Refined Estimates on Conjectures of Woods and Minkowski-I

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Abstract

Let \( \Lambda \) be a lattice in \( \mathbb{R}^n \) reduced in the sense of Korkine and Zolotarev having a basis of the form \((A_1, 0, \ldots, 0), (a_{21}, A_2, 0, \ldots, 0), \ldots, (a_{2n}, A_2, \ldots, A_n)\) where \( A_1, A_2, \ldots, A_n \) are all positive. A well known conjecture of Woods in Geometry of Numbers asserts that if \( A_1 A_2 \cdots A_{n-1} = 1 \) and \( A_i \leq A_{i+1} \) for each \( i \) then any closed sphere in \( \mathbb{R}^n \) of radius \( \sqrt{n/2} \) contains a point of \( \Lambda \). Woods’ Conjecture is known to be true for \( n \leq 9 \). In this paper we obtain estimates on the Conjecture of Woods for \( n=10; 11 \) and 12 improving the earlier best known results of Hans-Gill et al. These lead to an improvement, for these values of \( n \), to the estimates on the long standing classical conjecture of Minkowski on the product of \( n \) non-homogeneous linear forms.

MSC: 11H46; 11 - 04; 11J20; 11J37; 52C15.

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Introduction

Let \( L_1, a_{11} x_1 + \cdots + a_{1n} x_n; 1 \leq i \leq n \) be \( n \) real linear forms in \( n \) variables \( x_1; \ldots; x_n \) and having determinant \( \Delta = \det(a_{ij}) \neq 0 \). The following conjecture is attributed to H. Minkowski:

Conjecture I: For any given real numbers \( c_1; \ldots; c_n \), there exists integers \( x_1; \ldots; x_n \) such that

\[
|L_1(c_1) \cdots L_n(c_n)| = \frac{1}{2^n} |\Delta|
\]

Equality is necessary if and only if after a suitable unimodular transformation the linear forms \( L_i \) have the form \( 2c_i x_i \) for \( 1 \leq i \leq n \).

This result is known to be true for \( n=9 \). For a detailed history and the related results, Minkowski’s Conjecture is equivalent to saying that [1]

\[
M_s \leq \frac{1}{2^n} |\Delta|
\]

where \( M_s = M_\Lambda(\Delta) \) is given by

\[
M_s = \operatorname{Sup} \left\{ \operatorname{Inf} \left( L_i(u_1, \ldots, u_n) + c_i \right) \right\}
\]

Chebotarev proved the weaker inequality

\[
M_s \leq \frac{1}{2\sqrt{2}} |\Delta|
\]

Since then several authors have tried to improve upon this estimate. The bounds have been obtained in the form

\[
M_s \leq \frac{1}{\sqrt{v_n^2} |\Delta|}
\]

where \( V_n > 1 \). Clearly \( v_n \leq 2^{n/2} \) by considering the linear forms \( L_i = x_i \) and \( c_i = \frac{1}{2^n} \) for \( 1 \leq i \leq n \). During 1949-1986, many authors such as Davenport, Woods, Bombieri, Gruber, Skubenko, Andrijasjan, Lifin and Malyshiev obtained \( V_n \) for large \( n \).

\[
v_n = 4 - 2\sqrt{\sqrt{n/2} - 2^{-1/2}}
\]

for all \( n \geq 4 \). Since recently \( V_n = 2^{n/2} \) has been established by the authors [9], we study \( V_n > 1 \) for \( n \leq 33 \) in a series of three papers.

In this paper we obtain improved estimates on Minkowski’s Conjecture for \( n=10; 11 \) and 12. In next papers [10-12], we shall derive improved estimates on Minkowski’s Conjecture for \( n=13; 14; 15 \) and for \( 16 \leq n \leq 33 \) respectively [13-16]. For sake of comparison, we give results by our improved \( V_n \) in Table 1.

