrightarrow S$, target map $\rho : R^{\mathrm{op}} \hookrightarrow S$, coproduct*

$$f_2(\Delta(\alpha))(x \otimes_B y) = \sum_{(\alpha)} f_2(\alpha_{(1)} \otimes_R \alpha_{(2)})(x \otimes_B y) = \alpha(xy) \tag{2.1}$$

Also A under the natural action of S is a left S-module algebra with invariant subring $A^S \cong \operatorname{End}{}_E A$, where $E := \operatorname{End} A_B \xleftarrow{\cong} A \# S$ via $a \otimes_R \alpha \mapsto \lambda_a \circ \alpha$.

We note in passing the measuring axiom of module algebra action from Eq. (2.1): in Sweedler notation, $\sum_{(\alpha)} \alpha_{(1)}(x)\alpha_{(2)}(y) = \alpha(xy)$.

3 Amitsur complex of a coring with grouplike

An R-coring $\mathcal{C}$ has coassociative coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$ and counit $\varepsilon : \mathcal{C} \to R$, both mappings being (R, R)-bimodule homomorphisms. We assume that $\mathcal{C}$ also has a grouplike element $g \in \mathcal{C}$, which means that $\Delta(g) = g \otimes_R g$ and $\varepsilon(g) = 1$. The Amitsur complex $\Omega(\mathcal{C})$ of $(\mathcal{C}, g)$ has n-cochain modules $\Omega^n(\mathcal{C}) = \mathcal{C} \otimes_R \cdots \otimes_R \mathcal{C}$ (n times $\mathcal{C}$), the zero'th given by $\Omega^0(\mathcal{C}) = R$. The Amitsur complex is the tensor algebra $\Omega(\mathcal{C}) = \oplus_{n=0}^{\infty}\Omega^n(\mathcal{C})$ with a compatible differential $d = \{d^n\}$ where $d^n : \Omega^n(\mathcal{C}) \to \Omega^{n+1}(\mathcal{C})$. These are defined by $d^0 : R \to \mathcal{C}$, $d^0(r) = rg - gr$, and

$$d^n(c^1 \otimes \cdots \otimes c^n) = g \otimes c^1 \otimes \cdots \otimes c^n + (-1)^{n+1} c^1 \otimes \cdots \otimes c^n \otimes g + \sum_{i=1}^{n} (-1)^i c^1 \otimes \cdots \otimes c^{i-1} \otimes \Delta(c^i) \otimes c^{i+1} \otimes \cdots \otimes c^n$$

Some computations show that $\Omega(\mathcal{C})$ is a differential graded algebra [2], with defining equations, $d \circ d = 0$ as well as the graded Leibniz equation on homogeneous elements,

$$d(\omega\omega') = (d\omega)\omega' + (-1)^{|\omega|}\omega d\omega'$$

The name Amitsur complex comes from the case of a ring homomorphism $B \to A$ and A-coring $\mathcal{C} := A \otimes_B A$ with coproduct $\Delta(x \otimes_B y) = x \otimes_B 1_A \otimes_B y$ and counit $\varepsilon(x \otimes_B y) = xy$. The element $g = 1 \otimes_B 1$ is a grouplike element. We clearly obtain the classical Amitsur complex, which is acyclic if A is faithfully flat over B. In general, the Amitsur complex of a Galois A-coring $(\mathcal{C}, g)$ is acyclic if A is faithfully flat over the g-coinvariants $B = \{b \in A \,|\, bg = bg\}$ [2, 29.5].

The Amitsur complex of interest to this note is the following derivable from the left bialgebroid $S = \operatorname{End}{}_B A_B$ of a depth two ring extension $A \,|\, B$ with centralizer $A^B = R$. The underlying R-coring S has grouplike element $1_S = \mathrm{id}_A$, with (R, R)-bimodule structure, coproduct and counit defined in the previous section. In Sweedler notation, we may summarize this as follows:

$$\Omega(S) = R \oplus S \oplus S \otimes_R S \oplus S \otimes_R S \otimes_R S \oplus \dots$$
$$d^0(r) = \lambda_r - \rho_r, \quad d^1(\alpha) = 1_S \otimes_R \alpha - \alpha_{(1)} \otimes_R \alpha_{(2)} + \alpha \otimes_R 1_S, \quad \dots$$

4 Cup product in simplicial Hochschild cohomology

Let $A \,|\, B$ be an extension of K-algebras. We briefly recall the B-relative Hochschild cohomology of A with coefficients in A (for coefficients in a bimodule, see the source [4]). The zero'th cochain group $C^0(A, B; A) = A^B = R$, while the n'th cochain group

$$C^n(A, B; A) = \mathrm{Hom}_{B-B}(A \otimes_B \cdots \otimes_B A, A)$$

(n times A in the domain). In particular, $C^1(A, B; A) = S = \operatorname{End}{}_B A_B$. The coboundary $\delta^n : C^n(A, B; A) \to C^{n+1}(A, B; A)$ is given by

$$(\delta^n f)(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})$$

and $\delta^0 : R \to S$ is given by $\delta^0(r) = \lambda_r - \rho_r$. The mappings satisfy $\delta^{n+1} \circ \delta^n = 0$ for each $n \geq 0$. Its cohomology is denoted by $H^n(A, B; A) = \ker \delta^n / \mathrm{Im}\, \delta^{n-1}$, and might be referred to as a simplicial Hochschild cohomology, since this cohomology is isomorphic to simplicial cohomology if A is the poset algebra of a finite simplicial complex and B is the separable subalgebra of vertices [3].

