Semi-Simplicity of a Lie Algebra of Isometries

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Abstract

A spray S on the tangent bundle TM with a n dimensional differentiable manifold M defines an almost product structure Γ (Γ=±I, I being the identity vector 1-form) and decomposes the TTM space into a direct sum of horizontal space (corresponding to the eigenvalue +1) and vertical space (for the eigenvalue -1). The Lie algebra of projectable vector fields $\mathfrak{a}_e$ whose Lie derivative vanishes the spray $S$ is of dimension at most $n^2+n$. The elements of the algebra $\mathfrak{a}_e$ belonging to the horizontal nullity space of Nijenhuis tensor of $\Gamma$ form a commutative ideal of $\mathfrak{g}_A$. They are not the only ones for any spray $S$. If $S$ is the canonical spray of a Riemannian manifold, the symplectic scalar 2-form $\Omega$ which is the generator of the spray $S$ defines a Riemannian metric $g$ upon the bundle vertical space of TM. The Lie algebra of infinitesimal isometries which is written $\mathfrak{g}_A$ contained in $\mathfrak{a}_e$ is of dimension at most $\frac{n(n+1)}{2}$. The commutative ideal of $\mathfrak{g}_A$ is also that of $\mathfrak{a}_e$. The Lie algebra $\mathfrak{a}_e$ of dimension superior or equal to three is semi-simple if and only if the nullity horizontal space of the $\Gamma$ Nijenhuis tensor is reduced to zero. In this case, $\mathfrak{a}_e$ is identical to $\mathfrak{a}_j$.

Keywords: Differentiable manifolds • Lie algebra • Spray • Nijenhuis tensor • Riemannian manifolds • Infinitesimal isometry

Introduction

Frölicher-Nijenhuis and Rund H.’s works [1,2] have enabled Griffone J. [3,4] to show a connection as a vector 1-form $\Gamma$ such that $J\Gamma=J$ and $J\Gamma=J$ of which $J$ is the natural tangent structure of the tangent bundle TM to a differentiable manifold $M$. Such $\Gamma$ is an almost product structure ($\Gamma^2=\pm I$), $I$ being the identity vector 1-form. This formalism allows a more algebraic approach for the study of Lie algebras defined by $\Gamma$ [5].

Given a paracompatible differentiable manifold $M$ of $n \geq 2$ dimension of class $C^\infty$, $\Gamma$ is a connection [3]. In [5], we studied the associated Lie algebras to $\Gamma$, through their first Chevalley-Eilenberg cohomology spaces, namely, the Lie algebra of vector fields $\mathfrak{a}_e$ on TM, whose Lie derivative applied to $\Gamma$ is zero, and the horizontal nullity space $\mathcal{N}_\Gamma$ of the curvature $R$ of $\Gamma$ ($\mathcal{N}_\Gamma=\mathfrak{a}_e\cap \mathfrak{g}_A$). The Lie algebra $\mathfrak{a}_e$ is formed by projectable vector fields, and contains two ideals $\mathfrak{a}_h$ and $\mathfrak{a}_v$, where $\mathfrak{a}_h$ consists of projectable vector fields of the horizontal nullity space of the curvature $R$ and $\mathfrak{a}_v$ an ideal of the vertical space.

In [8], Loos O. considered a spray $S$ as a system of second order differential equations on $M$. If we denote by $A$, the Lie algebra of vector fields on TM which commute with $S$, the projectable vector fields of $A$, correspond to $A_e \cap \mathfrak{g}(M)$, $\mathfrak{g}(M)$ being the complete lift on the tangent bundle $TM$ of the set of all vector fields $\chi(M)$ on $M$. We will denote the set $\mathfrak{a}_e$ such that $\mathfrak{a}_e = A_e \cap \mathfrak{g}(M)$. The Lie algebra $\mathfrak{a}_e$ corresponds to a Lie group $G_e$ of transformations on $M$ of which the tangent linear mappings preserve the spray $S$. The group $G_e$ acts freely on the linear frame bundle of $M$. The dimension of $\mathfrak{a}_e$ is at most equals to $n^2+n$.

In the following, a connection $\Gamma$ is linear without torsion, in this case [3], $\Gamma$ is written: $\Gamma = [J, S]$ and $[C, S]=S$, where $C$ being the Lievi-Civita field on TM. We prove that $\mathfrak{a}_e = \mathfrak{g}_A$ by considering $\mathfrak{g}_A = A_e \cap \mathfrak{g}(M)$. For a linear connection without torsion such that the rank of the horizontal nullity space of the projectable vector fields of curvature $R$ is nonzero constant, the Lie algebra $\mathfrak{g}_A$ contains a nonzero commutative ideal $\mathfrak{h}$ formed by the horizontal elements of $\mathfrak{g}_A$. All the commutative ideals of $\mathfrak{h}$ do not always come from the horizontal nullity space of the curvature for any spray $S$. Let $E$ be an energy function [3], that is to say, a mapping $C^\infty$ on $TM$, null on the null section, homogeneous of degree two $(L_i E=2E)$, $L_i$ being the Lie derivative with respect to $C$, such that the 2-form $\Omega=ddE$ is of maximum rank. The canonical spray $S$ is defined [5] by $i_s ddE=de,$

where $g_{ij}(x^1,\ldots,x^n)$ are positive functions such that the symmetric matrix $(g_{ij}(x^1,\ldots,x^n))$ is invertible. We denote by $\mathfrak{h}$ the set of $\mathfrak{h} \subset \mathfrak{g}(M)$ such that $\mathfrak{h} \cap \mathfrak{g}(M)$ is a commutative ideal of $\mathfrak{g}_A$ and $\mathfrak{h}$. We then have the following algebra inclusions: $\mathfrak{a}_e \subset \mathfrak{a}_h \subset \mathfrak{h}$. The dimension of $\mathfrak{h}$ is at most equal to $\frac{n(n+1)}{2}$. The $\mathfrak{h}$ is also the Lie algebra of the Killing fields of projectable vectors which commute with the spray $S$. If the dimension of $\mathfrak{h}$ is greater than or equal to three, the Lie algebra $\mathfrak{h}$ is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of $\Gamma$ is reduced to zero. In this context, the algebra $\mathfrak{h}$ coincides with $\mathfrak{a}_e$. It follows that for the flat Riemannian manifolds (R=0), the Lie algebra $\mathfrak{a}_e$ is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of $\Gamma$ is reduced to zero. This study also makes it possible to construct numerous examples of Lie algebra containing a commutative ideal. In particular, for a semi-simple Lie algebra, the derivative ideal coincides with algebra. This examples show that this property is not sufficient to be a semi-simple Lie algebra.
Preliminaries

We will recall the bracket of two vector 1-form K and L on a manifold M [1]. If we denote χ(M) the set of vector fields on M,
\[ [L, K](X, Y) = L(KY) + K(LX) - [X, K]LY - [Y, K]XL \]
(2.1)
for all X, Y ∈ χ(M).