We shall follow the Remak-Davenport approach. For the sake of convenience of the reader we give some basic results of this approach. Minkowski’s Conjecture can be restated in the terminology of lattices as: Any lattice \( \Lambda \) of determinant \( d(\Lambda) \) in \( \mathbb{R}^n \) is a covering lattice for the set

\[
S = \{ x_1 x_2 \cdots x_n : |x_i| \leq \frac{d(\Lambda)}{2^n} \}
\]

The weaker result (1.3) is equivalent to saying that any lattice \( \Lambda \) of determinant \( d(\Lambda) \) in \( \mathbb{R}^n \) is a covering lattice for the set

\[
S = \{ x_1 x_2 \cdots x_n : |x_i| \leq \frac{d(\Lambda)}{V_n 2^n/2} \}
\]

Define the homogeneous minimum of \( \Lambda \) as

\[
m_h(\Lambda) = \inf \{ |x_1 x_2 \cdots x_n | : X = (x_1, x_2, \ldots, x_n) \in \Lambda, X = 0 \}
\]

Proposition 1. Suppose that Minkowski Conjecture has been proved for dimensions 1, 2, ..., \( n-1 \); then it holds for all lattices \( \Lambda \) in \( \mathbb{R}^n \) for which \( MH(\Lambda) = 0 \).

Proposition 2. If \( \Lambda \) is a lattice in \( \mathbb{R}^n \) for \( n \geq 3 \), then there exists an ellipsoid having \( n \) linearly independent points of \( \Lambda \) on its boundary and no point of \( \Lambda \) other than \( 0 \) in its interior.

It is well known that using these results, Minkowski’s Conjecture would follow from

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Conjecture II. If \( L \) is a lattice in \( \mathbb{R}^n \) of determinant 1 and there is a sphere \( |X| < R \) which contains no point of \( L \) other than \( O \) in its interior and has \( n \) linearly independent points of \( L \) on its boundary then \( L \) is a covering lattice for the closed sphere of radius \( \sqrt{n}/4 \). Equivalently, every closed sphere of radius \( \sqrt{n}/4 \) lying in \( \mathbb{R}^n \) contains a point of \( L \).

They formulated a conjecture from which Conjecture II follows immediately. To state Woods' conjecture, we need to introduce some terminology \([17,18]\).

Let \( L \) be a lattice in \( \mathbb{R}^n \). By the reduction theory of quadratic forms introduced by a cartesian co-ordinate system may be chosen in \( \mathbb{R}^n \) in such a way that \( L \) has a basis of the form \([19-22]\),

\[
(A_1; 0; \ldots; 0); (a_2;1;A_2; 0; \ldots; 0); \ldots; (an;1; an;2; \ldots; an;n-1,An);
\]

where \( A_1;A_2; \ldots;An \) are all positive and further for each \( i=1; 2; \ldots; n \) any two points of the lattice in \( \mathbb{R}^n-i+1 \) with basis 

\[
(A_i; 0; \ldots; 0); (a_{i+1};i;Ai+1; 0; \ldots; 0); \ldots; (an;i; an;i+1; \ldots; an;n-1;An)
\]

are at a distance atleast \( A_i \) apart. Such a basis of \( L \) is called a reduced basis \([23]\).

Conjecture III (Woods): If \( A_1A_2\ldots An=1 \) and \( \frac{1}{i}A_i \leq 1 \) for each \( i \), let \( 0 < l_i \leq A_i \) where \( l_i \) and \( m_i \) are real numbers. Then \( L \) is a covering lattice for the sphere \( |x| \leq \sqrt{n}/2 \) where \( W^n \) is defined inductively by 

\[
w_i = \max\{w_{i-1}^{-1/l_i} + l_i, w_{i-1}^{-1/m_i} + m_i\}
\]

Here we prove

**Theorem 1.** Let \( n=10; 11; 12 \). If \( d(L) = A_1; A_2; \ldots; A_n = 1 \) and \( A_i \leq A_i' \) for each \( i \) with \( 0 < l_i < A_i' \leq m_i \) where \( l_i \) and \( m_i \) are real numbers. Then \( L \) is a covering lattice for the sphere \(|X| < \sqrt{n}/4\) lying in \( \mathbb{R}^n \) containing a point of \( L \). Let \( w_1 = 10.3, w_2 = 11.62 \) and \( w_3 = 13.499927 \).

To deduce the results on the estimates of Minkowski's Conjecture we also need the following generalization of Proposition 1

**Proposition 4.** Suppose that we know

\[
M_j \leq \frac{1}{w_{j+1}^{2\gamma - 2}/|\Delta|} \quad \text{for} \, 1 \leq j \leq n - 1
\]
Let $v < \min V_{k_1} V_{k_2} \ldots V_{k_s}$, where the minimum is taken over all $(k_1, k_2, \ldots, k_s)$ such that $n=k_1+k_2+k_s$. Then for all lattices $L$ in $\mathbb{R}^n$ with homogeneous minimum $MH(<)=0$, the estimate $V_{k}$ holds for Minkowski's Conjecture.

