Journal of
Applied & Computational Mathematics
Gadjiev et al., J Appl Computat Math 2017, 6:3
DOI: 10.4172/2168-9679.1000364
Research Article
Open Access

# Regularity of Solutions of Degenerate Parabolic Non-linear Equations and Removability of Solutions

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**Abstract**

In this paper we prove regularity of solutions of degenerate parabolic nonlinear equations. We also the proof of a removability theorem for solutions to degenerate parabolic nonlinear equations.

**Keywords:** Degenerate; Nonlinear parabolic equations; Regularity; Removability

## Introduction

Let we are considered in cylindrical domains $Q_T = \Omega \times (0,T)$, where $\Omega \subset R^n$, $n \geq 2$ is a bounded Lipschitz domain, T>0, degenerate non-linear parabolic equations

$$u_t - div(\omega(x)|Du|^{p-2}Du) = 0 \tag{1.1}$$

$$u|_{\Gamma(Q_T)} = h, \tag{1.2}$$

where $\Gamma(Q_T) = (\Omega^- \times \{0\}) \cup (\partial\Omega \times [0,T])$ denote the parabolic boundary of $Q_T$, $h : Q_T \to R$ continuous function, $\omega(x)$-Makenxhoupt weight function [1].

To regularity of solutions to the degenerate parabolic non-linear operator introduced by DiBenedetto et al. [2,3]. Let $C_w^\alpha(Q_T)$ weighted space, where norm following:

$$\|f\|_{C^\alpha_{w(x)}(Q_T)} = \sup_{z_1, z_2 \in Qr} \frac{|f(z_1)w(x_1) - f(z_2)w(x_2)|}{\|z_1 - z_2\|\alpha} < \infty$$

where the parabolic metric is defined as

$$\|(x_1,t_1) - (x_2 t_2)\|_\alpha = \max\left[|x_1 - x_2|\right], |t_1 - t_2|^{\frac{1}{[p-\alpha(p-2)]}}, 0 < \alpha < 1$$

## Main Results

We are now ready to state our result which concerns regularity for solutions to the problem (1.1), (1.2).

**Theorem 2.1**

Let's consider problem (1.1), (1.2)and let u(x,t) solve this problem. Let $Q^0_T \subset Q_T$ be a bounded space time cylinder such that (interior regularity)

$Q^0_T \cap \Gamma(Q_T) = \emptyset$. Then $u \in C^\alpha_{w(x)}(Q'_T)$ and

$$\|u(x,t)\|_{u \in C^\alpha_{w(x)}(Q'_T)} \leq c(n, p, w(x), Q_T, Q'_T, |h(x) \in C^\alpha_{w(x)}(\Omega), osu(x,t)|) \tag{2.1}$$

Theorem 2.1 concerns optimal interior regularity. We also establish optimal regularity up to initial state. In particular, in this case we prove $C^\alpha_w(Q_r)$ estimates on $Q_r = \Omega \times (0,T)$ ) for every $\Omega_r \subset \Omega$. We doing remark that in this case $Q_T$ is not a compact subset of $Q_T$.

In this context hold following result [1-12].

**Theorem 2.2**

Let u(x,t) solve problem (1.1), (1.2) and(Initial time regularity)

$h(x) \in C^\alpha_{w(x)}(\Omega)$ ,$\Omega' \subset \Omega$ $Q'_T = \Omega'$ $X(0,T)$, $u \in C^\alpha_{w(x)}(Q'_T)$

And

$$\|u(x,t)\|_{u \in C^\alpha_{w(x)}(Q'_T)} \leq c(n, p, w(x), Q_T, Q'_T, |h(x) \in C^\alpha_{w(x)}(\Omega), osu(x,t)|) \tag{2.2}$$

We also can be is considered obstacle problem similarly to problem (1.1), (1.2). In the case of linear uniformly parabolic equations [4]. Optimal regularity problem of the solution is considered [5].

