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Riesz Triple Almost Lacunary χ^3 Sequence Spaces Defined by a Orlicz Function-II

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Abstract

The aim of this paper is to introduce a new concept for strong almost Pringsheim convergence with respect to an Orlicz function, combining with Riesz mean for triple sequences and a triple lacunary sequence. We also introduce and study statistics convergence of Riesz almost lacunary χ^3 sequence spaces and also some inclusion theorems are discussed.

Keywords: Analytic sequence; Modulus function; Double sequences; Chi sequence; Riesz space; Riesz convergence; Pringsheim convergence

2010 Mathematics subject classification: 40A05,40C05,40D05

Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m,n,k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy, Subramanian et al. [2-9], and many others. Later on investigated by some initial work on triple sequence spaces is found in Sahiner et al. [10], Esi et al. [11-15], Subramanian et al. [16-25] and many others. Some interesting results in this direction can be seen [26-29].

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence $(S_{mnk})$is convergent, where,

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} \, (m,n,k = 1,2,3,...) \cdot$$

A sequence $x=(x_{mnk})$ is said to be triple analytic if,

$$sup_{m,n,k} \left| x_{mnk} \right|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by ^{3}. A sequence $x=(x_{mnk})$ is called triple entire sequence if,

$$\left| x_{mnk} \right|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric,

$$d(x,y) = sup_{m,n,k} \left\{ \left| x_{mnk} - y_{mnk} \right|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\}, \qquad (1)$$

forall $x=(x_{mnk})$ and $y=(y_{mnk})$ in Γ^3. Let ϕ={*finite sequences*}.

Consider a triple sequence $x=(x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \delta_{ijq}$ for all m,n,k $\mathbb{N}$,

$$\delta_{mnk} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ & & & & \end{bmatrix}$$

with 1 in the $(m,n,k)^{th}$ position and zero otherwise.

A sequence $x=(x_{mnk})$ is called triple gai sequence if $\left((m+n+k)! \left| x_{mnk} \right| \right)^{\frac{1}{m+n+k}} \to 0$ as $m,n,k\infty$. The triple gai sequences will be denoted by χ^3.

Definitions and Preliminaries

A triple sequence $x=(x_{mnk})$ has limit 0 (denoted by $Plimx=0$)

(i.e) $\left((m+n+k)! \left| x_{mnk} \right| \right)^{1/m+n+k} \to 0$ as $m,n,k \to \infty$. We shall write more briefly as *P–convergent to* 0.

Definition

A modulus function was introduced by Nakano [30]. We recall that a modulus f is a function from $[0,\infty)[0,\infty)$, such that,

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(1) $f(x)=0$ if and only if $x=0$

(2) $f(x+y)\leq f(x)+f(y)$, for all $x\geq 0$, $y\geq 0$,

(3) f is increasing,

(4) f is continuous from the right at 0. Since $|f(x)-f(y)\leq f(|xy|)$, it follows from here that f is continuous on $[0,\infty)$.

Definition

Let $(q_{rst}),(\overline{q_{rst}}),(\overline{\overline{q_{rst}}})$ be sequences of positive numbers and

$$Q_r=\begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0\ldots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}=q_{11}+q_{12}+\ldots+q_{rs}\neq 0,$$

$$\overline{Q}_s=\begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0\ldots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}=\overline{q}_{11}+\overline{q}_{12}+\ldots+\overline{q}_{rs}\neq 0,$$

$$\overline{\overline{Q}}_t=\begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0\ldots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \cdots & \overline{\overline{q}}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix}=\overline{q}_{11}+\overline{q}_{12}+\ldots+\overline{q}_{rs}\neq 0.$$

Then the transformation is given by:

$$T_{rst}=\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}q_m\overline{q}_n\overline{\overline{q}}_k\left((m+n+k)!|x_{mnk}|\right)^{1/m+n+k}$$ is

called the Riesz mean of triple sequence $x=(x_{mnk})$. If $P-lim_{rst}T_{rst}(x)=0,0\mathbb{R}$, then the sequence $x=(x_{mnk})$ is said to be Riesz convergent to 0. If $x=(x_{mnk})$ is Riesz convergent to 0, then we write $P_R-limx=0$.