Since by arithmetic-geometric inequality the sphere $\{x \in \mathbb{R}^n | \|x\|_2^2 \leq \frac{1}{2} \sum k_i w_i^2 \}$ is a subset of $\{x \in \mathbb{R}^n | \|x\|_2^2 \leq \frac{1}{2} \sum (w_i)^2 \}$ Propositions 2 and 4 immediately imply

**Theorem 2**: The values of $V_n$ for the estimates of Minkowski's Conjecture can be taken as $(\frac{2n}{e})^{n}$

For $10 \leq n \leq 33$ these values are listed in Table 1. In Section 2 we state some preliminary results and in Sections 3-5 we prove Theorem 1 for $n=10, 11$ and 12.

**Preliminary Results and Plan of the Proof**

Let $L$ be a lattice in $\mathbb{R}^n$ reduced in the sense of Korkine and Zolotare. Let $(S_n)$ denotes the critical determinant of the unit sphere $\Delta S_n$ with centre $O$ in $\mathbb{R}^n$.

\[ \Delta(S_n) = \inf \{d(L) : x \notin L \} \]

Let $\gamma_n$ be the Hermite's constant i.e. $\gamma_n$ is the smallest real number such that for any positive definite quadratic form $Q$ in $n$ variables of determinant $D$, there exist integers $u_1, u_2, \ldots, u_n$ not all zero satisfying

\[ Q(u_1, u_2, \ldots, u_n) \leq \gamma_n D^{n/2} \]

It is well known that $\gamma_2=\sqrt{2}$.

We state below some preliminary lemmas. Lemmas 1 and 2 are due to Woods [25], Lemma 3 is due to Korkine and Zolotare [21] and Lemma 4 is due to Pendavingh and Van Zwam [24]. In Lemma 5, the cases $n=2$ and 3 are classical results of Lagrange and Gauss; $n=4$ and 5 are due to Korkine and Zolotare [21] while $n=6; 7$ and 8 are due to Blichfeldt [3].

**Lemma 1.** If $2\Delta(S_{n+1}) A^*_n > d(L)$ then any closed sphere of radius

\[ R = A \left(1 - \{A^*_n \Delta(S_{n+1}) / d(L)\}^{1/2} \right) \]

in $\mathbb{R}^n$ contains a point of $L$.

**Lemma 2.** For a Fixed integer $i$ with $1 \leq i \leq n-1$ denote by $L_i$ the lattice in $\mathbb{R}^n$ with reduced basis

\[ (A_{i,0}, \ldots, 0, a_{i2}, \ldots, a_{i,0}, \ldots, 0, a_{i1}, a_{i,1}, \ldots, a_{i,1-1}, A_{i}) \]

and denote by $L_2$ the lattice in $\mathbb{R}^n$ with reduced basis

\[ (A_{i,0}; 0; 0; \ldots; 0; a_{i2}; \ldots; a_{i,0}; \ldots; 0; 0; \ldots; 0; a_{i1}; a_{i,1}; \ldots; a_{i,1-1}; A_{i}) \]

If any closed sphere in $\mathbb{R}^n$ of radius $r_1$ contains a point of $L_i$ and if any closed sphere in $\mathbb{R}^n$ of radius $r_2$ contains a point of $L_i$, then any closed sphere in $\mathbb{R}^n$ of radius $(r_1 + r_2)^{1/2}$ contains a point of $L_i$.

**Lemma 3.** For all relevant $i$,\n
\[ B_{i+1} \geq \frac{3}{4} B_i \text{ and } B_{i+2} \geq \frac{2}{3} B_i \quad (2.1) \]

**Lemma 4.** For all relevant $i$,\n
\[ B_{i+1} \geq (0.46873) B_i \quad (2.2) \]

Throughout the paper we shall denote $0.46873$ by $\varepsilon$.

**Lemma 5.** $\Delta(S_n) = \sqrt{\frac{\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \frac{1}{2} \sum k_i w_i^2}}}}}{2}}$

**Lemma 6.** For any integer $s$,\n
\[ B_{i+1} \leq \frac{1}{\gamma_2} \gamma_n \gamma_{n-1} \gamma_{n-2} \ldots \gamma_3 \gamma_2 \gamma_1 \]

This is Lemma 4 of Hans-Gill et al. [12].