We are study weak solutions from $L^P(t_1 t_2, W^{1,p}_{w(x)}(Q_T))$ space. In the space

$W^{1,p}_{w(x)}(\Omega)$ the norm denote the space of equivalence classes of functions f with distributional gradient Df, both of which are p[th] power integral on $Q_T$. Let

$$\|f\|_{W^{1,p}_{w(x)}(\Omega)} = \|w(x) f(x)\|_{LP(\Omega)} + \|w(x)\||Df|_{LP(\Omega)}$$

be the norm in $W^{1,p}_{w(x)}(\Omega)$.

Given $t_1 < t_2$ we denote by $L^P(t_1 t_2, W^{1,p}_{w(x)}(Q_T))$ the space of functions such that for almost every t, $t_1 \leq t \leq t_2$ the function

$x \to u(x,t)$ belongs to $W^{1,p}_{w(x)}(\Omega)$ and $\|u\|_{L^P(t_1 t_2, W^{1,p}_{w(x)}(Q_T))}$

$$= \left( \int_{t_1}^{t_2} \int_\Omega (w(x)|u(x,t)|^p + w(x)|Du(x,t)|^p dxdt \right)^{\frac{1}{P}} \leq \infty$$

We say that a function u(x, t) is a weak solution to (1.1), (1.2) in an open set

$Q_T \subset R^{n+1}$ if whenever $Q^0_T = \Omega^0 \times (t_1, t_2) \subset Q_T$ with $\Omega^0 \subset \Omega \subset R^n$ and $t_1 < t_2$ then $u \in L^P(t_1 t_2, W^{1,p}_{w(x)}(\Omega))$ and

$$\int_{Q'_T} (w(x)|Du|^{p-2} Du D\varphi - u\varphi_t) dxdt = 0 \tag{2.3}$$

for all nonnegative $\varphi \in C^\infty_0(Q'_T)$.

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**Received** July 07, 2017; **Accepted** August 21, 2017; **Published** August 31, 2017

**Citation:** Gadjiev TS, Yangaliyeva A, Zulfalieva G (2017) Regularity of Solutions of Degenerate Parabolic Non-linear Equations and Removability of Solutions. J Appl Computat Math 6: 364. doi: 10.4172/2168-9679.1000364

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Using Theorem2.1 we are able to establish sharp removability conditions for compact sets. We of cylinders introduced

$$Q_T^{\lambda}(x_0,t_0)=\left\{(x,t)\in R^{n+1}:\left|x_0-x\right|<r,\left|t_0-t\right|<\lambda^{2-p\tau p}\right\}$$

And a concave modulus of continuity $\psi(\cdot)$. We let $\psi$: $R_+ \to R_+$ be a concave modulus of continuity, i.e., concave non-decreasing function such that $\psi(1)=1$ and $\psi(0)=\lim_{r\to 0}\psi(r)=0$. We also define Hausdorff measure as follows. We let for fixed $\delta$,0< $\delta<r_0$ and $E\subset R^{n+1}, L(\delta,\psi(.);E)=\left\{Q_{ri}^{\psi(ri)}(x_i,t_i)\right\}$ be a family of cylinders such that $E\subset\cup Q_{ri}^{\psi(ri)}(x_i,t_i)$ and $0<r_i<\delta$ for i=1,2,..

Using this notation we let

$$H^{\psi(.)}(E)=\lim_{\delta\downarrow 0}\inf_{L(\delta,\psi(.);E)}\left\{\sum r_i^n\psi(r_i):E\subset\cup Q_{ri}^{\psi(ri)}(x_i,t_i)\right\}$$

where the indium is taken with respect to all possible coverings L($\delta$,$\psi(\cdot)$; E) of E.

