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Revised Methods for Solving Nonlinear Second Order Differential Equations

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Abstract

In this paper, it has been tried to revise the solvability of nonlinear second order Differential equations and introduce revised methods for finding the solution of nonlinear second order Differential equations. The revised methods for solving nonlinear second order Differential equations are obtained by combining the basic ideas of nonlinear second order Differential equations with the methods of solving first and second order linear constant coefficient ordinary differential equation. In addition to this we use the property of super posability and Taylor series. The result yielded that the revised methods for second order Differential equation can be used for solving nonlinear second order differential equations as supplemental method.

Keywords: Solvability • Differential equations • Highest order derivatives

Introduction

Most problems in mathematical physics, engineering, astrophysics and many physical phenomena are governed by differential equations. The exact analytical solutions of such problems, except a few, are difficult to obtain. Many researchers have made attempts to rectify this problem and are able to develop new techniques for obtaining solutions which convincingly approximate the exact solution the difficulties that surround higher-order nonlinear differential equations and the few methods that yield analytic solutions [1-4]. Two of the solution methods considered in this section employ a change of variable to reduce a nonlinear second-order differential equations to a first-order differential equations by omitting the dependent and independent variable and these equations and more equations that can be easily solved by this method can be found in [5-10]. Nonlinear differential equations do not possess the property of super posability that is the solution is not also a solution. We can find general solutions of linear first-order differential equations and higher-order equations with constant coefficients even when we can solve a nonlinear first-order differential equation in the form of a one-parameter family, this family does not, as a rule, represent a general solution. Stated another way, nonlinear first-order differential equations can possess singular solutions, whereas linear equations cannot. But the major difference between linear and nonlinear equations of orders two or higher lies in the realm of solvability. Given a linear equation, there is a chance that we can find some form of a solution that we can look at an explicit solution or perhaps a solution in the form of an infinite series. On the other hand, nonlinear higher-order differential equations virtually challenge solution by analytical methods [11]. Although this might sound disheartening, there are still things that can be done. A nonlinear differential equation can be analyzing qualitatively and numerically. Let us make it clear at the outset that nonlinear higher-order differential equations are important even more important than linear equations because it is used to model a physical system, we also increase the possibility that this higher-resolution model will be nonlinear.

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Some Preliminary Concepts

Differential equation is an equation involving derivatives or differential of one or more dependent variables with respect to one or more independent variables.

Example:

a) $\frac{dy}{dx} = 2x + 5$ e) $\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} - z = 0$

b) $dy = (x + \sin x)dx$ f) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$

c) $\frac{d^4x}{dt^2} + \frac{d^2x}{dt^2} + (\frac{dx}{dt})^5$ g) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^4 + 4y = x^3$

d) $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{dy/_{dx}}$ h) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

i) $\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$ k) $\frac{dy}{dx} + 5y = e^x$

j) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Note: $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, \ldots\ldots\ldots\ldots, y^{(n)} = \frac{d^ny}{dx^n}$

Notation: prime notation $y', y'', \ldots\ldots\ldots\ldots, y^{(n)}$ or

Leibniz notation $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots\ldots\ldots, \frac{d^ny}{dx^n}$

To talk about them, Differential Equations are classified according to type, order, degree and linearity [12-14].

Classification by type

Ordinary differential equation: is a differential equation involving derivatives with respect to single independent variables. Example: a, b, c, d, g, i, k

Partial differential equation: is a differential equation involving partial derivatives with respect to more than one independent variables. Example: e, f, h, j

Classification by Order

The order of a differential equation (either ODE or PDE) defined as the order of the highest order derivative which appears in the equation.

Example: a, b, d, e, i, j, k are order one whereas f, g, h are order of two and c is order of four.

Classification by Degree

The degree of a differential equation is the degree of the highest order derivatives which occurs in it after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

Note that: the above definition of degree does not require variables x, t, y, etc to be free from fractions and radicals. Example: $a, b, c, e, f, h, i, j, k$ are of first degree and d, g are of second degree.

Linear and non-linear differential equation

The n^{th} order linear ODE is $a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)..(*)$ Two important special cases of $(*)$ are linear first order $(n=1)$ and linear second order $(n=2)$ DEs: $a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$ and $a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$. In the additive combination on the left hand side of equation $(*)$ we see that characteristic two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \ldots\ldots\ldots, y^{(n)}$ are of the first degree.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y', y'', \ldots\ldots\ldots, y^{(n)}$ depend at most on the independent variable x.