Definition

The triple sequence $\theta_{i,l,j}=\{(m_i,n_l,k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$m_0=0$, $h_i=m_{r-1}\to\infty$ as $i\infty$ and

$n_0=0,\overline{h_\ell}=n_\ell-n_{\ell-1}\to\infty$ as $l\infty$.

$k_0=0,\overline{h_j}=k_j-k_{j-1}\to\infty$ as $j\to\infty$.

Let $m_{i,\ell,j}=m_in_\ell k_j,h_{i,\ell,j}=h_i\overline{h_\ell}\overline{h_j}$, and $\theta_{i,l,j}$ is determine by

$$I_{i,\ell,j}=\{(m,n,k):m_{i-1}<m<m_i \quad and \quad n_{\ell-1}<n\leq n_\ell \quad and \quad k_{j-1}<k\leq k_j\},q_k=\frac{m_k}{m_{k-1}},\overline{q_\ell}=\frac{n_\ell}{n_{\ell-1}},\overline{q_j}=\frac{k_j}{k_{j-1}}.$$

Using the notations of lacunary sequence and Riesz mean for triple sequences.

$\theta_{i,l,j}=\{(m_i,n_l,k_j)\}$ be a triple lacunary sequence and $q_m\overline{q}_n\overline{\overline{q}}_k$ be sequences of positive real numbers such that

$$Q_{m_i}=\sum_{m\in(0,m_i]}p_{m_i},Q_{n_\ell}=\sum_{n\in(0,n_\ell]}p_{n_\ell},Q_{n_j}=\sum_{k\in(0,k_j]}p_{k_j}\text{and}$$

$$H_i=\sum_{m\in(0,m_i]}p_{m_i},\overline{H}=\sum_{n\in(0,n_\ell]}p_{n_\ell},\overline{\overline{H}}=\sum_{k\in(0,k_j]}p_{k_j}.$$

Clearly, $H_i=Q_{m_i}-Q_{m_{i-1}},\overline{H}_\ell=Q_{n_\ell}-Q_{n_{\ell-1}},\overline{\overline{H}}_j=Q_{k_j}-Q_{k_{j-1}}$.

If the Riesz transformation of triple sequences is RH-regular, and $H_i=Q_{m_i}-Q_{m_{i-1}}\to\infty$ as $i\to\infty,\overline{H}=\sum_{n\in(0,n_\ell]}p_{n_\ell}\to\infty$ as $\ell\to\infty,\overline{\overline{H}}=\sum_{k\in(0,k_j]}p_{k_j}\to\infty$ as $j\infty$, then $\theta'_{i,\ell,j}=\{(m_i,n_\ell,k_j)\}=\{(Q_{m_i}Q_{n_j}Q_{k_k})\}$ is a triple lacunary sequence. If the assumptions $Q_r\infty$. as $r\infty$, $\overline{Q}_s\to\infty$ as $s\infty$ and $\overline{\overline{Q}}_t\to\infty$ as $t\to\infty$. may be not enough to obtain the conditions $H_i\to\infty$. as $i\to\infty,\overline{H}_\ell\to\infty$ as $l\infty$ and $\overline{\overline{H}}_j\to\infty$ as $j\to\infty$ respectively. For any lacunary sequences $(m_i),(n_l)$ and $(_{kj})$ are integers.