**Lemma 7.**\n
\[ \{(8.5337)^{1/12} (\gamma_2)^{1/12} (\gamma_3)^{1/12} (\gamma_4)^{1/12} \ldots (\gamma_n)^{1/12} \} \leq \gamma_n \]

This is Lemma 6 of Hans-Gill et al. [14].

**Remark 1.** Let $\delta_n$ = the best centre density of lattice packings of unit spheres in $\mathbb{R}^n$; $\delta_n^*$ = the best centre density of packings of unit spheres in $\mathbb{R}^n$.

Then it is known that $\gamma_n = 4(\delta_n)^2 \leq 4(\delta_n^*)^2$.

$\delta_n^*$ and hence $\delta_n$ is known for $n \leq 8$. Also $\gamma_n$ =4 has been proved by Cohn and Kumar [6]. For $9 \leq n \leq 12$ using the bounds on $\delta_n$ given by Cohn and Elkies [5] and inequality (2.6) we find that $\gamma_n \geq 2.0266360$, $\gamma_{12} \geq 2.3933470$, $\gamma_{12} \leq 2.521787$.

We assume that Theorem 1 is false and derive a contradiction. Let $L$ be a lattice satisfying the hypothesis of the conjecture. Suppose that there exists a closed sphere of radius $\sqrt{w_n}/2$ in $\mathbb{R}^n$ that contains no point of $L$ in $\mathbb{R}^n$.

Since $B_k=A_k^2$ and $d(L)=1$, we have $B_1=A_1$.

We give some examples of inequalities that arise. Let $L_1$ be a lattice in $\mathbb{R}$ with basis $(A_1; 0; 0; 0; a_{i2}; A_1; a_{i1}; A_1)$ and $(A_1; 0; 0; a_{i1}; a_{i2}; A_1)$; and $L_2$ for $2 \leq i \leq n$ be lattices in $\mathbb{R}$ with basis $(A_i; 3)$. Applying Lemma 2 repeatedly and using Lemma 1 we see that if $2\Delta(S_n) A^*_n \geq A_1 A_2 A_4$ then any closed sphere of radius

\[ (A_1^2) \left(\frac{\Delta(S_n)}{A_1^2 A_2 A_3 A_4} + \frac{1}{4} + \frac{1}{4} A_2^2 + \ldots + \frac{1}{4} A_3^2 \right)^{1/2} \]

contains a point of $L$. By the initial hypothesis this radius exceeds $\sqrt{w_n}/2$. Since $\Delta(S_n) = 1/2\sqrt{2}$ and $B_i B_i \ldots B_i = 1$ this results in the conditional inequality if $B_k B_i B_k B_i \geq 2$ then

\[ 4B_i \geq B_1^2 B_2 B_3 B_4 + B_5 + B_6 + \ldots + B_8 > \sqrt{w_n} \quad (2.7) \]

We call this inequality (4; 1; 1; 1; 1; 1; 1; 1) since it corresponds to the ordered partition $(1; 1; 1; 1; 1; 1; 1; 1)$ of $n$ for the purpose of applying Lemma 2. Similarly the conditional inequality $(1; 1; 1; 1; 1; 1; 2; 1; i; 1)$ corresponding to the ordered partition $(1; 1; 1; 1; 1; 1; 1; 1; 1; 1)$ then

\[ B_i + B_2 + B_3 + B_4 + B_5 + B_6 + \ldots + B_8 > \sqrt{w_n} \quad (2.8) \]

Since $4B_i \geq B_1^2$ \quad (2.8) gives

\[ B_1 + B_2 + B_3 + B_4 + B_5 + \ldots + B_8 > \sqrt{w_n} \]

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One may remark here that the condition $2B_i \geq B_{i+1}$ is necessary only if we want to use inequality (2.8), but it is not necessary if we want to use the weaker inequality (2.9). This is so because if $2B_i < B_{i+1}$, using the partition (1; 1) in place of (2) for the relevant part, we get the upper bound $2B_i + B_{i+1}$ which is clearly less than $2B_{i+1}$. We shall call inequalities of type (2.9) as weak inequalities and denote by (1; 1; 1; 1; 2; 1; 1; 1) $A^w$.

