Regularity of Solutions of Degenerate Parabolic Non-linear Equations and Removability of Solutions

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Abstract

In this paper we prove regularity of solutions of degenerate parabolic nonlinear equations. We also prove the proof of a removability theorem for solutions to degenerate parabolic nonlinear equations.

Keywords: Degenerate; Nonlinear parabolic equations; Regularity; Removability

Introduction

Let Q0 ⊂ Ω be a bounded Lipschitz domain, T>0, degenerate parabolic equations
\[ u_0 - \text{div}(w(x)|Du|^{p-2}Du) = 0 \]
\[ u|_{t=0} = h, \]
where \( \Gamma(Q_0) = (\Omega \times \{0\}) \cup (\partial \Omega \times [0,T]) \) denote the parabolic boundary of Q0, h : \( Q_0 \times \mathbb{R} \rightarrow \mathbb{R} \) a continuous function, \( w(x) \)-Makianxhoup weight function [1].

To regularity of solutions to the degenerate parabolic non-linear operator introduced by DiBenedetto et al. [2,3]. Let \( C^\alpha_\omega(Q_\tau) \) be a weighted space, where the norm is defined as:
\[ \|f\|_{C^\alpha_\omega(Q_\tau)} = \sup_{x,y \in Q_\tau} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} \]
where the parabolic metric is defined as:
\[ \|x,t\|_h = \max\{|x_1|, t^{\frac{1}{\alpha}}\} \]

Main Results

We are now ready to state our result which concerns regularity for solutions to the problem (1.1), (1.2).

Theorem 2.1

Let's consider problem (1.1), (1.2) and let \( u(x,t) \) solve this problem. Let \( Q_{t_0} \subset Q_\tau \) be a bounded space time cylinder such that (interior regularity)
\[ \Omega_{t_0} \cap \Gamma(Q_\tau) = \emptyset. \]
Then \( u \in C^\alpha_\omega(Q_\tau) \) and
\[ \|u(x,t)\|_{C^\alpha_\omega(\Omega_{t_0})} \leq c(n,p,w(x),Q_\tau,\Omega,\|u(x,t)\|_{C^\alpha_\omega(\Omega,\|h(x)\|_{C^\alpha_\omega(\Omega,\|u(x,t)\|_{C^\alpha_\omega(\Omega)})}}) \leq \infty \]
(2.1)

Theorem 2.1 concerns optimal interior regularity. We also establish optimal regularity up to initial state. In particular, in this case we prove \( C^\alpha_\omega(Q_\tau) \) estimates on \( Q_\tau = (\Omega \times (0,T)) \) for every \( \Omega \subset \Omega_\tau \). We do not remark that in this case \( Q_\tau \) is not a compact subset of \( Q_{t_0} \).

In this context hold following result [1-12].

Theorem 2.2

Let \( u(x,t) \) solve problem (1.1), (1.2) and (initial time regularity)
\[ h(x) \in C^\alpha_\omega(\Omega, \Omega \subset \Omega_\tau = \Omega \times (0,T), u \in C^\alpha_\omega(Q_\tau^T) \]
And
\[ \|u(x,t)\|_{C^\alpha_\omega(Q_\tau^T)} \leq c(n,p,w(x),Q_\tau,\Omega,\|h(x)\|_{C^\alpha_\omega(\Omega,\|u(x,t)\|_{C^\alpha_\omega(\Omega)})}} \]
(2.2)

We also consider an obstacle problem similarly to problem (1.1), (1.2). In the case of linear uniformly parabolic equations [4]. Optimal regularity problem of the solution is considered [5].