Throughout the paper, we assume that

$$Q_r=q_{11}+q_{12}+\ldots+q_{rs}\to\infty(r\to\infty),\overline{Q}_s=\overline{q}_{11}+\overline{q}_{12}+$$
$$\ldots+\overline{q}_{rs}\to\infty(s\to\infty),\overline{\overline{Q}}_t=\overline{\overline{q}}_{11}+\overline{\overline{q}}_{12}+\ldots+\overline{\overline{q}}_{rs}\to\infty(t\to\infty),$$

such that $H_i=Q_{m_i}-Q_{m_{i-1}}\to\infty$ as $i\to\infty,\overline{H}_\ell=Q_{n_\ell}-Q_{n_{\ell-1}}\to\infty$ as $l\infty$ and $\overline{\overline{H}}_j=Q_{k_j}-Q_{k_{j-1}}\to\infty$ as $j\infty$.

Let

$$Q_{m_i,n_\ell,k_j}=Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j},H_{i\ell j}=H_i\overline{H}_\ell\overline{\overline{H}}_j,I'_{i\ell j}=$$

$$\{(m,n,k):Q_{m_{i-1}}<m<Q_{m_i},\overline{Q}_{n_{\ell-1}}<n<Q_{n_\ell} \quad and \quad \overline{Q}_{k_{j-1}}<k<\overline{Q}_{k_j}\},V_i=\frac{Q_{m_i}}{Q_{m_{i-1}}},\overline{V}_\ell=\frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$$

and $\overline{\overline{V}}_j=\frac{Q_{k_j}}{Q_{k_{j-1}}}$. and $V_{i\ell j}=V_i\overline{V}_\ell\overline{\overline{V}}_j$.

If we take $q_m=1,\overline{q}_n=1$ *and* $\overline{\overline{q}}_k=1$ for all m,n and k then $H_{i\ell j},Q_{i\ell j},V_{i\ell j}$ and $I'_{i\ell j}$ reduce to h_{ilj}, q_{ilj}, v_{ilj} and I_{ilj}.

Let f be an Orlicz function and $p=(p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:

$$[\chi^3_R,\theta_{i\ell j},q,f,p]=$$
$$\left\{P-lim_{i,\ell,j\to\infty}\frac{1}{H_{i,\ell j}}\sum_{i\in I_{i\ell j}}\sum_{\ell\in I_{i\ell j}}\sum_{j\in I_{i\ell j}}q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left((m+n+k)!|x_{m+i,n+\ell,k+j}|\right)^{p_{mnk}}\right]=0\right\},$$

uniformly in i,l and j.

$$[\Lambda^3_R,\theta_{i\ell j},q,f,p]=$$
$$\left\{x=(x_{mnk}):P-sup_{i,\ell,j}\frac{1}{H_{i,\ell j}}\sum_{i\in I_{i\ell j}}\sum_{\ell\in I_{i\ell j}}\sum_{j\in I_{i\ell j}}q_m\overline{q}_n\overline{\overline{q}}_k\left[f|x_{m+i,n+\ell,k+j}|^{p_{mnk}}\right]<\infty\right\},$$

uniformly in i,l and j.

Let f be an Orlicz function, $p=pmnk$ be any factorable double sequence of strictly positive real numbers and and $q_m,\overline{q}_n$ and $\overline{\overline{q}}_k$ be

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sequences of positive numbers and $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$,

If we choose $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all *m*,*n* and *k*, then we obtain the following sequence spaces.

$$\left[\chi_R^3, q, f, p\right] = \left\{P - lim_{i,\ell,j\to\infty} \frac{1}{Q_i \overline{Q}_\ell \overline{\overline{Q}}_j} \sum_{m=1}^{i} \sum_{n=1}^{\ell} \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0\right\},$$

uniformly in *i*,*l* and *j*.

$$\left[\Lambda_R^3, q, f, p\right] = \left\{P - sup_{i,\ell,j} \frac{1}{Q_i \overline{Q}_\ell \overline{\overline{Q}}_j} \sum_{m=1}^{i} \sum_{n=1}^{\ell} \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] < \infty\right\},$$

uniformly in *i*,*l* and *j*.

Main Results

Theorem

If *f* be any Orlicz function and a bounded factorable positive triple number sequence p_{mnk} then $\left[\chi_R^3, \theta_{i\ell j}, q, f, p\right]$ is linear space.