The bracket N = [I, L] is called the Nijenhuis tensor of L. The Lie derivative with respect to X applied to L is written:
\[ \text{d}_X L = \{X, L\} \]
which is equal to
\[ \text{d}_X L = \{X, L\} = \{X, L\} \]
(2.2)
When X is a horizontal vector field,
\[ \text{d}_X L = \{X, L\} \]
(2.3)
for all X, Y ∈ χ(M).

We will recall the bracket of two vector 1-form K and L on a manifold M [3] as follows:
\[ [K, L](X, Y, Z) = [KX, LY] - [LX, KY] + [KLX, Y] - [LY, KX] ]
(2.4)
This means that the horizontal part of \([X, Y]\) is zero, we find
\[ \text{d}_X L = \{X, L\} \]
(2.5)
for all X, Y, Z ∈ χ(M).

Properties of the Vertical Ideal of \( A_r^v \)

A vertical vector field can be written in the form JX, J being the tangent structure on TM, and X a horizontal vector field.

**Proposition 1** [5]

A vertical vector field JX is an element of \( A_r^v \) if and only if JX commutes with all horizontal and projectable vector fields.

**Proposition 2**

Let \( \Gamma \) be a zero weak torsion connection, X and Y two horizontal vector fields such that JX and JY are elements of \( A_r^v \). So we have
\[ J[X, Y] = 0. \]

In particular,
\[ [X, Y] = R(X, Y). \]

**Proof:** The nullity of the weak torsion of \( \Gamma \) allows to write
\[ v[JX, Y] + v[X, JY] = J[X, Y], \]
(3.1)
for all \( X, Y \) horizontal vector fields. If JX and JY are two elements of \( A_r^v \), we get \( v[JX, Y] = 0 \) and \( v[X, JY] = 0 \), that is to say
\[ J[X, Y] = 0. \]

This means that the horizontal part of \([X, Y]\) is zero, we find
\[ [X, Y] = v[X, Y] = R(X, Y). \]

**Proposition 3**

Let \( \Gamma \) be a zero weak torsion connection, \( X \) a horizontal vector field. The two following conditions are equivalent:

a) \( X \) is a projectable vector field and \( [X, h] = 0 \);

b) \( [X, J] = 0 \).

**Proof:** The nullity of the weak torsion of \( \Gamma \) is written:
\[ [X, J]Y = 0, \]
(3.2)
for all horizontal vector fields \( X, Y \). If \( X \) is projectable, the term \( h [X, JY] \) is zero; and if \( [JX, h] = 0 \), we get \( [X, JY] = 0 \), for all horizontal vector fields \( Y \). If \( Y \) is a vertical vector field, we have \( [X, JY] = 0 \), for all horizontal vector fields \( Y \).

**Proposition 4**

Let \( \Gamma \) be a zero weak torsion connection, \( X \) a horizontal vector field. The two following conditions are equivalent:

a) \( X \) is a projectable vector field and \( [X, h] = 0 \);

b) \( [X, J] = 0 \).

**Proof:** The nullity of the weak torsion of \( \Gamma \) is written:
\[ [X, J]Y = 0, \]
(3.3)
for all horizontal vector fields \( X, Y \). If \( X \) is projectable, the term \( h [X, JY] \) is zero; and if \( [JX, h] = 0 \), we get \( [X, JY] = 0 \), for all horizontal vector fields \( Y \). If \( Y \) is a vertical vector field, we have \( [X, JY] = 0 \), for all horizontal vector fields \( Y \).

**Proposition 5**

Let \( \Gamma \) be a zero weak torsion connection, \( X \) a horizontal vector field. The two following conditions are equivalent:

a) \( X \) is a projectable vector field and \( [X, h] = 0 \);

b) \( [X, J] = 0 \).

**Proof:** The nullity of the weak torsion of \( \Gamma \) is written:
\[ [X, J]Y = 0, \]
(3.4)
for all horizontal vector fields \( X, Y \). If \( X \) is projectable, the term \( h [X, JY] \) is zero; and if \( [JX, h] = 0 \), we get \( [X, JY] = 0 \), for all horizontal vector fields \( Y \). If \( Y \) is a vertical vector field, we have \( [X, JY] = 0 \), for all horizontal vector fields \( Y \).
In particular, 
\[ \mathcal{H}(X, Y) S = - R(X, Y). \]

**Proposition 4**

Let \( JX \in A^r \) such that \( X \) be a projectable vector field, then we have
\[ \mathcal{H}(Y, Z) X = 0, \]
for all \( Y, Z \in \mathcal{H}(TM) \).

**Proof:** Let \( JX \in A^r \). This means \( [JX, h] = 0 \). According the expression of \( R\) one finds \( [JX, R] = 0 \). In other words, for all \( Y, Z \in \mathcal{H}(TM) \), we have
\[ [JX, R(Y, Z)] = R([JX, Y], Z) + R(Y, [JX, Z]) = 0. \]

So the expression of \( \mathcal{H} \) becomes
\[ \mathcal{H}(Y, Z) X = J(JX, R(Y, Z)). \]

The vector field \( X \) being projectable and \( R(Y, Z) \) vertical, the second member of the equality is null. So we find \( \mathcal{H}(Y, Z) X = 0 \).

Let \( N^r \) be the nullity space of \( \mathcal{H} \), \( Ker \mathcal{H} \) the kernel of \( \mathcal{H} \):
\[ N^r = \{ X \in \mathcal{H}(TM) \mid [JX, h] = 0, \forall Y, Z \in \mathcal{H}(TM) \}; \]
\[ Ker \mathcal{H} = \{ X \in \mathcal{H}(TM) \mid \mathcal{H}(Y, Z) X = 0, \forall Y, Z \in \mathcal{H}(TM) \}. \]

**Proposition 5**

The horizontal ideal \( A^h \) of \( A_r \) becomes
\[ A^h = N^r \cap H^r \subset Ker \mathcal{H} \cap H^r. \]

**Proof:** According to the relation \( \mathcal{H}(X, Y) S = - R(X, Y) \) for all \( X, Y \in \mathcal{H}(TM) \), we have
\[ N^r \subset N^h. \]

If \( X \in N^r \), according to the previous relationship between \( \mathcal{H} \) and \( R \), we get
\[ \mathcal{H}(X, Y) Z = R([JZ, X], Y); \]
if \( X \) is projectable, \( [JZ, X] \) is vertical and \( R([JZ, X], Y) \) is therefore zero. This results in
\[ N^r \cap H^r \subset N^h \cap H^r. \]

In conformity with result of \([5]\), we have
\[ N^r \cap H^r = N^h \cap H^r = A^h. \]

Bianchi identity on \( \mathcal{H} \) is written \([4]\):
\[ \mathcal{H}(X, Y) Z + \mathcal{H}(Y, Z) X + \mathcal{H}(Z, X) Y = 0, \]
for all \( X, Y, Z \in \mathcal{H}(TM) \), if \( X \in N^r \), we find
\[ \mathcal{H}(Y, Z) X = 0, \]
for all \( Y, Z \in \mathcal{H}(TM) \).