If $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an ordered partition of $n$, then the conditional inequality arising from it, by using Lemmas 1 and 2, is also denoted by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. If the conditions in an inequality $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are satisfied then we say that $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ holds. Sometimes, instead of Lemma 2, we are able to use induction. The use of this is indicated by putting (*) on the corresponding part of the problem. For example, if for $n = 10$, $B_5$ is larger than each of $B_6, B_7, \ldots, B_{10}$, and if $\frac{B_i}{B_i B_{i+1}} > 2$, then the inequality (4; 6*) gives $4B_i - 1 + 6B_i B_{i+1} > 0$. In particular the inequality ((n-1); 1) always holds. This can be written as $w_{n-1}(B_i) = \frac{1}{(n-1)} + B_i > W_s$.

Also we have $B_i \geq 1$ because if $B_i < 1$, then $B_i \leq B_i < 1$ for each $i$ contradicting $B_{12} = B_{10} = 1$.

Using the upper bounds on $\epsilon$ and the inequality (2.5), we obtain $\epsilon^2 B_i - 2B_i + 2 + \epsilon B_i - B_i - B_i > 10.3$ and $\epsilon^2 B_i - B_i - B_i > 10.3$. Now the left side is a decreasing function of $B_i$. Hence we have $B_i < 1.8815$. Applying AM-GM inequality we get $\frac{B_i}{B_i B_{i+1}} \geq \epsilon B_i$, and $B_i < 1.8815$. Hence we have $B_i < 1.8815$.

Claim (ii) $B_i < 1.8815$

Suppose $B_i \geq 1.8815$ then using (3.1) and that $B_i \geq \epsilon B_i$ we find that $\frac{B_i}{B_i B_{i+1}} > 2$ and $\frac{B_i}{B_i B_{i+1}} > 2$. So the inequality (1; 4; 4; 1) holds, i.e. $B_i + 4B_{i+1} - 1 - B_i - B_i > 10.3$ and $B_i + 4B_{i+1} - 1 - B_i - B_i > 10.3$. Hence we have $B_i < 10.3$ for $1 < B_i < 2.263632$, a contradiction. Hence we must have $B_i < 1.8815$.

Claim (iii) $B_i < 1.6562$

Suppose $B_i \geq 1.6562$. From (3.1) we have $B_i + 4B_{i+1} - 1 - B_i - B_i > 10.3$ and $B_i + 4B_{i+1} - 1 - B_i - B_i > 10.3$. Hence we must have $B_i < 1.6562$.

Claim (iv) $B_i < 1.9378$

Suppose $B_i \leq 1.9378$. Using (3.1) and that $B_i < 1.6562$, we find that $B_i$ is larger than each of $B_2, B_3, \ldots, B_{12}$. So the inequality (1; 9,*; 1) holds. This gives $B_i + 9(B_i)^{1} > 10.3$ which is not true for $B_i < 1.9378$. So we must have $B_i > 1.9378$.

Claim (v) $B_i < 1.5485$

Suppose $B_i \geq 1.5485$. We proceed as in Claim (iii) and replace $B_i$ by $1.9378$ and $B_i$ by $\epsilon B_i$ to get that $\epsilon B_i - 2B_i + 2 + \epsilon B_i - B_i - B_i > 10.3$. One easily checks that $\epsilon B_i < 10.3$ for $1.5485 \leq B_i < 1.6562$ and $1.7046 < B_i < 1.8815$. Hence we have $B_i < 1.5485$.

Claim (vi) $B_i < 0.2087$

Suppose $B_i \geq 0.2087$. Using (3.1) and Claims (ii), (v) we have $B_i + 4B_{i+1} - 1 - B_i - B_i > 10.3$ and $B_i + 4B_{i+1} - 1 - B_i - B_i > 10.3$. Hence we must have $B_i < 0.2087$.
using (3.1) that $B_3$ is larger than each of $B_9 B_{10}$. Hence the inequality (4; 6; *') holds. This gives $\phi_5 = 4 B_1 + \frac{1}{2} \frac{B_i}{B_i B_j B_k} + 6 (B_i B_j B_k) \gamma^{\gamma/2} > 10.3$

Left side is an increasing function of $B_2 B_3 B_4$, decreasing function of $B_1$. So we can replace $B_3 B_4 B_5$ by $2.3793$, hence we have $B_2 < 1.766$.