We are study weak solutions from \( L^p(t_1,t_2, C^\alpha_\omega(Q_\tau)) \) space. In the space \( C^\alpha_\omega(\Omega) \) the norm denote the space of equivalence classes of functions \( f \) with distributional gradient \( Df \), both of which are \( p \)-th power integral on \( Q_\tau \). Let
\[ \|f\|_{C^\alpha_\omega(\Omega)} = \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} \]

Given \( t_1 < t_2 \) we denote by \( L^p(t_1,t_2, C^\alpha_\omega(Q_\tau)) \) the space of functions such that for almost every \( t, t_1 \leq t \leq t_2 \) the function \( x \rightarrow u(x,t) \) belongs to \( C^\alpha_\omega(\Omega) \) and
\[ \|u(x,t)\|_{L^p(t_1,t_2, C^\alpha_\omega(Q_\tau))} \]

We say that a function \( u(x,t) \) is a weak solution to (1.1), (1.2) in an open set \( Q_\tau \subset R_{t} \times R_{x} \) if whenever \( Q_\tau \times (t_1,t_2) \subset Q_\tau \) with \( \Omega^{t_1,t_2} \subset Q_\tau \subset Q \) and \( t_1 \leq t_2 \) then \( u \in L^p(t_1,t_2, C^\alpha_\omega(Q_\tau)) \) and
\[ \int_{Q_\tau} (w(x)|Du|^{p-2}Du) \cdot \varphi \, dx \, dt = 0 \]
for all nonnegative \( \varphi \in C^\alpha_\omega(Q_\tau^T) \).

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Using Theorem 2.1 we are able to establish sharp removability conditions for compact sets. We of cylinders introduced
\[
Q_{x,t}^*(r) = \left\{ (x,t) \in \mathbb{R}^{n+1} : |x| < r \right\}
\]
and a concave modulus of continuity \( \psi(\cdot) \). We let \( \psi : \mathbb{R} \to \mathbb{R} \) be a concave modulus of continuity, i.e., concave non-decreasing function such that \( \psi(1) = 1 \) and \( \psi(0) = \lim_{r \to 0} \psi(r) = 0 \). We also define Hausdorff measure as follows. We let for fixed \( 0 < \delta < \delta_0 \) and \( E \subset \mathbb{R}^{n+1} \), \( L(\delta, \psi(\cdot); E) = \left\{ Q_{x,t}^*(r) \right\} \) be a family of cylinders such that \( E \subset \bigcup Q_{x,t}^*(r) \) and 0 < \( r_i < \delta \) for \( i = 1, 2, \ldots \).

Let \( \bar{u} \) and \( u \) be a concave modulus of continuity \( \psi \). We let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a concave modulus of continuity, i.e., concave non-decreasing function such that \( \psi(1) = 1 \) and \( \psi(0) = \lim_{r \to 0} \psi(r) = 0 \). We also define Hausdorff measure as follows. We let for fixed \( 0 < \delta < \delta_0 \) and \( E \subset \mathbb{R}^{n+1} \), \( L(\delta, \psi(\cdot); E) = \left\{ Q_{x,t}^*(r) \right\} \) be a family of cylinders such that \( E \subset \bigcup Q_{x,t}^*(r) \) and 0 < \( r_i < \delta \) for \( i = 1, 2, \ldots \).

Using this notation we let
\[
H^{\psi}(E) = \lim_{\delta \to 0} \inf_{\delta < \delta_0} \left\{ \sum r_i^p \psi(\cdot) : E \subset \bigcup Q_{x,t}^*(r) \right\}
\]
where the indium is taken with respect to all possible coverings \( L(\delta, \psi(\cdot); E) \) of \( E \).

**Theorem 2.3**

Let \( Q_\beta \) be a cylindrical domain and let \( E \subset \mathbb{R}^{n+1} \) be closed set. Let \( u(x,t) \) be a weak solution to eqn. (2.1) in \( Q_\beta \) and \( E \subset \mathbb{R}^{n+1} \).

We define function
\[
\lambda(\tau) = \max\left\{ \frac{\lambda_{\inf}(\cdot)}{\psi(\cdot)} : (x, t) \in (\Omega \times (0, \infty)) \right\}
\]
where \( \lambda_{\inf}(\cdot) \) is a concave modulus of continuity \( \psi(\cdot) \).