Proof: The proof is easy. Therefore omit the proof.

Theorem

For any Orlicz function *f*, we have $\left[\chi_R^3, \theta_{i\ell j}, q, f, p\right] \subset \left[\chi_R^3, \theta_{i\ell j}, q, p\right]$.

Proof: Let $x \in \left[\chi_R^3, \theta_{i\ell j}, q, p\right]$ so that for each *i*,*l* and *j*.

$$\left[\chi_R^3, \theta_{i\ell j}, q, f, p\right] = \left\{P - lim_{i,\ell,j\to\infty} \frac{1}{H_{i,\ell j}} \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0\right\},$$

uniformly in *i*,*l* and *j*.

Since *f* is continuous at zero, for $\varepsilon > 0$ and choose δ with $0<\delta<1$ such that $f(t)<\varepsilon$ for every *t* with $0 \le t\delta$. We obtain the following,

$$\frac{1}{h_{i\ell j}}\left(h_{i\ell j}\varepsilon\right) + \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j} \text{ and } \left|x_{m+i,n+\ell,k+j} - 0\right| > \delta} f \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right] \frac{1}{h_{i\ell j}}\left(h_{i\ell j}\varepsilon\right) + \frac{1}{h_{i\ell j}} K \delta^{-1} f(2) h_{i\ell j} \quad \left[\chi_R^3, \theta_{i\ell j}, q, p\right].$$

Hence *i*,*l* and *j* goes to infinity, we are granted $x \in \left[\chi_R^3, \theta_{i\ell j}, q, f, p\right]$.

Theorem

Let $\theta_{i,l,j} = \{m_i n_\ell k_j\}$ be a triple lacunary sequence and $q_i, \overline{q}_\ell \overline{\overline{q}}_j$ with $liminf_i V_i > 1$, $liminf_\ell \overline{V}_\ell > 1$ *and* $liminf_j V_j > 1$ then for any Orlicz function $f, \left[\chi_R^3,, f, q, p\right] \subseteq \left[\chi_R^3, \theta_{i\ell j}, q, p\right]$

Proof: Suppose $liminf_i V_i > 1$, $liminf_\ell \overline{V}_\ell > 1$ and $liminf_j \overline{\overline{V}}_j > 1$ then there exists $\delta>0$, such that $V_i > 1+\delta$, $\overline{V}_\ell > 1+\delta$ and $\overline{\overline{}} > 1$.

This implies $\frac{H_i}{Q_{m_i}} \ge \frac{\delta}{1+\delta}$, $\frac{\overline{H}_\ell}{\overline{Q}_{n_\ell}} \ge \frac{\delta}{1+\delta}$ and $\frac{\overline{\overline{H}}_j}{\overline{\overline{Q}}_{k_j}} \ge \frac{\delta}{1+\delta}$ Then for $x \in \left[\chi_R^3, f, q, p\right]$, we can write for each *i*,*l* and *j*.

$$\begin{aligned}
A_{i,\ell,j} &= \frac{1}{H_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} \\
&= \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} \\
&- \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{i-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} \\
&- \frac{1}{H_{i\ell j}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} \\
&- \frac{1}{H_{i\ell j}} \sum_{k=k_j+1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} \\
&= \frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{Q}_{k_j}}{Hh_{i\ell j}} \left(\frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) \\
&- \frac{Q_{m_{k-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}} \left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{kj-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) \\
&- \frac{Q_{k_{j-1}}}{H_{i\ell j}} \left(\frac{1}{Q_{k_{j-1}}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) \\
&- \frac{\overline{Q}_{n_{\ell-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{\overline{Q}}_{n_{\ell-1}}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) \\
&- \frac{\overline{\overline{Q}}_{m_{k-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{\overline{Q}}_{m_{k-1}}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right).
\end{aligned}$$