**Theorem 2.3**

Let $Q_T$ be a cylindrical domain and let E ⊂ $Q_T$ be a closed set. Let u(x,t) is a weak solution to eqn. (1.1) in $Q_T\backslash E$ and that $u(x,t)\in C^{\alpha}_{w(x)}(Q_T)$

Assume also that $H^{\psi(\cdot)}(E)=0$. Then the set E is removable, i.e., u(x,t) can be extended to be a weak solution in $Q_T$.

Similarly result the fundamental work [6], under assumption Holder continuity of the solution can be found [7-12].

**Proof of theorem 2.1**

We assume $Q_{,T}\subset Q_T$ such that $Q^0_T\cap\Gamma(Q_T)=\emptyset$.

We define function

$$\underset{\Omega X[0,\infty]}{osc}\overline{h}=\underset{\Omega X[0,T]}{osc}h$$

Then $\underset{\Omega X[0,\infty]}{osc}\overline{h}=\underset{\Omega X[0,T]}{osc}h$ . Let $\bar{u}$ be the unique solution to

$$\bar{u}_t-div(w(x)\left|D\bar{u}\right|^{p-2}D\bar{u})=0 \text{ in } \Omega\times(0,\infty)$$

$\bar{u}(x,t)=\bar{h}(x,t)$ on $\Gamma(\Omega\times(0,\infty))$.

By the uniqueness $\bar{u}=u$ in $\Omega\times[0,T]$ and hence $\bar{u}$ is an extension of $u$. Let

$R=\max\{1,diam\Omega,T^{1/2}\}$. As clearly

$T\le(\psi(R))^{2-p}R^p\le R^2$.

Whenever $R\ge 1$. By maximum and minimum principle implies that

$$\underset{Qr}{osc}u\le\underset{Qr}{osc}u\le\overline{c}\left(\Omega,T,\underset{Qr}{osc}u\right) \tag{2.4}$$

We may assume that $Q'_{T,\gamma=}\Omega' x(\tau,T)$, where $\Omega^0\subset\Omega$ and $\tau>0$. We let $R$ be

a number subject to the restrictions

$R\le dist(\Omega^0,\partial\Omega),\tau\ge R^p\max\{osch,\psi(R),s\cdot R\}^{2-p}$.

$Q_T$

As so $\psi(1)=1$, we see that these conditions are satisfied if we take

$$R\le dist(\Omega',\partial(\Omega',\partial\Omega)),\max\left\{T^{\frac{1}{P}}(\Omega,T,\underset{QT}{osch})^{\frac{p-2}{p}},\tau^{\frac{1}{P}},\tau^{\frac{1}{r}},\tau^{\frac{1}{2}},S^{\frac{P-2}{P}}\right\}$$

Taking correspondingly $\lambda$ it follows that $Q_R^{\lambda\psi(R)}(z)\subset Q_T$ whenever $z\in Q^0_{T,\tau}$.

Now we prove that the following holds whenever $z\in Q'_T$

$$\underset{Q_{T,r}^{c-\lambda\psi(r)}(z_0)\cap Qr}{osc}u\le\underset{QT}{osc}u=\frac{\underset{QT}{osc}u}{\psi(r)}\le\frac{\underset{QT}{osc}u}{\psi(\frac{R}{2})}\le 2\lambda\psi(r)$$

This completes the proof of Theorem 2.1.

**Proof of theorem 2.2**

After extending $u(x,t)$ as in the above we choose

$R=dist(\Omega^0,\partial\Omega)$ and define

$$\lambda=\max\left(\overline{c}/\psi(R),\left|b\right|_{C_{w(x)}(\Omega)},s.R/\psi(R)\right)\quad\text{where}\quad\overline{c}=\overline{c}\left(\Omega,T,\underset{Qr}{osc}u\right)$$

We let $Z=\Omega^{-0}\times(0)$ then

$\underset{Q_r^{2\lambda\psi(r)}(z_1)}{osc}u\le c\,\lambda\psi(r)$ for every $r\in(0,R)$,

$Q\lambda\psi r\,(r)(z)\cap Q0T$.