**Proposition 6**

If the module \( N^r \cap H^r \) on the ring of constant functions to the fibers is of nonzero and constant rank, there is locally a non-trivial vector field such that \( X \in A^h \) and \( JX \in A^h \).

**Proof:** We will solve the equation
\[ X \in N^r \cap H^r \quad \text{and} \quad [JX, h] = 0. \]

This leads to solving, according to the proposition 3, the equations
\[ X \in N^r \cap H^r \quad \text{and} \quad [X, J] = 0. \]
it possible to write
\[ [J, [X, S]] + [X, [S, J]] + [S, [J, X]] = 0. \]
If \([X, S] = 0\), the above relation becomes
\[ [J, [X, S]] = [X, [S, J]] + [S, [J, X]] = 0. \]
But we have \( \Gamma = [J, S], \Gamma = h = v \) and \( h + v = I \), we find
\[ \Gamma, [Y] = [\Gamma, Y] + v \Gamma [Y], \]
for all \( X \) and \( Y \) vector fields of \( \Gamma \). On the other hand, we have
\[ [S, [J, X]] = [S, [J, X]] \]
as \([J, X] = [S, [J, X]] \) and \([J, X] = [S, [J, X]] \). As \([J, X] = [S, Y] \) and \([J, X] = [S, Y] \) are vertical since \( X \) is projectable, by applying \( J \) to the relation (4.1) we find
\[ [J, [X, S]] = [X, [S, J]] + [S, [J, X]] = 0. \]

\[ [J, X] = 0, \]
for all \( X \) horizontal vectors field. If \( Y \) is vertical, it is immediate to note that
\[ [J, Y] = 0. \]
The relations (4.2) and (4.3) prove that \([J, X] = 0\). In other words, \( X \in A_{\Gamma} \). But \( A_{\Gamma} = \{ [M, J] + J[M], \} \) and taking into account the proposition 8, therefore, the set of projectable vector fields of \( A_{\Gamma} \) coincides with \( A_{\Gamma} \cap \mathcal{X}(M) \).

With \([J, X] = 0\), the relation (4.1) proves that \( X \in A_{\Gamma} \). That is to say
\[ A_{\Gamma} \cap \mathcal{X}(M) = A_{\Gamma} \cap \mathcal{X}(M). \]
Let us now prove that \( A_{\Gamma} \cap \mathcal{X}(M) \) is a commutative ideal of the Lie algebra \( A_{\Gamma} \).

\[ A_{\Gamma} \cap \mathcal{X}(M) = A_{\Gamma} \cap \mathcal{X}(M). \]
Let be \( \bar{X} \in A_{\Gamma} \cap \mathcal{X}(M) \). This results in \([\bar{X}, h] = 0\) with \( S = hS \). If we apply to \( S \) the above relation, we get
\[ [\bar{X}, hS] - h[\bar{X}, S] = 0 = v[\bar{X}, S]. \]
From the relation \([\bar{X}, J] = 0 \), we find \( J[\bar{X}, S] = 0 \). In other words, taking into account that \( h \) is semi-basic
\[ h[\bar{X}, S] = 0, \]
which proves that
\[ A_{\Gamma} \cap \mathcal{X}(M) = A_{\Gamma} \cap \mathcal{X}(M). \]

**Remark 1**: The proposition 9 is contained in [12] in another way.
In the following, we assume \( \overline{A_{\Gamma}} = A_{\Gamma} \cap \mathcal{X}(M) \) and \( \overline{A_{\Gamma}} = A_{\Gamma} \cap \mathcal{X}(M), \overline{A_{\Gamma}} \) the horizontal part of \( A_{\Gamma} \).

**Proposition 10**
The horizontal part of \( \overline{A_{\Gamma}} \) is such that
\[ \overline{A_{\Gamma}} = \{ X \in A_{\Gamma} \text{ and } JX \in A_{\Gamma} \}. \]
and \( \overline{A_{\Gamma}} \) is a commutative ideal of the Lie algebra \( A_{\Gamma} \).

**Proof**: This is to prove that the horizontal part of \( A_{\Gamma} \) is only the B of the proposition 7, which is immediate according to the proposition 3.

**Theorem 1**
For all differentiable manifold of \( n \) dimension with a linear connection without torsion such that the rank of the nullity space of the projectable vector fields of the curvature is nonzero constant, the Lie algebra \( \overline{A_{\Gamma}} \) contains a nonzero commutative ideal formed by the horizontal part of \( A_{\Gamma} \).

**Proof**: This is the consequence of previous studies. The existence of the nonzero horizontal part of \( \overline{A_{\Gamma}} \) is given by the proposition 6.

**Automorphisms**
We denote by \( G_{\Gamma} \) the set of diffeomorphisms \( \phi \) of \( M \) such that the induced diffeomorphisms \( \phi_{\Gamma} \) on \( TM \) preserve a spray \( S, G_{\Gamma} \) forms a group. From the study of the equations of \( A_{\Gamma} \) and \( G_{\Gamma} \) on a local coordinate system of \( M \), using the classical theorem of Palais, Loos O. [6] deduces the following result:

**Theorem 2** [6]

a) \( G_{\Gamma} \) acts freely on the linear frame bundle of \( M \). In particular, the dimension of \( \overline{A_{\Gamma}} \) is less than or equals to \( n^2 + n \).

b) There exists a unique Lie group structure on \( G_{\Gamma} \) such that \( G_{\Gamma} \) is a Lie transformation group of \( M \) with Lie algebra the set of complete vector fields in \( \overline{A_{\Gamma}} \).

c) If the dimension of \( \overline{A_{\Gamma}} = n^2 + n \) and \( M \) a manifold simply connected, then \((M, S)\) is isomorphic to \((\mathbb{R}^n, Z)\) for an unique \( \lambda \in \mathbb{R}; Z_{\lambda} \) is given by the equations
\[ \frac{dy^j}{dt} = \lambda y^j, \]
for an unique \( \lambda \in \mathbb{R}; Z_{\lambda} \) is given by the equations
\[ \frac{dy^j}{dt} = \lambda y^j, \]

**Examples**

**Example 1**
Let be \( \mu = \mathbb{R}^1 \) and the connection \( \Gamma = [J, S] \) with
\[ S = y^j \frac{\partial}{\partial x^j} + y^j \frac{\partial}{\partial y^j} + y^j \frac{\partial}{\partial y^j}, \]
then the coefficients of \( \Gamma \) being \( \Gamma = \frac{\partial y^j}{\partial x^i} \).

The horizontal space is generated by
\[ \frac{\partial}{\partial x^i} - 2e^x y^j y^i \frac{\partial}{\partial y^j} - 2e^x y^j y^i \frac{\partial}{\partial y^j}. \]

We notice that \( A_{\Gamma} = \mathbb{R} \) and \( \frac{\partial}{\partial y^j} \) on \( \mathbb{R} \).
We note that all the subalgebras of \( \{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \} \) are commutative ideals of \( A_{\Gamma} \).