Since $x \in \left[\chi_R^3, f, q, p\right]$, the last three terms tend to zero uniformly in *m*,*n*,*k* in the sense, thus, for each *i*,*l* and *j*:

$$\begin{aligned}
A_{i,\ell,j} &= \frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}}{H_{i\ell j}} \left(\frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) - \\
&\frac{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}} \left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) + O(1).
\end{aligned}$$

Since $H_{i\ell j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j} - Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}$ we are granted for each *i*,*l* and *j* the following:

$$\frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}}{H_{i\ell j}} \le \frac{1+\delta}{\delta} \text{ and } \frac{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}} \le \frac{1}{\delta}.$$

The terms

$$\left(\frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

and

$$\left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

are both gai sequences for all *r*,*s* and *u*. Thus A_{ilj} is a gai sequence for each *i*,*l* and *j*. Hence $x \in \left[\chi_R, \theta_{i\ j}, q, p\right]$.

Theorem

Let $\theta_{i,l,j} = \{m_i n_\ell k_j\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ with $limsup_i V_i < \infty$, $limsup_\ell \overline{V}_\ell < \infty$ and $limsup_j \overline{\overline{V}}_j < \infty$ then for any Orlicz function f, $\left[\chi_R^3, \theta_{i\ell j}, q, f, p\right] \subseteq \left[\chi_R^3, q, f, p\right]$.

Proof: Since $limsup_i V_i < \infty$, $limsup_\ell \overline{V}_\ell < \infty$ and $limsup_j \overline{\overline{V}}_j < \infty$ there exists $H>0$ such that $V_i < H$, $\overline{V}_\ell < H$ and $\overline{\overline{V}} \quad H$ for all *i*,*l* and *j*. Let $x \in \left[\chi_R^3, \theta_{i\ell j}, q, f, p\right]$ and $\varepsilon>0$. Then there exist $i_0>0$, $l_0>0$ and $j_0>0$ such that for every ai_0, $b \ge l_0$ and $c \ge j_0$ and for all *i*,*l* and *j*.

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$$A'_{abc}=\frac{1}{H_{abc}}\sum_{m\in I_{a,b,c}}\sum_{n\in I_{a,b,c}}\sum_{k\in I_{a,b,c}}q_m\bar{q}_n\bar{\bar{q}}_k$$
$$\left[f\left(\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right)\right]^{p_{mnk}}\to 0 \quad as \quad m,n,k\to\infty.$$

Let $G'=max\left\{A'_{a,b,c}:1\le a\le i_0,\ \ 1\le b\le \ell_0\ \ and\ \ 1\le c\le j_0\right\}$ and p,r and t be such that $m_{i-1}p\le m_i$, $n_{l-1}<r\le n_l$ and $k_{j-1}<tk_j$. Thus we obtain the following:

$$\frac{1}{Q_p\bar{Q}_r\bar{\bar{Q}}_t}\sum_{m=1}^{p}\sum_{n=1}^{r}\sum_{k=1}^{t}q_m\bar{q}_n\bar{\bar{q}}_k\left[f\left(\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right)\right]^{p_{mnk}}$$
$$\le\frac{1}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j}\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$\le\frac{1}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}\sum_{a=1}^{i}\sum_{b=1}^{\ell}\sum_{c=1}^{j}\left(\sum_{m\in I_{a,b,c}}\sum_{n\in I_{a,b,c}}\sum_{k\in I_{a,b,c}}q_m\bar{q}_n\bar{\bar{q}}_k\left[f\left(\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right)\right]^{p_{mnk}}\right)$$
$$=\frac{1}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}\sum_{a=1}^{i_0}\sum_{b=1}^{\ell_0}\sum_{c=1}^{j_0}H_{a,b,c}A'_{a,b,c}+\frac{1}{Q_{m_{k-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}\sum_{(i_0<a\le i)\bigcup(\ell_0<b\le\ell)\bigcup(j_0<c\le j)}H_{a,b,c}A'_{a,b,c}$$
$$\le\frac{G'Q_{m_{i_0}}\bar{Q}_{n_{i_0}}\bar{\bar{Q}}_{k_{i_0}}}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}+\frac{1}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}\sum_{(i_0<a\le i)\bigcup(\ell_0<b\le\ell)\bigcup(j_0<c\le)}H_{a,b,c}A'_{a,b,c}$$
$$\le\frac{G'Q_{m_{i_0}}\bar{Q}_{n_{\ell_0}}\bar{\bar{Q}}_{k_{j_0}}}{m_{i-1}n_{\ell-1}k_{j-1}}+\frac{1}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{j_{j-1}}}\sum_{(i_0<a\le i)\bigcup(\ell_0<b\le\ell)\bigcup(j_0<c\le j)}H_{a,b,c}A'_{a,b,c}$$
$$\le\frac{G'Q_{m_{i_0}}\bar{Q}_{n_{\ell_0}}\bar{\bar{Q}}_{k_{j_0}}}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}+\left(sup_{a\ge i_0\bigcup b\ge\ell_0\bigcup c\ge j_0}A'_{a,b,c}\right)\frac{1}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}\sum_{(i_0<a\le i)\bigcup(\ell_0<b\le\ell)\bigcup(j_0<c\le j)}H_{a,b,c}$$
$$\le\frac{G'Q_{m_{i_0}}\bar{Q}_{n_{\ell_0}}\bar{\bar{Q}}_{k_{j_0}}}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}+\frac{\varepsilon}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}\sum_{(i_0<a\le i)\bigcup(\ell_0<b\le\ell)\bigcup(j_0<c\le j)}H_{a,b,c}=\frac{G'Q_{m_{i_0}}\bar{Q}_{n_{\ell_0}}\bar{\bar{Q}}_{k_{j_0}}}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}+V_i\bar{V}_\ell\bar{\bar{V}}_j\varepsilon$$
$$\le\frac{G'Q_{m_{i_0}}\bar{Q}_{n_{\ell_0}}\bar{\bar{Q}}_{k_{j_0}}}{Q_{m_{i-1}}\bar{Q}_{n_{\ell-1}}\bar{\bar{Q}}_{k_{j-1}}}+\varepsilon H^3.$$

Since $Q_{m_{i-1}}\ \bar{Q}_{n_{\ell-1}}\ \bar{\bar{Q}}_{k_{j-1}}\to\infty$ as $i,l,j\to\infty$ approaches infinity, it follows that

$$\frac{1}{Q_p\bar{Q}_r\bar{\bar{Q}}_t}\sum_{m=1}^{p}\sum_{n=1}^{q}\sum_{k=1}^{t}q_m\bar{q}_n\bar{\bar{q}}_k$$
$$\left[f\left(\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right)\right]^{p_{mnk}}=0,\ \ uniformly\ \ in\ \ i,\ell\ \ and\ \ j.$$

Hence $x\in\left[\chi^3_R,q,f,p\right]$.

Corollary

Let $\theta_{i,l,j}=\{m_i,n_l,k_j\}$ be a triple lacunary sequence and $q_m\bar{q}_n\bar{\bar{q}}_k$ be sequences of positive numbers. If $1<lim_{ilj}V_{ilj}lim_{ilj}supV_{ilj}<\infty$, then for any Orlicz function f, $\left[\chi^3_R,\theta_{i\ell j},q,f,p\right]=\left[\chi^3_R,q,f,p\right]$.