A non-linear ODE is simply one that is not linear. Non-linear functions of the dependent variable or its derivative, such as $\sin y$ or e^y cannot appear in the linear equation.

Example:

i) The equations $(y-x)dx+4xdy=0$, $y''-2y+y=0$ and $x^3\frac{d^3y}{dx^3}+x\frac{dy}{dx}-5y=e^x$ are in turn, linear first-, second-, and third-order ODEs. We have just demonstrated that the first equation is linear in the variable y by writing it in the alternative form $4xy'+y=x$.

ii) The equations $(1-y)y'+2y=e^x$ (non-linear term that is the coefficient depends on), $\frac{d^2y}{dx^2}+\sin y=0$ (non-linear term that is non-linear functions of y) and $\frac{d^4y}{dx^4}+y^2=0$ (non-linear term that is the power is not one) are examples of non-linear first-, second-,and fourth-order ODEs respectively.

The principle of superposition

If $y_1(x)$ is a solution of the linear second order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f_1(x)$ and $y_2(x)$ is a solution of the linear second order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f_2(x)$ (with the same left hand side), then the function $y(x)=k_1y_1(x)+k_2y_2(x)$ Where k_1 and k_2 are any constants, is a solution of the differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=k_1f_1(x)+k_2f_2(x)$ [15-16].

Definition: a solution of a differential equation in the unknown function y and the independent variable x on the interval I is a function $y(x)$ that satisfies the differential equation identically for all x in I.

Note: A particular solution of a DE is any one solution and the general solution of a differential equation is the set of all solutions.

Results of the Methods

We illustrated revised methods enables us to find explicit/implicit solutions of special kinds of nonlinear second-order differential equations.

Method 1: substitution method

We apply substitution method for nonlinear second order differential equations of the form $y''(x)=p(x)(y')^n$ and $y^my''(x)=(y')^n$.

i) **Nonlinear second-order differential equations of the form $y''(x)=p(x)(y')^n$** where $p(x)$ is the function of x and $n\in z$. If $u=y'$ then we can solve the differential equation for u, we can find y by integration. Since we are solving a second-order equation, its solution will contain two arbitrary constants.

Example: solve

a) $y''=2x(y')^2$

Solution: let $u=y'$ then $\frac{du}{dx}=y''$ after substituting this second order equation reduces to a first order equation with a separable variable, x is the independent variable and u is the dependent variable.

$\frac{du}{dx}=2xu^2$ or $\frac{du}{u^2}=2xdx$

$\int u^{-2}du=\int 2xdx$ by the method of separation of variable

$-u^{-1}=x^2+{c_1}^2$

$\frac{dy}{dx}=\frac{-1}{x^2+c_1^2}$ or $y=\frac{-1}{c_1}\tan^{-1}\frac{x}{c_1}+c_2$

b) $xy''=y'+(y')^3$

Let $u=y'$ then $\frac{du}{dx}=y''$ after substituting this second order equation reduces to a first order equation with a separable variable, x is the independent variable and u is the dependent variable.

$xu'=u+u^3$

$xu'-u=u^3$

$u'-\frac{1}{x}u=\frac{1}{x}u^3$ it is a Bernoulli differential equation ....(*)

Let $m=u^{-2}$, then $\frac{dm}{dx}=-2u^{-3}\frac{du}{dx}$

Implies $\frac{du}{dx}=\frac{-1}{2}u^3\frac{dm}{dx}$ .... (**)

Now substituting equation (**) into equation (*) yields

$$\frac{-1}{2}u^3\frac{dm}{dx}-\frac{1}{x}u=\frac{1}{x}u^3$$

$$\Rightarrow\frac{dm}{dx}+\frac{2}{x}uu^{-3}=\frac{1}{x}u^3$$

$\frac{dm}{dx}+\frac{2}{x}m=\frac{-2}{x}$...linear first order differential equations. The general standard form of linear first order differential equations is $\frac{dy}{dx}+p(x)y=Q(x)$.