**Example 2**
Let \( E \) be the energy function such that
\[ E = \frac{1}{2}(e^x y^j y^j + e^x y^j y^i + e^x y^j y^i), \]
The canonical spray is written:
\[ S = y^j \frac{\partial}{\partial x^j} + y^j \frac{\partial}{\partial y^j} + \frac{1}{2} (y^j y^i - 2e^x y^j y^i) \frac{\partial}{\partial y^j} + \frac{1}{2} (y^j y^i - 2e^x y^j y^i) \frac{\partial}{\partial y^j}. \]

The nonzero coefficients of the connection are
\[ \Gamma_1 = \frac{y^j}{2}, \Gamma_2 = \frac{y^j}{2} \]

The horizonal vector fields are generated by
\[ \frac{\partial}{\partial y^j} - \frac{y^j}{2} \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial y^j}. \]
The horizontal nullity space of the curvature is generated by \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \).  
The elements of \( \mathcal{A}_E \) are generated by  
\[
\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right), \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)^2 - \frac{\partial^3}{\partial x^3} - \frac{\partial^3}{\partial y^3}.
\]

The commutative ideal is generated by  
\[
\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right).
\]
We note that for any linear connection (see the example 1), the commutative ideals do not always come from the horizontal nullity space of the curvature.

The example 2 shows that the commutative ideal is of the horizontal nullity space of the curvature. The question is to know if it is true for all canonical connection coming from an energy function in the language of [3]. This is the object of our study.

**Connections on Riemannian Manifolds**

In this paragraph, we are content to state the well-known results on the Riemannian manifolds which will be used.

**Definition 2**

We call the Riemannian manifold, the couple \((M, E)\):

- \( M \) is a differentiable manifold;
- \( E \) a function of \( T^M = TM \setminus \{0\} \) into \( \mathbb{R}^+ \), with \( E(0) = 0 \), \( C^\infty \) on \( TM \), \( C^2 \) on the null section, and homogeneous of degree two, such that \( \dd_J E \) has a maximum rank.

We notice that if \( E \) is of class \( C^1 \) on the null section, we will have a Finsler manifold. Imposing \( E \) of class \( C^2 \) on the null section implies that the function \( E \) is of class \( C^\infty \) on \( TM \).

The application \( E \) is called energy function, \( \Omega = \dd_J E \) its fundamental form defines a spray \( S \) [13]:

\[
\Omega = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right).
\]

The fundamental form \( \Omega \) allows to define a metric \( g \) on the tangent bundle by

\[
g(X, Y) = \Omega(X, FY),
\]

\( X \) and \( Y \) being two vector fields on \( TM \), \( F \) the almost complex structure associated with \( \Gamma \), \( (FJ = h \) and \( Fh = -J \) where \( h = \frac{I + J}{2} \). There is, [4], one and only one metric lift \( D \) of the canonical connection such that:

1. \( \mathcal{J}^T(kX, hY) = 0 \),
2. \( \mathcal{T}(JX, JY) = 0 \), \( \mathcal{T}(X, Y) = D_J Y - D_Y X - [X, Y] \),
3. \( D_J = 0 \),
4. \( DC = v \),
5. \( DI = 0 \),
6. \( DG = 0 \),
7. \( DF = 0 \).

The linear connection \( D \) is called Cartan connection. For a Riemannian manifold, the Cartan connection \( D \) and the Berwald connection \( D \) are identical. We have

\[
D_{\partial x} Y = [J, Y] X, \quad D_{\partial y} Y = [h, JY] X.
\]

To the linear connection \( D \), we associate a curvature

\[
\mathcal{R}(X, Y) Z = D_{\partial x} D_{\partial y} J Z - D_{\partial y} D_{\partial x} J Z - D_{[\partial x, \partial y]} J Z, \quad (9.3)
\]

where \( g(x_1, \ldots, x_n) \) are symmetric positive functions such that the matrix \( g(x_1, \ldots, x_n) \) is invertible and the relation \( \Omega = \dd_J E \) gives the following classic results: for all \( X, Y, Z \),

\[
\mathcal{R}(X, Y) Z = - \mathcal{R}(X, Y) Z.
\]

From the properties of the curvatures given above, we have:

\[
g(\mathcal{R}(X, Y) Z, JT) = -g(\mathcal{R}(X, Y) T, JZ)\]

(8.1)

\[
\mathcal{R}(X, Y) Z + \mathcal{R}(Y, Z) X + \mathcal{R}(Z, X) Y = 0;
\]

(8.2)

\[
g(\mathcal{R}(T, X) Y, JZ) + g(\mathcal{R}(T, Y) Z, JX) + g(\mathcal{R}(T, Z) X, JY) = 0;
\]

(8.3)

\[
g(\mathcal{R}(X, Y) Z, JT) = g(\mathcal{R}(Z, X) T, JY).
\]

(8.4)

**Properties of the Horizontal Nullity Space of Curvatures**

From the properties of the curvatures given above, we have:

\[
JX \perp \text{Im} R^- \iff X \in \ker R^- \quad (R^- = i_J R) \in \chi(TM);
\]

(9.1)

\[
JX \perp \text{Im} R \iff \mathcal{R}(S, X) Y = 0, X \in \chi(TM), X \in \chi(TM);
\]

(9.2)

\[
X \in N_{h} \iff JX \perp \mathcal{R}(S, Y) Z, X \in \chi(TM), \forall Y, Z \in \chi(TM);
\]

(9.3)

\[
X \in N_{h} \iff JX \perp \mathcal{R}(Y, Z) T, X \in \chi(TM), \forall Y, Z, T \in \chi(TM);
\]

(9.4)

\[
X \in N_{h} \iff \mathcal{R}(Y, Z) X = 0, \forall Y, Z \in \chi(TM).
\]

(9.5)

**Proposition 11**

On a Riemannian manifold, the horizontal ideal \( \mathcal{A}_E \) of the Lie algebra \( \mathcal{A}_T \) satisfies

\[
\mathcal{A}_E = N_{h} \cap H^* = \ker \mathcal{R} \cap H^* = N_{h} \cap H^* = \ker R^* \cap H^*.
\]

Proof: According to the proposition 5, the horizontal ideal \( \mathcal{A}_E \) is written...
We have
\[ J^2 = N^2 \cap H^* = N^2 \cap H^* \subset \text{Ker R} \cap H^*. \]

From the relation \((8.4)\), we get \( \text{Ker R} \cap H^* \subset N^2 \cap H^* \). It remains to prove that
\[ \text{Ker R} \cap H^* \subset J^2 \cap H^*. \]

Let be \( X \in \text{Ker R} \cap H^* \). According to the relation \((9.1)\), we have for all \( Y \in \chi(TM) \),
\[ g(R^* (Y), JX) = 0. \]

As \( Dg = 0 \), we get
\[ g([J, R^* (Y)], JX) = g(R^* (Y), [J, JX]Z). \]

The vector field \( X \) being projectable, \([J, JX] = 0\), we have
\[ g([J, R^* (Y)]Z, JX) = 0. \quad \text{(9.6)} \]

From the relation \((9.2)\), namely:
\[ \Omega((S, Y)Z) + \Omega((Y, Z)S) + \Omega((Z, S)Y) = 0, \]
we conclude
\[ g([J, R^* (Y)]Z, JX) = g(3R(Y, Z) + R(J, Y), Z), JX). \]

According to the relation \((9.6)\) and that \( JX \) is orthogonal to \( \text{Im R} \), we find
\[ g(R(Y, Z), JX) = 0, \]
that is, \( JX \perp \text{Im R} \). Taking into account \( Dg = 0 \), and \( X \) a projectable vectors field, we have
\[ g(\Omega(Y, Z)T, JX) = 0, \]
that is, \( \text{Ker R} \cap H^* \subset \text{Ker} \Omega \cap H^* \).