Definition

Let $\theta_{i,l,j}=\{m_i,n_l,k_j\}$ be a triple lacunary sequence. The triple number sequence x is said to be $S_{\left[\chi^3_R,\theta_{i\ell j}\right]}-P$ convergent to 0 provided that for every $\varepsilon>0$,

$$P-lim_{i\ell J}\frac{1}{H_{i\ell j}}sup_{i\ell j}\left|\left\{(m,n,k)\in I'_{i\ell j}:q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{mnk}\right|\right)^{1/m+n+k},\bar{0}\right]\right\}\ge\varepsilon\right|=0.$$

In this case we write $S_{\left[\chi^3_R,\theta_{i\ell j}\right]}-P-limx=0$

Theorem

Let $\theta_{i,l,j}=\{m_i,n_l,k_j\}$ be a triple lacunary sequence. If $I'_{i,\ell,j}\subseteq I_{i,\ell,j}$, then the inclusion $\left[\chi^3_R,\theta_{i\ell j},q\right]\subset S_{\left[\chi^3_R,\theta_{i\ell j}\right]}$ is strict and $\left[\chi^3_R,\theta_{i\ell j},q\right]-P-limx=S_{\left[\chi^3_R,\theta_{i\ell j}\right]}-P-limx=0.$

Proof: Let

$$K_{Q_{i\ell j}}(\varepsilon)=\left|\left\{(m,n,k)\in I'_{i\ell j}:q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]\right\}\ge\varepsilon\right| \quad (2)$$

Suppose that $x\in\left[\chi^3_R,\theta_{i\ell j},q\right]$. Then for each i,l and j

$$P-lim_{i\ell J}\frac{1}{H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k$$
$$\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]=0.$$

Since

$$\frac{1}{H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]$$
$$\ge\frac{1}{H_{i\ell j}}\sum_{m\in I'_{i\ell j}}\sum_{n\in I'_{i\ell j}}\sum_{k\in I'_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]=\frac{\left|K_{Q_{i\ell j}}(\varepsilon)\right|}{H_{i\ell j}}$$

for all i,l and j, we get

$P-lim_{i,\ell,j}\dfrac{\left|K_{Q_{i\ell j}}(\varepsilon)\right|}{H_{i\ell j}}=0$ for each i,l and j. This implies that $x\in S_{\left[\chi^3_R,\theta_{i\ell j}\right]}$.

To show that this inclusion is strict, let $x=(x_{mnk})$ be defined as

$$(x_{mnk})=\begin{bmatrix}1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & \frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & \frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & 0 & \ldots\\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ \frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & 2 & 3 & \ldots\frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!} & \frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ . & & & & & & \\ \frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & \frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & \frac{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots & & \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots & \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \end{bmatrix};$$

and $q_m=1;\bar{q}_n=1;\bar{\bar{q}}_k=1$ for all m,n and k. Clearly, x is unbounded sequence. For $\varepsilon>0$ and for all i,l and j we have

$$\left|\left\{(m,n,k)\in I'_{i\ell j}:q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]\right\}\ge\varepsilon\right|$$
$$=P-lim_{i\ell j}\left(\frac{(m+n+k)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(m+n+k)!}\right)^{1/m+n+k}=0.$$

Therefore $x\in S_{\left[\chi^3_R,\theta_{i\ell j}\right]}$ with the P–lim=0. Also note that

$$P-lim_{i\ell J}\frac{1}{H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]$$
$$=P-\frac{1}{2}\left(lim_{i\ell j}\left(\frac{(m+n+k)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(m+n+k)!}\right)^{1/m+n+k}+1\right)=\frac{1}{2}.$$

Hence $x\notin\left[\chi^3_R,\theta_{i\ell j},q\right]$.

Theorem

Let $I'_{i\ell j}\subseteq I_{i\ell j}$. If the following conditions hold, then $\left[\chi^3_R,\theta_{i\ell j},q\right]_\mu\subset S_{\left[\chi^3_R,\theta_{i\ell j}\right]}$ and

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$\left[\chi^3_R,\theta_{i\ell j},q\right]_\mu - P-limx = S_{\left[\chi^3_R,\theta_{i\ell j}\right]} - P-limx = 0.$

(1). $0<\mu<1$ and $0\le\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]<1.$

(2). $1<\mu<\infty$ and $1\le\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]<\infty.$