Claim (vii) $B_i < 1.337$

Suppose $B_i \geq 1.337$ then using (3.1) we get

$$B_i^2 > 2$$

Applying AMGM to inequality (1,2,4,2,1) we have

$$B_i + 2B_i = \frac{2B_i}{B_i B_j B_k B_l} + 4B_i + 4B_i + 2 < 10.3$$

Since $B_i \geq 3/4$, $B_i \geq 1.337 B_i \geq B_3$ and $B_9 \geq 2B_5 B_4$ we find that left side is a decreasing function of $B_2$, $B_9$ and $B_5$. So we can replace $B_3$ by $1.7046$; $B_9$ by $\varepsilon B_9$ and $B_5$ by $\frac{2}{3} B_9$ to get

$$\phi_5 = B_1 + 4(1.7046) - \frac{20.740f_5}{B_9} (1 + 4 + \frac{2}{3} B_9) > 10.3$$

Now left side is a decreasing function of $B_4$, replacing $B_4$ by $1.337$, we find that $\phi_5 < 10.3$ for $1 < B_1 < 0.2187$ and $1 < B_3 < 1.5485$, a contradiction. Hence we have $B_i < 1.337$.

Claim (viii) $B_3 < 1.1492$

Suppose $B_3 \geq 1.1492$ Using (3.1), we get $B_3 B_9 B_5 < 0.5445$.

Therefore $B_i^2 > 2$ Also using Lemma 3 & 4; 2 $B_i \geq 2(\varepsilon B_i)$

$$1.0077 B_i^2 = 10.3$$

So the inequality (4; 4; 2) holds, i.e. 4

$$B_i + 2B_i = \frac{2B_i}{B_i B_j B_k B_l} + 4B_i + 4B_i + 2 < 10.3$$

Now left side is a decreasing function of $B_3$ and $B_5$. So we replace $B_3$ by $1.1492$ and $B_5$ by $\frac{1.1492}{x}$ and get that $\phi_5 (x, B_3) = \frac{4}{(1-(1.1492)^x)} B_3^{3/4} + 4(1 + \varepsilon)$

$$\frac{1.1492}{x} = 2 \frac{1.1492}{x}$$

and $\phi_5 (x, B_3) = \frac{4}{(1-(1.1492)^x)} B_3^{3/4} + 4(1 + \varepsilon)$

which is not true for $1 < 0.5445$ and $\frac{1}{4}(1-(1.1492)^x) < B_2 < 0.4398$ giving thereby a contradiction. Hence we must have $B_i < 1.1492$.

Claim (ix) $B_i < 1.766$.

Suppose $B_i \geq 1.766$ We have $B_3 B_4 B_5 < 2.3793$. So $B_i^2 > 2$ Also $B_i \geq B_3 > 0.8277$. Therefore $B_6$ is larger than each of $B_i B_2 B_4 B_5 B_7$ and $B_8 B_9 B_{10}$. Hence the inequality (1; 4; 5; *') holds. This gives $B_i + 2B_i = \frac{1}{2} \frac{B_i}{B_i B_j B_k B_l} + 6 (B_i B_j B_k B_l) \gamma^{\gamma/2} > 10.3$ Left side is an increasing function of $B_3 B_4 B_5$, a decreasing function of $B_2$ and an increasing function of $B_3$. One easily checks that this inequality is not true for $B_i < 2.0187$.

$$B_i \geq 1.766$$

and $B_3 B_4 B_5 < 2.3793$. Hence we have $B_i < 1.766$.

Final contradiction

As $2(B_2 + B_4 + B_9 + B_{10})\leq 2(1.766 + 1.337 + 0.9383 + 0.6597 + 0.4398)$<10.3,
$B_1 \leq \frac{B_1}{e^2} < 2.0068, B_2 \leq \frac{B_2}{e^2} < 2.13557 \quad (4.1)$

Claim (ii) B2>1.913
The inequality (2; 2; 2; 2; 2; 1) gives $2B_2 + 2B_3 + 2B_4 + 2B_6 + B_{11} > 11.62$. Using (4.1) we find that this inequality is not true for $B_2 \leq 1.913$ so we must have $B_2>1.913$.