Whenever $z\in Z$. Consider $z_1\in\left(Q'_T\cap(\overline{\Omega}'X(0)\right)$ and define $\overline{r}=\overline{r}(z_1)=\sup\left[r\le R:Q_r^{\lambda\psi(r)}(z_1)\cap Z=\theta\right]$ If $r>R/2$, then

$\underset{Q_r^{2\lambda\psi(r)}(z_1)}{osc}u\le c\,\lambda\psi(r)$ for every $r\in(0,R)$.

In the final

$$\bar{\lambda}=\max\left\{4\lambda\psi(\bar{r}),s\cdot r/\psi^{-}(\bar{r})\right\}4\max\left\{\lambda,sR/\psi(R)\right\}=c\cdot\lambda,$$

implies that

$\underset{Q_r^{2\lambda\psi(r)}(z_1)}{osc}u\le c\,\lambda\psi(r)$ for every $r\in[0,\bar{r}]$.

Whenever $z_1\in\left(Q'_T\cap(\overline{\Omega}'X(0)\right)$.

This completes the proof of Theorem 2.2.

**Proof of theorem 2.3**

Let $u(x,t)$ weakly solve of eqn. (1.1) in $Q_T\backslash E$ and assume that $u(x,t)\in C^{\alpha}_{w(x),loc}(Q_T)$ and $H^{\psi(\cdot)}(E)=0$. $Q^2_T\subset Q^1_T\subset Q_T$ be arbitrary space-time smooth cylinders. Our only need to prove the conclusion in $Q^1_T$ since the one of being a weak solution is a local property. By the assumption

$u(x,t)\in C^{\alpha}_{w(x),loc}(Q_T)$ there exists $M>0$ such that

$$\underset{Q_T^1}{osc}u(x,t)\le M\ \text{and}\ \underset{Q_r^{\mu-\psi(r)}\cap Q_T^1}{osc}u(x,t)\le M\psi(r) \tag{2.5}$$

If we using the existence result, then see that there exist a unique solution $v(x,t)$ of problem

$$u_t-div(w(x)\left|Dv\right|^{p-2}Dv)=0 \tag{2.6}$$

$v|_{\Gamma(}QT^1)=u$

Let $\mu$ be the nonnegative Riesz measure associated to $v(x,t)$. Note that from existence $\mu$ follows $v(x,t)$ is a supersolution [7]. Let $F=\{(x,t)\in Q^1_T: v(x,t)=u(x,t)\}$. Now prove that the support of $\mu$ is contained in $F\cap E$. For these is sufficient to show that $v(x,t)$ is a weak solution to (2.6) in $Q^1_T\backslash(F\cup E)$. We already know that (2.6) satisfy in $Q^1_T\backslash F$ and it therefore remains to show that (2.6) satisfy in $Q^{-}_{T=}Q^1_T\backslash E$. To this aim, we show that if $Q^{-*}_T\subset Q^{-}_T$ is a cylinder and $\alpha\in C^0(Q^{*}_T)$ is a weak solution to $\alpha_t-(w(x)\left|D\alpha\right|^{P-2}D\alpha)$ witk $\alpha=u$ on $\Gamma(Q_T)$, then actually $v$ must coincide with $\alpha$ (x,t)in the( $Q_T$ ) . Note that such a unique solution $\alpha(x,t)$ exists. We immediately see by the comparison principle that $v\ge\alpha$ in $Q^{-*}_T$, because $v(x,t)$ is a weak supersolution. To show that $v\le\alpha$ we

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instead argue as follows: since $u(x,t) \le v(x,t)$, we also have $u(x,t) \le \alpha(x,t)$ on $\Gamma\, Q_T$ and as $u(x,t)$ solves eqn. (1.1) in $Q_T$, the comparison principle holdsu(x,t)≤ $\alpha$ (x,t)in. We thus conclude that $v(x,t) \le \alpha(x,t)$ on $\Gamma(Q^{\tilde{}*}_T) \cup F$. Therefore $v(x,t)=\alpha(x,t)$ and consequently also eqn. (2.6) yields in $Q^{\tilde{}*}_T$. This completes the proof that support of $\mu$ is contained in $F \cap E$.