Now using the working rule of linear first order differential equations

Here $P(x)=\frac{2}{x}$ and $Q(x)=\frac{-2}{x}$ and let $\mu(x)$ be the Integrating factor, then

$\mu(x)=e^{\int p(x)dx}$

$=e^{\int\frac{2}{x}dx}$

$=e^{2\ln x}$

$=x^2$

Then, $m(x).x^2=\int\left[\frac{-2}{x}.x^2\right]dx+c$, where c is arbitrary constant

$\Rightarrow m(x).x^2=\int[-2x]dx+c$

$\Rightarrow m(x).x^2=-x^2+c$

$\Rightarrow m(x)=-1+cx^{-2}$

Now $\frac{1}{u^2}=-1+cx^{-2}$

$u^2=-1+cx^{-2}$

ii) **Nonlinear second-order differential equations of the form $y^my''(x)=(y')^n$** where $n, m\in z$ the dependent variable omitting. If $u=y'$ and the independent variable x is missing, we use this substitution to transform the differential equation into one in which the independent variable is y and the dependent variable is u. To this end we use the Chain Rule to compute the second derivative of y that is $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$.

In this case the first-order equation that we must now solve is $F\left(y, u, u\frac{du}{dy}\right) = 0$

Example: solve

a) $yy'' = (y')^2$

Let $u = y'$ then $y\left(u\frac{du}{dy}\right) = u^2$ or $\frac{du}{u} = \frac{dy}{y}$ integrating the last equation gives

$$\ln|u| = \ln|y| + c_1$$

$$u = c_2 y$$

We now resubstitute $u = \frac{dy}{dx}$

$\int \frac{dy}{y} = c_2 \int dx$ by the method of separation of variable

$$\ln|y| = c_2 x + c_3$$

$$y = c_4 e^{c_2 x}$$

Method 2: Using Methods Applied for Finding the Solutions of Linear Second Order Differential Equations

The explicit solutions of some nonlinear second order ordinary differential equation of the form $(y')^n = (y)^n$ can be found by using methods applied for finding the solutions of linear second order differential equations.

Example: solve

$$(y')^2 - y^2 = 0$$

Solution: $(y')^2 - y^2 = 0$

$$\Rightarrow (y')^2 = y^2$$

$$\Rightarrow y'' = \pm y$$

Case 1: if $y'' = y$

Implies $y'' - y = 0$, by using the method of linear second order differential equation with constant coefficients [17-18].

The auxiliary /characteristics equations for this differential equations is

$m^2 - 1 = 0$ or $(m-1)(m+1) = 0$

Implies $m = \pm 1$

Therefore the solution of this differential equation is $y_1(x) = c_1 e^{-x} + c_2 e^{x}$

Case 2: if $y'' + y = 0$

Implies $y'' + y = 0$, by using the method of linear second order differential equation with constant coefficients

The auxiliary /characteristics equations for this differential equations is

$m^2 + 1 = 0$ or $(m-i)(m+i)$

Implies $m = \pm i$

Therefore the solution of this differential equation is $y_2(x) = c_3 \cos x + c_4 \sin x$

The overall general solution for the differential equation $(y')^2 - y^2 = 0$ using superposition principle,

$$y(x) = y_1(x) + y_2(x)$$

Implies $y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x$ where $c_1, c_2, c_3 \,\&\, c_4$ arbitrary constants are

Method 3: Using Taylor's Series

Definition (Taylor series): If f has a power series representation at a, then any expression of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}.(x-a)^n = f(a) + \frac{f'(a)}{1!}.(x-a) + \frac{f''(a)}{2!}.(x-a)^2 + \frac{f'''(a)}{3!}.(x-a)^3 + \cdots$$

is called the Taylor series of the function f at a.

If $a = 0$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}.x^n = f(0) + \frac{f'(0)}{1!}.x + \frac{f''(0)}{2!}.x^2 + \frac{f'''(0)}{3!}.x^3 + \cdots$ is called the Maclaurin series representation of $f(x)$ and a function $f(x)$ is said to be analytic at appoint a if f is differentiable at a and at every point in some neighborhood of a [19].

Now let us approximate the solution of the initial value problem in the case of nonlinear second order ordinary differential equation using Taylor's series.