**Proposition 12**

On a Riemannian manifold, the three following relations are equivalent:

i. the horizontal nullity of the curvature \( R \) is reduced to zero

ii. the horizontal nullity space of the projectable vector fields of \( R \) is reduced to zero

iii. the dimension of the image space of the curvature \( R \) is equal to \( n - 1 \)

**Proof:** If the horizontal nullity space of the projectable vector fields is not reduced to zero, there will be a nonzero horizontal and projectable vector field \( X \), according to the proposition 11, such that \( X \in N^2 \), and according to the relation \((9.4)\), \( X \) is orthogonal to \( \text{Im} R \), therefore orthogonal to \( \text{Im} R \). According to the relation \((7.4)\), the image space of the curvature \( R \) is both orthogonal to \( JX \) and to the Liouville field \( C = JS \) with \( JX \wedge C \neq 0 \). This is only possible if the dimension of the image space of \( R \) is strictly less than \( n - 1 \). So the relation (ii) implies (i). We notice by \((7.5)\) that \( d_JE = 0 \). That is, the image space of \( R \) is in the kernel of \( d \). If the dimension of the image space of the curvature \( R \) is strictly less than \( n - 1 \), there will be a horizontal vector field \( X \) such that \( JX = \text{Ker} E \) and \( JX \perp \text{Im} R \). According to the relation \((9.2)\), we have \( \Omega(S, X)Y = 0, \forall Y \in \chi(TM) \). By developing this equality and taking into account the relation \((9.1)\), we have \( R(X, Y) = R(J, Y) \) \( \forall Y \in \chi(TM) \). This is only possible if \( X = S \) or \( X \in N^2 \). Since, \( JX \perp C \), therefore, we have \( X \in N^2 \).

This last calculation proves at the same time that the horizontal nullity space of \( R \) is generated by projectable vectors field belonging to \( N^2 \). Consequently, we have \((i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)\).

**Conformal Infinitesimal Transformations of a Riemannian Manifold**

**Definition 3**

Let \( \omega \) be a \( p \)-differential form defined on a manifold \( M \). A conformal infinitesimal transformation of \( \omega \) is a vector field \( X \) defined on \( M \), such that for all elements \((t, x)\) of the domain \( D(X) \) of the flow \( \Phi \) of \( X \), we have
\[ \Phi^* \omega(x) = \rho(t, x) \omega(x), \]
where \( \rho \) is a differentiable numerical function, with positive values, defined on \( D(X) \).

We have the following result:

**Proposition 13**

For a vector field \( X \) to be a conformal infinitesimal transformation of \( \rho \)-differential form \( \omega \), it is necessary and sufficient that we have,
\[ L_x \omega = \lambda \omega, \]
where \( L_x \) is the Lie derivative with respect to \( X \) and \( \lambda \) a differentiable numerical function defined on \( M \). When this is the case, the function \( \lambda \) is linked to the function \( \rho \) which appears in the definition 3, by the relation
\[ \rho(t, x) = \exp \left( \int_0^t \lambda(x) ds \right). \]

In the following, we are interested in conformal transformations of the fundamental form \( \Omega \) of a Riemannian manifold.

**Proposition 14**

If \( X \) is a conformal infinitesimal transformation of \( \Omega \), i.e.
\[ L_x \Omega = \lambda \Omega, \]
then \( \lambda \) is a constant function.

**Proof:** Let be \( X \) a vector field belonging to \( \chi(TM) \) such that
\[ L_x \Omega = \lambda \Omega, \]
with \( \lambda \in \mathcal{F}(TM) \) the set of differentiable functions on \( TM \). Since \( \Omega = \text{dd} \xi \), we have
\[ \lambda \text{dd} \xi = dL_x \xi. \]
The expression \( \lambda \text{dd} \xi \) is an exact 2-form, this implies
\[ d\lambda \wedge \text{dd} \xi = 0. \quad \text{(10.1)} \]
as \( \text{dd} \xi \) is of maximum rank and the dimension of the manifold \( M \) is assumed to be greater than or equal to two, the relation \((10.1)\) results in
\[ d\lambda = 0. \]

In other words, the function \( \lambda \) is a constant.

In the following, we will denote \( A^c \) the set of conformal infinitesimal transformations of \( \Omega \) and \( A^c = A^c \cap \chi(M) \).

**Proposition 15**

If \( \nabla x \in \overline{A^c} \) if and only if
\[ L_{\nabla x} E = \lambda E, \]
where \( \lambda \) is a constant given by the relation \( L_{\nabla x} \Omega = \lambda \Omega \).

**Proof:** Let \( \nabla x \in \overline{A^c} \), we have
\[ L_{\nabla x} \text{dd} \xi = \lambda \text{dd} \xi, \]
where \( \lambda \) is a constant according to the proposition 14. As the two derivations \( L_{\nabla x} \) and \( d \) commute, we obtain
\[ d \lambda \text{dd} \xi = \lambda d \xi, \]
By deriving by \( i_x \) the two 2-forms above, \( C \) being Liouville field, we find
\[ d_x L_{\nabla x}E = \lambda d_x E. \]
The application of \( i_x \) to the above equality gives
\[ L_\chi E = \lambda E \]

Conversely, if \( L_\chi E = \lambda E \) with \( \lambda \in \mathbb{R} \), it is immediate to note that by continuing the previous calculation, we have \( L_\chi dE = \lambda dE \).