Later using Theorem 2.1 and a covering argument we can conclude that there exists $C$ depending only on $n,p,v,L,M,\psi(\cdot),Q^1_T,Q^2_T$ such that

$$\underset{Q_T^{\mu-\psi(\mathrm{r})} \cap Q_T^1}{osc} u(\mathrm{x},\mathrm{t}) \le M\,\psi(\mathrm{r}) \qquad (2.7).$$

Whenever $\underset{Q_T^1}{osc}\, u(\mathrm{x},\mathrm{t}) \le M$. Consider concentric cylinders $\underset{Q_T^{\mu-\psi(\mathrm{r})} \cap Q_T^1}{osc} u(\mathrm{x},\mathrm{t}) \le M\,\psi(\mathrm{r})$. In the following we will use the short notation $\bar{Q}_T = \mathrm{Q}_T^1 \setminus \mathrm{E}$. Let $\bar{Q}_T = \mathrm{Q}_T^1 \setminus \mathrm{E}$ be such 0 and $\phi \equiv 1$ on $Q^{\tilde{}}_\tau$. Let $k=\sup v(x,t)$. Using eqn. (2.6) we have

$Q\tilde{}2\tau$

$$0 \le \mu\left(\bar{Q}_r\right) \le \int_{Q2r} \varphi^p \, d\mu =$$

$$\int_{Q2r}\left[-\left(\varphi^p\right)_{t^v} + \left(w(x)|Dv|^{p-2}Dv\right)D\varphi^p\right]dxdt \le$$

$$c\int_{Q2r} w(x)|Dv|^{p-1}|D\varphi|\varphi^{p-1}dxdt + \int_{Q2r}(\varphi^p)_t\, vdxdt \le$$

$$c\left(\int_{Q2r} w(x)|Dv|^p\varphi^p dxdt\right)^{\frac{p-1}{p}}\left(\int_{Q2r} w(x)|D\varphi|\varphi^p dxdt\right)^{\frac{1}{p}} +$$

$$\int_{Q2r}(\varphi^p)_t\, vdxdt \qquad (2.8)$$

For the nonnegative weak sub solution $k-v(x,t)$ we see that

$$c\int_{Q2r}[\mathrm{w}(\mathrm{x})(k-v)^p|D\varphi|^{p-1} + |k-v|^2\left|(\varphi^p)_t\right| + S^p\varphi^p]dxdt$$

for some const $c=c(n,p,v,L) \ge 1$. By eqn. (2.7)

$$\sup_{Q_{2r}}|k-v| \le \underset{Q_T^1}{osc}\, v(x,t) \le C\psi(r)$$

and putting the estimates (2.8) we obtain that

$$\mu(Q_T) \le c\left[\left(\psi^2(\tau)\tau^n + s^p\psi^{2-p}(\tau)\tau^{2-p}\right)\right]^{\frac{p-1}{p}} \cdot \left[|\psi(\tau)|^{2-p}\tau^n\right]^{\frac{1}{p}}$$

$$c\psi(\tau)\tau^n \le c(1+s)^{p-1}\psi(\tau)\tau^n. \qquad (2.9)$$

Here we also used the estimate $|\psi(\tau)|^{2-p} \le \tau^{2-p}$ for $\tau \le 1$. Now we consider cylinder $Q^3_T \subset Q^2_T$. We will prove that $\mu(Q^3_T)=0$. We first note using eqn. (2.9) we have