By the uniqueness \( \bar{u} = u \) in \( \Omega \times (0, \infty) \) and hence \( \bar{u} \) is an extension of \( u \). Let
\[
R = \max\{1, \text{diam}\Omega, T^{1.3} \}.
\]
As clearly
\[
T \leq \psi(R)^{1.3} R \leq R
\]
Whenever \( R \geq 1 \). By maximum and minimum principle implies that
\[
osc_{\frac{\psi}{\psi}} \leq osc_{\frac{\psi}{\psi}} \leq \tau\left( \Omega, T, osc_{\frac{\psi}{\psi}} \right)
\]
We may assume that \( \bar{T} \subset \Omega \setminus \{ x \in \Omega : \Omega \setminus \Omega \setminus \Omega \setminus \Omega \} \), where \( \Omega \subset \Omega \) and \( \Omega > 0 \). We let \( R \) be a number subject to the restrictions
\[
R \leq \text{dist} (Q_\beta, \partial \Omega), \tau \geq R \max\{osc_{\psi}(\psi(R), s) \cdot R\}^{1.3} = R
\]
As \( \psi(1) = 1 \), we see that these conditions are satisfied if we take
\[
R \leq \text{dist} (\Omega \setminus \partial \Omega, \Omega \setminus \partial \Omega, \Omega) \max\left\{ T^{1.3}(\Omega, \Omega, osc_{\psi}(\psi(R), s) \cdot R) ^{1.3} \right\}
\]
Taking correspondingly \( \alpha \) it follows that \( Q_{x,t}^*(\alpha, \psi) \subset \Omega \), whenever \( \alpha \in Q_{x,t}^* \). Now we prove that the following holds whenever \( z \in \Omega \)
\[
osc_{\frac{\psi}{\psi}} u \leq osc_{\frac{\psi}{\psi}} \leq \frac{\psi}{\psi} \leq \frac{\psi}{\psi} \leq 2 \lambda \psi(r)
\]
This completes the proof of Theorem 2.1.

**Proof of theorem 2.2**

After extending \( u(x,t) \) as in the above we choose
\[
R = \text{dist}(Q_\beta, \partial \Omega) \text{ and define } \lambda = \max\left\{ \frac{T}{\psi(R)}, \lambda_{\inf}(\cdot), s, R \right\}
\]
We let \( Z = \Omega \times (0, \infty) \) then
\[
osc_{\frac{\psi}{\psi}} u \leq \lambda \psi(r) \text{ for every } r \in (0, R), \lambda \psi(r) \text{ for every } r \in (0, R).
\]
In the final
\[
\lambda = \max\left\{ 4 \lambda \psi(r) \right\}
\]
implies that
\[
osc_{\frac{\psi}{\psi}} u \leq \lambda \psi(r) \text{ for every } r \in (0, R).
\]
This completes the proof of Theorem 2.2.

**Proof of theorem 2.3**

Let \( u(x,t) \) weakly solve of eqn. (1.1) in \( Q_\beta \setminus \partial \Omega \) and assume that \( u(x,t) \in C^{0,1}(\Omega \setminus \partial \Omega) \) and \( H^{\psi}(E) = 0 \). Let \( E \subset \mathbb{R}^{n+1} \) be arbitrary space-time smooth cylinders. Our only need to prove the conclusion in \( Q_\beta \), since the one of being a weak solution is a local property. By the assumption
\[
u(x,t) \in C^{0,1}(\Omega \setminus \partial \Omega)
\]
there exists \( M > 0 \) such that
\[
osc_{\frac{\psi}{\psi}} u \leq M \text{ and } osc_{\frac{\psi}{\psi}} u \leq M \psi(r)
\]
If we using the existence result, then see that there exist a unique solution \( v(x,t) \) of problem
\[
0 \leq \psi R^{1.3} = 0
\]
instead argue as follows: since \( u(x,t) \leq \psi(x,t) \), we also have \( u(x,t) \leq a(x,t) \) on \( \Omega_0 \) and as \( u(x,t) \) solves eqn. (1.1) in \( Q_0 \), the comparision principle holds \( u(x,t) \leq a(x,t) \) in \( \Omega \). We thus conclude that \( v(x,t) \leq a(x,t) \) on \( Q'(r) \cup F \). Therefore \( v(x,t) = u(x,t) \) and consequently also eqn. (2.6) yields in \( Q'(r) \). This completes the proof that support of \( \mu \) is contained in \( F \cup E \).