**Proposition 16**
\[
\overline{A}_F \subset \overline{A}_t.
\]

**Proof:** Let \( \overline{X} \in \overline{A}_t \). Since \( i_\chi dE = dE \), we can write successively
\[
i_\chi L_\chi dE = \lambda i_\chi dE = -\lambda dE
\]

and
\[
i_\chi L_\chi dE = -\lambda dE.
\]

From the formula \( [i_\chi, L_\chi] = i_{\{S, \chi\}} \), we get
\[
i_{\{S, \chi\}} dE = 0.
\]

As \( dd E \) is a symplectic form, we find
\[
[\{S, X\}, \chi] = 0.
\]

This means that \( \overline{X} \in \overline{A}_t \). For a Riemannian manifold, \( \overline{A}_t \) coincides with \( \overline{A}_F \). Thus,
\[
\overline{A}_g \subset \overline{A}_t.
\]

**Proposition 17**

Let \( F \) be the almost complex structure associated to \( \Gamma \). So by noting
\[
[\{F, X\} \chi = \chi
\]

we have
\[
\overline{A}_F = \overline{A}_t \cap \chi(M).
\]

**Proof:** Let \( \overline{X} \in \overline{A}_t \). According to the definition of \( F \), namely, \( FJ = h \) and \( FH = J \),

we can write successively for all \( Y \in \chi(TM) \)
\[
\{X, F\} Y = \{X, FY\} - F\{X, Y\} = \{X, hY\} - F\{X, Y\} = h\{X, Y\} - h\{X, Y\} = 0.
\]

Similarly, we have
\[
\{X, F\} hY = \{X, FH\} - F\{X, hY\} = -\{X, JY\} - F\{X, Y\} = 0.
\]

for all \( Y \in \chi(TM) \), the two relations (10.2) and (10.3) lead to
\[
[\{X, F\} = 0.
\]

In other words,
\[
\overline{A}_t \subset \overline{A}_F.
\]

**Proposition 18**

Let \( \overline{X} \in \overline{\chi}(M) \), \( \lambda \in \mathbb{R} \). So
\[ L_\chi \Omega = \lambda \Omega \] is equivalent to \( L_\chi g = \lambda g \).

**Proof:** a) Suppose \( L_\chi \chi = \lambda \chi \), we have
\[
(L_\chi g)(Y, Z) = L_\chi g(Y, Z) - \lambda g(Y, L_\chi Z).
\]

for all \( Y, Z \in \chi(TM) \). Given the relation \( g(Y, Z) = \Omega(Y, FZ) \), the second member of the above equality becomes
\[
(L_\chi g)(Y, Z) = L_\chi \Omega(Y, FZ) - \lambda \Omega(Y, FZ).
\]

By taking into account the propositions 16 and 17, we obtain
\[
(L_\chi g)(Y, Z) = L_\chi \Omega(Y, FZ),
\]
for all \( Y, Z \in \chi(TM) \). If \( L_\chi \Omega = \lambda \Omega \), we find
\[ L_\chi g = \lambda g \].

b) Suppose \( L_\chi \chi = \lambda \chi \). We have
\[
(L_\chi \Omega)(Y, Z) = L_\chi \Omega(Y, Z) - \Omega(L_\chi Y, Z) - \Omega(Y, L_\chi Z),
\]
for all \( Y, Z \in \chi(TM) \). The expression of \( \Omega \) in function of \( g \) is \( \Omega(Y, Z) = g(Y, JZ) + g(JY, Z) \),

\[ (L_\chi \Omega)(Y, Z) = L_\chi (g(Y, JZ) + g(JY, Z) - g(L_\chi Y, Z) - g(L_\chi J, Z)) - g(JL_\chi Y, Z) - g(Y, JL_\chi Z) - g(JY, L_\chi Z).
\]

As \( \overline{X} \), belongs to \( \overline{A}_t \), we get
\[
(L_\chi \Omega)(Y, Z) = (L_\chi g)(Y, JZ) + (L_\chi g)(JY, Z).
\]

If \( L_\chi g = \lambda g \), taking into account the relation (10.5) we find
\[ L_\chi \chi = \lambda \chi \].

**Automorphisms of a Riemannian Manifold**

**Definition 4**

A vector field \( X \) on a Riemannian manifold \((M, E)\) is said to be infinitesimal automorphism of the symplectic form \( \Omega \) if
\[
L_\chi \chi = 0.
\]

We notice the canonical spray \( S \) of \((M, E)\) is an infinitesimal automorphism of the symplectic form \( \Omega \).

The set of infinitesimal automorphisms of \( \Omega \) forms a Lie algebra. We will denote this Lie algebra by \( A_g \).

**Proposition 19**

By supposing \( \overline{A}_g = \overline{A}_F \cap \chi(M) \), we have
\[
\overline{A}_t \subset \overline{A}_F
\]

- the horizontal elements of \( \overline{A}_t \) form a commutative ideal of \( \overline{A}_g \).

- c) The elements of \( \overline{A}_t \) are Killing fields of projectable vectors of the metric \( g \) belonging to \( \overline{A}_s \). The dimension of \( \overline{A}_t \) is at most equal to
\[
\frac{2}{n(n+1)}.
\]

**Proof:** It is clear that if \( X \in \overline{A}_t \), then \( L_\chi E = 0 \). This is a particular case of the proposition 15 for \( \lambda = 0 \).

Let \( X \) be a projectable vector field such that \( L_\chi \chi = 0 \) and \( L_\chi E = 0 \), we will prove that \( X \in \overline{A}_t \). The fundamental form \( \Omega \) is written \( \Omega = dd E \), as \( L_\chi \chi = 0 \), we have
\[
i_\chi L_\chi E = 0.
\]

Besides, according to the relation (7.1) and the hypothesis \( L_\chi E = 0 \)
\[
i_\chi L_\chi E = dL_\chi E = 0.
\]

According to the formula \( i_{\{S, X\}} i_{\chi} L_\chi \chi = \lambda g \), we find
\[ i_{\omega, \Omega} \Omega = 0. \]

Since the fundamental form \( \Omega \) is symplectic, we get \([S, X]=0\). The vector field \( \mathbf{X} \) being projectable by hypothesis, according to the proposition 9, \( \mathbf{X} \in \mathfrak{A}_h \cap \mathfrak{X}(M) \).

- The result of b) is a consequence of the proposition 18 because \( \mathfrak{A}_s \) is a particular case of \( \mathfrak{A}_c \) for \( \lambda = 0 \). Since, we have by (7.5), the horizontal elements of \( \mathfrak{A}_h \) belong to \( \mathfrak{A}_c \), and form a commutative ideal of \( \mathfrak{A}_c \), according to the theorem 1.

- Finally, the property c) is given by the proposition 18 by taking \( \lambda = 0 \).

In addition, \( g \) defines a metric on the vertical bundle which is of \( n \) dimension. The flow \( \varphi_\lambda \) of a vector field \( \mathbf{X} \in \mathfrak{X} \) is a local isometry for \( g \). Consequently, \( \varphi_\lambda \) preserves the orthogonal bases of \( g \) on the vertical bundle. With the method used in theorem 3.3 of [14] vol.1 p.238, the dimension of \( \mathfrak{A}_c \) is at most equal to \( \frac{n(n+1)}{2} \).

**Theorem 3**

Let \((M, E)\) be a Riemannian manifold with a nonzero constant sectional curvature. Then \( \mathfrak{A}_c \) coincides with \( \mathfrak{A}_s \).