$$\mu\left(Q_\tau^{w(\tau)}\right) \le c\tau^n\varphi(\tau)\cdot \qquad (2.10)$$

Whenever $Q_{2\tau}^{w(2\tau)} \subset Q_\tau^2$. Since $H^{\psi(\cdot)}(E)=0$ we obtain for $\varepsilon>0$ and $\delta>0$ given (to be taken smaller that $\mathrm{dist}(\Gamma(Q^3_T),Q^2_T)/4$), then there exists a countable family

$$\left\{Q_{\tau_i}^{\psi(\tau_i)}\right\} = \left\{Q_{\tau_i}^{\psi(\tau_i)}(x_i,t_i)\right\}$$

of cylinders with $0 < \tau_i < \delta, i=1,2,...$, such that $Q_{2\tau_i}^{\psi(2\tau_i)} \subset Q_\tau^2$ and

$$E \cap Q3T \subset [Q\tilde{}\psi\tau i(\tau i) \text{ and } X\tau in\psi(\tau i)<\varepsilon. \qquad (2.11)$$

Later using eqn. (2.10) we is obtain

$$\mu\left[F \cap \left(E \cap Q_T^3\right)\right] \le \sum_i \mu\left(Q_{\tau_i}^{\psi(\tau_i)}\right) \le \sum_i \tau_i^n\psi(\tau_i) < C.\in \qquad (2.12)$$

proving that $\mu[F \cap (E \cap Q^3_T)]=0$. The fact that both $Q^2_T$ and $Q^3_T$ are arbitrary, we can conclude that $\mu(Q^1_T)=0$. Thus $v(x,t)$ is a solution in $Q^1_T$. Finally applying the above argument with $u(x,t)$ replaced by $-u(x,t)$ we deduce that there exist two solutions $v_1(x,t)$ and $v_2(x,t)$ i.e., eqn. (2.6) for $v_1$ equal to eqn. (2.6) for $v_2$. Such that $v_1(x,t) \le u(x,t) \le v_2(x,t)$ and $v_1(x,t)=v_2(x,t)$ on $\Gamma(Q^1_T)$. It follows that $v_{1=}v_{2=}u$. Theorem is proof.

## References


1. Chamillo S, Wheeden RL (1985) Weighted Poincare and Sobolev inequalities. Amer J Math 107: 1191-1226.
2. Di Benedetto E (1993) Degenerate parabolic equations. Springer-Verlag.
3. Di Benedetto E, Gianazza U, Vespri V (2008) Intrinsic Harnack Estimates for Non-Negative Solutions of Quasilinear Degenerate Parabolic Equations. Acta Mathematica 200: 181-209.
4. Friedman A (1995) Parabolic variational inequalities in one-space dimension and smoothness of the free boundary. J Func Anal 18: 151-76.
5. Caffarelli LA (1998) The obstacle problem revisited. J Fourier Anal Appl 4: 383-402.
6. Serrin J (1964) Local behavior of solutions of quasi-linear equations. Acta Mathe-matica 3: 247-302.
7. Heinonen J, Kilpelainen T (1998) A superharmonic functions and supersolutions of degenerate elliptic equations. Arkiv Matematik 26: 87-105.
8. Kilpelainen T, Zhong X (2002) Removable sets for continuous solutions of quasi-linear elliptic equations. Proceedings of the American Mathematical Society 130: 1681-1688.
9. Gadjiev TS, Sadigova NR, Rasulov RA (2011) Removable singularities of solutions of degenerate nonlinear elliptic equations on the boundary of a domain. Nonlinear Analysis: Theory, Methods and Applications 74: 5566-5571.
10. Gadjiev T, Bayramova N (2014) The removability of compact of solutions in classes bounded functions. Ukrayne Mathematics Journal 8: 38-44.
11. Gadjiev T, Aliev O (2013) On Removable Sets of Solutions of Neuman Problem for Quasilinear Elliptic Equations of Divergent Form. Applied Mathematics 4: 290-298
12. Gadjiev T (2013) On Removable Sets for Generated Elliptic Equations. British Journal of Mathematics and Computer Science 3: 290-298.