The cup product $\cup : C^m(A, B; A) \otimes_K C^n(A, B; A) \to C^{n+m}(A, B; A)$ makes use of the multiplicative stucture on A and is given by

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{n+m})$$

which satisfies [3] the equation

$$\delta^{n+m}(f \cup g) = (\delta^m f) \cup g + (-1)^m f \cup \delta^n g$$

Cup product therefore passes to a product on the cohomology. We note that $(C^*(A, B; A), \cup, +, \delta)$ is a differential graded algebra we denote by $C(A, B)$.

Theorem 4.1. *Suppose $A \,|\, B$ is a right or left D2 algebra extension. Then the relative Hochschild A-valued cochains $C(A, B)$ is isomorphic as a differential graded algebra to the Amitsur complex $\Omega(S)$ of the R-coring S.*

Proof. We define a mapping f by $f_0 = \mathrm{id}_R$, $f_1 = \mathrm{id}_S$, and for $n > 1$,

$$f_n : \; S \otimes_R \cdots \otimes_R S \xrightarrow{\cong} \mathrm{Hom}_{B-B}(A \otimes_B \cdots \otimes_B A, A)$$

by $f_n(\alpha_1 \otimes \cdots \otimes \alpha_n) = \alpha_1 \cup \cdots \cup \alpha_n$. (Note that f_2 is consistent with our notation in section 2.) We proved by induction on n in [6] that f is an isomorphism of graded algebras. We complete the proof by noting that f is a cochain morphism, i.e., commutes with differentials. For $n = 0$, we note that $\delta^0 \circ f_0 = f_1 \circ d^0$, since $d^0 = \delta^0$. For $n = 1$,

$$\begin{aligned}\delta^1(f_1(\alpha))(a_1 \otimes_B a_2) &= a_1\alpha(a_2) - \alpha(a_1 a_2) + \alpha(a_1)a_2 \\ &= f_2(1_S \otimes_R \alpha - \alpha_{(1)} \otimes_R \alpha_{(2)} + \alpha \otimes_R 1_S)(a_1 \otimes_B a_2) \\ &= f_2(d^1(\alpha))(a_1 \otimes_B a_2)\end{aligned}$$

using Eq. (2.1). The induction step is carried out in a similar but tedious computation: this completes the proof that $C(A, B) \cong \Omega(S)$. □

5 Applications of the theorem

We immediately note that the cohomology rings of the two differential graded algebras are isomorphic.

Corollary 5.1. *Relative A-valued Hochschild cohomology is isomorphic to the cohomology of the A^B-coring $S = \mathrm{End}\,{}_B A_B$:*

$$H^*(A, B; A) \cong H^*(\Omega(S), d)$$

if $A \,|\, B$ is a left or right depth two extension.

For example, we recover by different means the known result,

Corollary 5.2. *If the ring extension $A \,|\, B$ is H-separable and one-sided faithfully flat, then the relative Hochschild cohomology is given by*

$$H^n(A, B; A) = \begin{cases} Z(A^B) & n = 0 \\ 0 & n > 0 \end{cases}$$

Proof. Note that the extension is necessarily proper by faithful flatness. Note that $S \cong R \otimes_Z R$ is a Galois R-coring, since $\{r \in R \,|\, r \cdot 1_S = 1_S \cdot r\} = Z$, the center of A and the isomorphism $r \otimes s \mapsto \lambda_r \circ \rho_s$ is clearly a coring homomorphism. Whence $\Omega(S)$ is acyclic by [2, 29.5].

Finally,

$$H^0 = \ker d^0 = \{x \in R \,|\, rx - xr = 0, \forall r \in A^B\}$$

which is the center of the centralizer. □

This will also follow from proving that an H-separable is a separable extension, a condition of trivial cohomology groups.

Corollary 5.3. *Suppose $A \,|\, B$ is a finite Hopf-H^*-Galois extension. Then relative Hochschild A-valued cohomology is isomorphic to the Cartier coalgebra cohomology of H with coefficients in the bicomodule $A^B \otimes H$:*

$$H^*(A, B; A) \cong H^*_{\mathrm{Ca}}(H, R \otimes H)$$

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