**Proof:** If the sectional curvature \( K \) is constant, we have [2]

\[ R^c = 2EKE - KdEDC, \]

where \( R^c = i_\mathbf{R} R \). For all \( \mathbf{X} \in \mathfrak{A}_c \), we have \( L_\mathbf{X} R^c = 0 \). If \( K \) is not zero, we get

\[ 2L_\mathbf{X} EJ = (L_\mathbf{X} dE) \otimes C, \]

this is only possible if \( L_\mathbf{X} E = 0 \), that is, according to the proposition 19 a), \( \mathbf{X} \in \mathfrak{A}_c \). In other words,

\[ \mathfrak{A}_h \subset \mathfrak{A}_c. \]

The proposition 19 b) gives

\[ \mathfrak{A}_s \subset \mathfrak{A}_c. \]

**Riemannian Manifolds with Regular Curvature**

**Definition 5**

We will say that a connection \( \Gamma \) has a regular curvature at a point \( z \) of \( TM \) if the vector space generated by the image of the curvature \( R_\Gamma \) is of \( n-1 \) dimension.

**Proposition 20**

Let \((M, E)\) be a Riemannian manifold such that

\[ \mathfrak{R}(\mathbf{X}, \mathbf{Y}) \neq 0 \]

for all horizontal vector fields \( \mathbf{X}, \mathbf{Y} \) linearly independent, then \((M, E)\) has a regular curvature.

**Proof:** The dimension of \( \text{Im} R \) is less than or equal to \( n-1 \), since the canonical field \( C = JS \) is orthogonal to \( \text{Im} R \).

If the dimension of \( \text{Im} R \) is less than or equal to \( n-2 \), there will be a horizontal vector field \( Z \) such that \( JZ \perp \text{Im} R \) and \( S \times Z \neq 0 \), and according to the relation (9.2), we would have

\[ \mathfrak{R}(S, Z) = 0 \]

this contradicts the hypothesis.

**Remark 2:** The converse of the proposition 20 is not true in general.

**Theorem 4**

Let \((M, E)\) be a Riemannian manifold, \( \Gamma \) a canonical connection of \((M, E)\) with regular curvature. So

\[ \mathfrak{A}_c = \mathfrak{A}_s. \]

**Proof:** The proposition 16 gives

\[ \mathfrak{A}_s \subset \mathfrak{A}_c. \]

It remains to prove that \( \mathfrak{A}_s \) is contained in \( \mathfrak{A}_c \) and for that, it is simply sufficient to prove \( L_\mathbf{X} E = \lambda E \) for all \( \mathbf{X} \in \mathfrak{A}_c \), \( \lambda \in \mathbb{R} \), according to the proposition 15.

Let \( \mathbf{X} \in \mathfrak{A}_c \), as \( \mathfrak{A}_c \) is identical to \( \mathfrak{A}_s \), we have

\[ d(\mathfrak{X}(\mathbf{X})) = 0. \]

According to the formula \( \{L_\mathbf{X}E, d(\mathfrak{X}(\mathbf{X}))\} = d(\mathfrak{X}(\mathbf{X})) \) and by (7.5), we obtain

\[ d(\mathfrak{X}(\mathbf{X})) = 0. \]

That implies

\[ d(\mathfrak{X}(\mathbf{X})) = 0 \text{ or } d(\mathfrak{X}(\mathbf{X})) = 0. \]

Let \( H \) be the horizontal space of \( \Gamma \), \( H \oplus \text{Im} R \) is contained in the kernel of \( dE \). By hypothesis, the curvature \( R \) is regular, so \( H \oplus \text{Im} R \) is of \( 2n-1 \) dimension, and the kernel of \( dE \). Therefore, \( H \oplus \text{Im} R \) is a completely integrable distribution. According to Frobenius theorem, at each point \( z \) of \( TM \), it passes a maximum integral manifold which is the solution of the equation (12.1). We notice that \( L_\mathbf{X} E \) is homogeneous of degree two and that the function \( L_\mathbf{X} E \) is continuous and null on the null section.

This imposes the initial conditions for the solutions of the equation (12.1). Hence the result

\[ L_\mathbf{X} E = \lambda E, \]

with \( \lambda \) a constant.

**Remark 3:** A Riemannian manifold with a constant and nonzero sectional curvature has a regular curvature, but the value of \( \lambda \) is zero. The theorem 3 excludes the case of \( \lambda \neq 0 \).

**Proposition 21**

Let \((M, E)\) be a Riemannian manifold with regular curvature, \( R^c = i_\mathbf{R} R \), we have the following relation:

\[ \{\mathbf{X}, R^c(\mathbf{Y})\} = R^c\{\mathbf{X}, \mathbf{Y}\} = -[\mathbf{Y}, R^c(\mathbf{X})], \]

for all \( \mathbf{X}, \mathbf{Y} \in \mathfrak{A}_c \).

**Proof:** For all \( \mathbf{X}, \mathbf{Y} \in \mathfrak{A}_c \), we have

\[ \{\mathbf{X}, \mathbf{Y}\} = 0. \]

That is, \( \{\mathbf{X}, R^c(\mathbf{Y})\} - R^c(\{\mathbf{X}, \mathbf{Y}\}) = 0 \), or

\[ \{\mathbf{X}, R^c(\mathbf{Y})\} - R^c(\{\mathbf{X}, \mathbf{Y}\}) = \{\mathbf{Y}, R^c(\mathbf{X})\}. \]

**Theorem 5**

The Lie algebra \( \mathfrak{X}(\mathfrak{K}) \) of the Killing fields contained in \( \mathfrak{X}(\mathfrak{K}) \) of dimension greater than or equal to three is semi-simple if and only if the Riemannian manifold is with regular curvature.

**Proof:** If the Riemannian manifold is not with regular curvature, there is the horizontal nullity space of the projectable vector fields of the curvature \( R \) which provides the commutative ideal of \( \mathfrak{X}(\mathfrak{K}) \), according to the propositions 19 b) and 12, which is also that of \( \mathfrak{X}(\mathfrak{K}) \). Then \( \mathfrak{X}(\mathfrak{K}) \) is not semi-simple.