Claim (iii) B3<1.761
Suppose $B_3 \geq 1.761$ then we have $\frac{B_1}{B_2B_3} > 2$ and $\frac{B_1}{B_2B_6B_{10}} > (\varepsilon B_3)^2$.
Applying AM-GM to the inequality (2,4,4,1) we get
\[4B_3 + \frac{2B_3}{B_1} \leq 4B_3 + 4B_2 + B_4 + \sqrt{4B_1B_2B_3B_4} < 11.62 \] One easily finds that it is not true for $B_3 \geq 1.913$, $B_2 \geq e^{B_3}$, $B_3 \geq 2.012$, $B_1 \leq 1.913 < B_2 < 2.13557$ and $1.761 < B_3 < 2.0068$ so we must have $B_3<1.761$.

Claim (iv) B1<2.2436
Suppose $B_1 \geq 2.2436$ As $B_1B_2 > 1.3557x1.761x1.5015 < 5.6468$, we have $\frac{B_1}{B_2B_3B_4} < 2$. Also $B_1 \geq e^{B_2}$, so B5 is larger than each of B7, B8, B11. Hence the inequality (1, 4; 6) holds. Proceeding as in Claim (viii) we find that this inequality is not true for $B_2 < 2.1669$, $B_4 > 1.36$, $B_2 > 1.9686$ and $B_2 > 1.9888$.

Claim (v) B4<1.4465 and B2>1.9686
Suppose $B_2 \geq 2.2436$ we have $B_2B_3B_4 > 1.3557x1.761x1.5015 < 5.6468$. Hence we have $B_2 > 2.012$. Therefore using Lemmas 3 & 4 we have
\[\frac{B_1}{B_2B_3B_4} > 2 \quad \text{and} \quad \frac{B_1}{B_2B_6B_{10}} > (\varepsilon B_3)^2 \]
So the inequality (1,2,4,4) holds. Applying AM-GM to inequality (1,2,4,4), we get $B_4 + 4B_2 + 2B_1 > 4B_3 + 4B_2 + B_4 + \sqrt{4B_1B_2B_3B_4} > 11.62$
A simple calculation shows that this is not true for $B_2 \geq 1.913$, $B_3 \geq 1.4465$, $B_4 \geq 1.4465$, $B_1 \geq 2.2436$ and $B_2 \leq 1.761$. Hence we have $B_4 < 1.4465$.

Further if $B_2 \leq 1.913$ then $2B_2 + 2B_3 + 2B_4 + 2B_6 + B_{11} < 11.62$. So the inequality (2; 2; 2; 2; 2; 2) gives a contradiction.

Claim (vi) B4<1.4265 and B2>1.9888
Suppose $B_2 \geq 1.9888$ we proceed as in Claim (v) and get a contradiction with improved bounds on $B_1$, $B_2$, and $B_4$.

Claim (vii) B1<2.012
Suppose $B_1 \leq 2.012$ we proceed as in Claim (v) and get a contradiction with improved bounds on B2 and B4.

Claim (viii) B2>1.9686
We have $B_2B_3B_4 > 1.3557x1.761x1.5015 < 5.6468$. So the inequality (2, 4; 6) holds. Proceeding as in Claim (vii) we find that this inequality is not true for $B_2 < 2.1669$, $B_4 > 1.36$, $B_2 > 1.9686$.

Claim (ix) B1<2.1669
Suppose $B_1 \geq 2.1669$ we proceed as in Claim (iv) and get a contradiction with improved bounds on $B_1$, $B_2$, and $B_4$.

Claim (x) B4>1.403 and B2>2.012
Suppose $B_2 \geq 2.012$ we proceed as in Claim (vii) and get a contradiction with improved bounds on B2 and B4.

Final Contradiction:
As now $B_3B_4B_5 \leq 1.761 < 1.403$, $1.3347 < 3.2977$, we have $\frac{B_1}{B_2B_3B_4} > 2$ for $B_2 > 2.012$. Also $B_2 \geq e^{B_3}$, $B_3 \geq 0.943$ for each of B7, B8, B11. Hence the inequality (1; 4; 6) holds. Proceeding as in Claim (v) we find that this inequality is not true for $B_1 < 2.1669$, $B_4 > 1.36$, $B_2 > 1.9686$ and $B_2 > 1.9888$ giving thereby a contradiction.