Proof: Let $x=(x_{mnk})$ be strongly $\left[\chi^3_R,\theta_{i\ell j},q\right]_\mu -$ almost P convergent to the limit 0. Since

$$q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]^\mu \ge q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]$$

for (1) and (2), for all i,l and j, we have

$$\frac{1}{H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]^\mu$$

$$\ge\frac{1}{H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]\ge\frac{\varepsilon\left|K_{Q_{i\ell j}}(\varepsilon)\right|}{H_{i\ell j}}$$

where $K_{Q_{i\ell j}}(\varepsilon)$ is as mentioned above. Taking limit $i,l,j\to\infty$ in both sides of the above inequality, we conclude that $S_{\left[\chi^3_R,\theta_{i\ell j}\right]} - P-limx = 0.$

Definition

A triple sequence $x=(x_{mnk})$ is said to be Riesz lacunary of χ almost P– convergent 0 if $P-lim_{i,\ell,j}w^{i\ell j}_{mnk}(x)=0,$ uniformly in i,l and j, where $w^{i\ell j}_{mnk}(x)=w^{i\ell j}_{mnk}=\frac{1}{H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right].$

Definition

A triple sequence (x_{mnk}) is said to be Riesz lacunary χ almost statistically summable to 0 if for every $\varepsilon>0$ the set

$K_\varepsilon=\left\{(i,\ell,j)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\left|w^{i\ell j}_{mnk},\bar{0}\right|\ge\varepsilon\right\}$ has triple natural density zero, (i.e) $\delta_3(K_\varepsilon)=0$. In this we write $\left[\chi^3_R,\theta_{i\ell j}\right]_{st_2} - P-limx=0.$ That is, for every $\varepsilon>0$,

$P-lim_{rst}\frac{1}{rst}\left|\left\{i\le r,\ell\le s,j\le t:\left|w^{i\ell j}_{mnk},\bar{0}\right|\ge\varepsilon\right\}\right|=0,$ uniformly in i,l and j.

Theorem

Let $I'_{i\ell j}\subseteq I_{i\ell j}$. and $q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]\le M$ for all $m,n,k\in\mathbb{N}$ and for each i,l and j. Let $x=(x_{mnk})$ be $S_{\left[\chi^3_R,\theta_{i\ell j}\right]} - P-limx=0.$

Let $K_{Q_{i\ell j}}(\varepsilon)=\left|\left\{(m,n,k)\in I'_{i\ell j}:q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]\right\}\ge\varepsilon\right|.$ Then,

$$\left|w^{i\ell j}_{mnk},\bar{0}\right|=\left|\frac{1}{H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]\right|$$

$$\le\left|\frac{1}{H_{i\ell j}}\sum_{m\in I'_{i\ell j}}\sum_{n\in I'_{i\ell j}}\sum_{k\in I'_{i\ell j}}q_m\bar{q}_n\bar{\bar{q}}_k\left[\left((m+n)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k},\bar{0}\right]\right|\le\frac{M\left|K_{Q_{i\ell j}}(\varepsilon)\right|}{H_{i\ell j}}+\varepsilon$$

for each i,l and j, which implies that $St_2-P-lim_{i\ell j}w^{i\ell j}_{mnk}=0$ uniformly i,l and j. Hence, $St_2-P-lim_{i\ell j}w^{i\ell j}_{mnk}=0$ uniformly in i,l, j. Hence $\left[\chi^3_R,\theta_{i\ell j}\right]_{st_2} - P-limx=0.$

Conclusion

To see that the converse is not true, consider the triple lacunary sequence $\theta_{i\ell j}\left\{\left(2^{i-1}3^{\ell-1}4^{j-1}\right)\right\},q_m=1,\bar{q}_n=1,\bar{\bar{q}}_k=1$ for all m,n and k, and the triple sequence $x=(x_{mnk})$ defined by $x_{mnk}=\frac{(-1)^{m+n+k}}{(m+n+k)!}$ for all m,n and k.

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