Example: solve the following IVP using Taylor's series.

$$y''(x) = x + y - y^2, y(0) = -1, y'(0) = 1$$

Solution: a) suppose the solution *y*(x) of the problem is analytic at 0 .then *y*(x) is represented by a Taylor's series centered at 0.

$$y(x) = y(0) + \frac{y'(0)}{1!}.x + \frac{y''(0)}{2!}.x^2 + \frac{y'''(0)}{3!}.x^3 + \frac{y^{(4)}(0)}{4!}.x^4 + \frac{y^{(5)}(0)}{5!}.x^5 + \cdots$$

The first and second terms of the series are known from the initial conditions that is $y(0) = -1, y'(0) = 1$. Since the differential equation defines the values of the second derivative at 0 that is $y''(0) = 0 + y(0) - y(0)^2 = 0 + (-1) - (-1)^2 = -2$

Now we can find $y''', y^{(4)}$,etc.by calculating the successive derivative of the differential equation:

$$y'''(x) = \frac{d}{dx}(x + y - y^2) = 1 + y' - 2yy'$$

$$\Rightarrow y'''(0) = 4$$

$$y^{(4)}(x) = \frac{d}{dx}(1 + y' - 2yy') = y'' - 2yy'' - 2(y')^2$$

$$\Rightarrow y^{(4)}(0) = -8$$

$$y^{(5)}(x) = \frac{d}{dx}(y'' - 2yy'' - 2(y')^2) = y''' - 2yy''' - 6y'y''$$

$$\Rightarrow y^{(5)}(0) = 24$$

etc.

Therefore the first six terms of a series solution of the given IVP are

$$y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \cdots$$

Conclusion

In this paper, the methods for solving nonlinear second order differential equations are illustrated and revised methods for solving nonlinear second order differential equations is formulated. This revised methods for solving nonlinear second order differential equations is investigated by starting with basic ideas of nonlinear second order differential equations and combining with the the second order linear differential equations.

References

1. Chowdhury MSH, Hashim I. "Solutions of a class of singular second-order IVPs by homotopy-perturbation method." *Phys Lett A* 365 (2007): 439-447.
2. Cveticanin L. "The homotopy-perturbation method applied for solving complex-valued differential equations with strong cubic nonlinearity." *JSV* 285 (2005):1171-1179.
3. Rana MA, Siddiqui AM, Ghori QK and Qamar R. "Application of He's homotopy perturbation method to sumudu transform." *Int J Nonlinear Sci Numer Simul* 8 (2007): 185-190.
4. Yildirim A, Ozis T. "Solutions of singular IVPs of Lane-Emden type by homotopy perturbation method." *Phys Lett A* 369 (2007):70-76.

5. Polyanin AD, Zaitsev VF. "Handbook of Exact Solutions for Ordinary Differential Equations." 2nd Edition, *Chapman & Hall/CRC Press* Boca Raton(2003).
6. Polyanin AD,Manzhirov AV. "Handbook of Mathematics for Engineers and Scientists." *Chapman & Hall/ CRC Press* Boca Raton(2006).
7. Zwillinger D. "Handbook of Differential Equations." 3rd Edition, *Academic Press* Boston(1997).
8. Polyanin AD, Zaitsev VF and Moussiaux A. "Handbook of First Order Partial Differential Equations." *Taylor & Francis* London(2002).
9. Kamke E. Differential gleichungen: Losungsmethoden und Losungen. I. Gewohnliche Differential gleichungen. B. G. Teubner, Leipzig.
10. Boyce WE, Prima RC. "Elementary Differential Equations and Boundary Value Problems." *John Wiley and Sons* Inc. Hoboken (2000).
11. Dennis G. "Zill-A First course in Differential Equations with Modeling Applications." *Brooks cole* (2012).
12. Boyce WE, Prima RC. "Elementary differential equations and boundary valueproblems." 7thed. *John Wiley & Sons* INC.(2001).
13. Nageletal. "Fundamentals ofdifferentialequations." 5thed, *Addison Wesley Longman* (2004).
14. Martin Braun. "Differential equations and their applications." *Springer-Verlag* (1993).
15. Rainville ED, Bedient PE. "Elementary differential equations." *MacMillan publishing company*(1999).
16. Ross RL." Introduction to ordinary differential equations." 4thed, *Wiley*(1989).
17. Erwin Kreyzing. "Advanced engineering mathematics." 10thed, *Wiley*(2000).
18. John Bird. "Higher engineering mathematics." 5thed, *Elsevier* (2006).
19. Alan Jeffrey. "Advanced engineering mathematics." 1sted, *Acadamic press* (2002).

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