Later using Theorem 2.1 and a covering argument we can conclude that there exists \( C \) depending only on \( n, p, v, L, \mu(\cdot) \), \( Q_0', Q_i' \), such that

\[
\text{osc}_{Q_0' \times [0,T]} u(x,t) \leq M \psi(t).
\]  

(2.7)

Whenever \( \text{osc}_{Q_0' \times [0,T]} u(x,t) \leq M \). Consider concentric cylinders \( \Omega_i = Q_i' \setminus E \). Let \( \tilde{Q}_i = Q_i' \setminus E \) be such 0 and \( \phi \equiv 1 \) on \( Q_i' \). Let \( k = \sup_{\Omega_i} \phi(x,t) \). Using eqn. (2.6) we have

\[
Q \tau_2 \int_{\tilde{Q}_i} \phi^p d\mu = 
\int_{\tilde{Q}_i} \left[ \phi^p + \int_{\tilde{Q}_i} \int_{\tilde{Q}_i} \frac{w(x)}{D_1} \left[ D_1 \psi \phi^p - D_1 \psi \phi \right] dx dt \right] \leq 
\int_{\tilde{Q}_i} \phi^p \int_{\tilde{Q}_i} \psi \phi^p dx dt + \int_{\tilde{Q}_i} \left( \phi^p \right) \psi dx dt \leq 
\int_{\tilde{Q}_i} \phi^p \psi dx \int_{\tilde{Q}_i} \phi^p dx dt
\]

(2.8)

For the nonnegative weak solution \( k - \psi(x,t) \) we see that

\[
\int_{\tilde{Q}_i} \left[ \psi^p \left( (k - \psi)^p + \int_{\tilde{Q}_i} \psi \bigg( \psi^p \bigg) \right) \right] dx dt \leq C \psi(r)
\]

and putting the estimates (2.8) we obtain that

\[
\mu(Q_i) \leq c \left[ \psi^{p - \tau} + \int_{\tilde{Q}_i} \rho^{p - \tau} \right] \left[ \psi^{p - \tau} + \int_{\tilde{Q}_i} \rho^{p - \tau} \right] \psi(r)
\]

(2.9)

Here we also used the estimate \( \psi(r) \leq t^{-\sigma} \tau \leq t^{-\sigma} \) for \( r \leq 1 \). Now we consider cylinder \( Q_i' \subset Q_i' \). We will prove that \( \mu(Q_i') = 0 \). We first note using eqn. (2.9) we have

\[
\mu(Q_i' \times [0,T]) \leq C \frac{r^{p - \tau}}{\psi(r)}
\]

(2.10)

Whenever \( Q_i' \subset Q_i' \). Since \( H^{p - \tau}(E) \) we obtain for \( \epsilon > 0 \) and \( \delta > 0 \) given (to be taken smaller that dist(\( \Gamma \cap Q_i', Q_i' \))/4), then there exists a countable family

\[
\left\{ Q_i'^{\epsilon} \right\} = \left\{ Q_i'^{\epsilon}(x,t) \right\}
\]

of cylinders with \( 0 < \tau_i < \delta_i = 1, 2, \ldots \), such that \( Q_i'^{\epsilon}(x,t) \) and

\[ E \cap Q_i'^{\epsilon} \subset \left[ Q_i' \psi^{(r)}(t) \right] \text{ and } X_t(r)(t) < \epsilon. \]

(2.11)

Later using eqn. (2.10) we is obtain

\[
\mu(F \cap (E \cap Q_i')) = \sum \mu(Q_i'^{\epsilon}(x,t)) \leq \sum \tau_i \psi(r) < C, \epsilon
\]

(2.12)

proving that \( \mu(F \cap (E \cap Q_i')) = 0 \). The fact that both \( Q_i' \) and \( Q_i' \) are arbitrary, we can conclude that \( \mu(Q_i') = 0 \). Thus \( v(x,t) \) is a solution in \( Q_i' \). Finally applying the above argument with \( u(x,t) \) replaced by \( -u(x,t) \) we deduce that there exist two solutions \( v_1(x,t) \) and \( v_2(x,t) \) i.e., eqn. (2.6) for \( v_1 \) equal to eqn. (2.6) for \( v_2 \). Such that \( v_1(x,t) \leq u(x,t) \leq v_2(x,t) \) and \( v_1(x,t) = v_2(x,t) \) on \( \Gamma'(r) \). It follows that \( v_1, v_2 u \). Theorem is proof.

References