It is assumed that the Riemannian manifold has a regular curvature. Let \( \mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\mathfrak{K}) \) such that \( \mathfrak{X}(\mathfrak{K}) \) and \( \mathfrak{X}(\mathfrak{K}) \) are linearly independent. As the mapping \( R \) is injective on the projectable vector fields according to the proposition 11 and the definition 5, \( R(\mathbf{X}), R(\mathbf{Y}) \) are also linearly independent.
According to the proposition 21, 
\[ [\pi, R(\pi)] = R[\pi, \pi] = [\pi, R(\pi)]. \]

As \( \pi, \pi \) are elements of \( \mathcal{A}_e \), we have the following system of equations
\[
\left\{ \begin{align*}
\pi g(R(\pi), \pi Y) &= g(R(\pi), \pi Y) + g(R(\pi), J[\pi, \pi Y]), \\
g(R(\pi), \pi Y) &= g(R(\pi), J[\pi, \pi Y]).
\end{align*} \right. \\
(12.2)
\]

Taking \( JY_2 \perp R(\pi) \) and \( JY_2 \perp R(\pi) \), we have
\[
\left\{ \begin{align*}
g(R(\pi), \pi Y) &= g(R(\pi), J[\pi, \pi Y]) = g(R(\pi), J[\pi, \pi Y]), \\
g(R(\pi), JY_2) &= g(R(\pi), J[\pi, JY_2]).
\end{align*} \right.
\]

If we have \( \pi, \pi, \pi \in \mathcal{A}_e \) linearly independent, we have \( R(\pi), R(\pi), R(\pi) \) linearly independent, and we have a system of six equations as (12.2). For \( \pi, \pi, \pi \in \mathcal{A}_e \), we have a system of 2p equations as (12.2). Such a system of equations do not allow us to have a commutative ideal of \( \mathcal{A}_e \) other than zero.

We conclude that if \( \mathcal{A}_e \) is of dimension superior or equals to three, then the manifold has a Riemannian manifold with regular curvature, \( \mathcal{A}_e \) is semi-simple.

**Corollary 1**

On a Riemannian manifold \( (M, E) \) of \( n \geq 2 \) dimension, the Lie algebra \( \mathcal{A}_e \) of infinitesimal isometries contained in \( \mathcal{A}_e \) of dimension superior or equals to three is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of \( \Gamma \) is reduced to zero. For this case, the Lie algebra \( \mathcal{A}_e \) coincides with \( \mathcal{A}_e \).

**Proof:** The reasoning used in the theorem 5 is based on the existence of two linearly independent elements of \( \mathcal{A}_e \). So the dimension of \( \mathcal{A}_e \) is assumed to be superior or equals to two. We know that there is no semi-simple Lie algebra of one or two dimension. Hence, the condition of the dimension of \( \mathcal{A}_e \) superior or equals to three.

We notice the interlocking of the algebras \( \mathcal{A}_e \subset \mathcal{A}_e \subset \mathcal{A}_e \). So if the dimension of \( \mathcal{A}_e \) is superior or equals to three, then the manifold is Riemannian manifold with regular curvature, to the theorem 5, \( \mathcal{A}_e \) is semi-simple. In this case, according to the theorem 4, \( \mathcal{A}_e = \mathcal{A}_e \) and \( \mathcal{A}_e \) is an ideal of \( \mathcal{A}_e \). Consequently \( \mathcal{A}_e = \mathcal{A}_e \) otherwise we have a contradiction.

**Remark 4:** If the Riemannian manifold is flat (\( R=0 \)), the horizontal elements of \( \mathcal{A}_e \) constitute a nonzero commutative ideal of \( \mathcal{A}_e \). If the Riemannian manifold has a nonzero constant sectional curvature, according to the theorem 3, \( \mathcal{A}_e = \mathcal{A}_e \). It is obvious that this manifold has a regular curvature.

**Remark 5:** The result of the theorem 5 is not true in general if we only impose the fields of projectable vectors to be those of Killing, that is to say to belong to \( \mathcal{A}_e \) without being elements of \( \mathcal{A}_e(M) \).

The elements of \( \mathcal{A}_e \) even projectable form in general an algebra of infinite dimension.

**Examples**

**Example 3**

We assume \( M = \mathbb{R}^2 \), \( E = \{1/2(e^e(y^2)^2 + (y^2)^2) \} \). The spray \( S \) is written
\[
S = y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y} - y^3 y^2 \frac{\partial}{\partial y^2} + \frac{1}{2} e^e(y^2)^2 \frac{\partial}{\partial y^2}.
\]

The nonzero coefficients of the connection are:
\[
\Gamma_1^1 = y^2, \quad \Gamma_1^2 = y^2, \quad \Gamma_2^1 = -\frac{y}{2} e^e y^2.
\]

The horizontal vector fields are generated by
\[
\frac{\partial}{\partial x^1} = \frac{1}{2} y^2 \frac{\partial}{\partial y^2} + \frac{1}{2} y^2 \frac{\partial}{\partial y^3} + \frac{1}{2} y^3 \frac{\partial}{\partial y^3}.
\]

The spray \( S \) is written:
\[
S = y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y} - y^3 y^2 \frac{\partial}{\partial y^2} + \frac{1}{2} e^e(y^2)^2 \frac{\partial}{\partial y^2}.
\]

The nonzero coefficients of the connection are:
\[
\Gamma_1^1 = y^2, \quad \Gamma_1^2 = y^2, \quad \Gamma_2^1 = -\frac{y}{2} e^e y^2.
\]

The horizontal vector fields are generated by
\[
\frac{\partial}{\partial x^1} = \frac{1}{2} y^2 \frac{\partial}{\partial y^2} + \frac{1}{2} y^2 \frac{\partial}{\partial y^3} + \frac{1}{2} y^3 \frac{\partial}{\partial y^3}.
\]

The curvature \( R \) is zero.
The elements of $A^\Gamma$ are generated by:

$$g_1 = e^{\frac{\partial}{\partial x^1}} \cdot \frac{x^2}{x^1}, \quad g_2 = e^{\frac{\partial}{\partial x^2}} \cdot \frac{y^1}{x^2},$$

$$g_3 = e^{\frac{\partial}{\partial x^3}} \cdot \frac{y^2}{x^1}, \quad g_4 = e^{\frac{\partial}{\partial x^4}} \cdot \frac{y^3}{x^2},$$

$$g_5 = e^{\frac{\partial}{\partial x^5}} \cdot \frac{y^4}{x^1}, \quad g_6 = e^{\frac{\partial}{\partial x^6}} \cdot \frac{y^5}{x^2}.$$  

The horizontal vectors fields which form the commutative ideal are $\{g_3, g_7, g_12\}$. The elements in $A^g$, according to the proposition 19 (Table 1), are

$$e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = e_7 = e_8 = e_9 = e_{10} = e_{11} = e_{12}.$$ 

| Table 1. Multiplication table of $A^g$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $e_1$          | $e_2$          | $e_3$          | $e_4$          | $e_5$          | $e_6$          | $e_7$          |
| $e_1$          | 0              | $e_1$          | $-e_6$         | $-e_2$         | $-e_3$         | 0              |
| $e_2$          | $-e_1$         | 0              | $e_2$          | $-e_3$         | $-e_4$         | 0              |
| $e_3$          | $e_6$          | $-e_2$         | 0              | $e_3$          | $-e_4$         | $-e_5$         |
| $e_4$          | $e_1$          | 0              | 0              | $e_2$          | $-e_3$         | 0              |
| $e_5$          | $e_6$          | 0              | 0              | $-e_2$         | 0              | $-e_3$         |
| $e_6$          | 0              | 0              | $e_3$          | 0              | 0              | $e_4$          |

We see that the derivative ideal from $A^g$ coincides with $A^g$. The commutative ideal is generated by $\{e_2, e_4, e_6\}$.

References