2018<11B1<2.1016019
Here $B_1 \geq B_1 > 2.018$ Therefore using Lemmas 3 & 4 we have
$B_1 = (B_1B_2B_3B_4B_5)^{\frac{1}{10}} \leq \frac{B_1}{3} \leq \frac{1}{3} B_6 \leq \frac{1}{3} B_6 \leq 0.8804$
and $B_1 \leq \frac{B_1}{e^2} < 0.3934$

Claim (ii) B7<0.678
Suppose $B_7 \geq 0.678$ Then $\frac{B_1}{B_2B_3B_4B_5} > 2$ so (6*; 4; 1) holds. This gives $\varphi(x) = 6x(x)^{1/3} + 4B_4 - \frac{1}{2} B_2B_3x + B_1 > 11.62$ where $x = B_3$.

Therefore $\varphi(x) > \varphi(\frac{1}{3} B_1) + 2 B_4 \left(\frac{2}{B_1} B_1\right)^{1/3}$ which is less than 11.62 for $0.768 < B_1 < 0.8084$.

Claim (iii) B1<1.795
Suppose $B_1 \geq 1.795$ then $\frac{B_1}{B_2B_3B_4B_5} > 2$ and $\frac{B_1}{B_2B_3B_4B_5} > 2$. Applying AM-GM to the inequality (1,4,4,11) we get $B_1 + 4B_2 + 4B_4 + B_5 > \sqrt{4B_1B_2B_3B_4B_5} > 11.62$.

We proceed as in Claim (ii) and get a contradiction with improved bounds on $B_1$, $B_2$, and $B_4$.

Claim (iv) B1<1.403 and B2>2.012
Suppose $B_2 \geq 2.012$ we proceed as in Claim (vii) and get a contradiction with improved bounds on B2 and B4.

Final Contradiction:
As now $B_3B_4B_5 < 1.761 < 1.403$, $1.3347 < 3.2977$, we have $\frac{B_1}{B_2B_3B_4B_5} > 2$ for $B_2 > 2.012$. Also $B_2 \geq e^{B_3}$, $B_3 \geq 0.943$ for each of B7, B8, B11. Hence the inequality (1; 4; 6) holds. Proceeding as in Claim (v) we find that this inequality is not true for $B_1 < 2.1669$, $B_4 > 1.36$, $B_2 > 1.9686$ and $B_2 > 1.9888$ giving thereby a contradiction.
Now $\phi_2(B_2) > 0$ so $\phi_2(B_2) < \max\{\phi_2(2.018), \phi_2(2.1016019)\}$ which can be verified to be at most 11.62 for $\varepsilon(1.795) < B_2 < 0.4402$ and 2.018<B_2<2.939477, giving thereby a contradiction.

Claim (iv) $B_5 < 0.98392$

Suppose $B_5 \geq 0.98392$. We have $\frac{B_1}{B_2 B_3 B_4} > 2$ and $\frac{B_1}{B_2 B_3 B_4} > 2$. Also $2B_5 \geq 2B_5 < B_3$. Applying AM-GM to the inequality (4; 2; 1) we get $4B_1 + 4B_2 + 4B_3 - 2B_5B_1 - \sqrt{B_5^2 B_1 B_2 B_3} > 11.62$ One can easily check that left side is a decreasing function of $B_5$ and $B_3$, so we can replace $B_3$ by $\varepsilon B_5$ and $B_1$ by $B_1$ to get $\phi_5 = 4B_1 + 4(1+\varepsilon)B_2 - 2(1+\varepsilon)B_5 - \sqrt{B_5^2 B_1 B_2 B_3} + 11.62$ Now the left side is a decreasing function of $B_5$, so replacing $B_5$ by $0.98392$ we see that $\phi_5 < 11.62$ for $0.98392 < B_5 < 0.4402$ and 2.018<B_2<2.939477, a contradiction.

Final Contradiction:

As in Claim (iv), we have $\frac{B_1}{B_2 B_3 B_4} > 2$. Also $B_5 \geq 0.98392$ each of $B_5 B_3 B_2 B_1$. Therefore the inequality (4; 6; 1) holds, i.e. $\phi_6 = 4B_1 + 4B_2 + 4B_3 - 2B_5B_2 - \sqrt{B_5^2 B_2 B_3 B_4} + 11.62$ Left side is an increasing function of $B_2 B_3$ and $B_4$ and decreasing function of $B_5$. Using $B_5 < 0.98392$, we have $B_5 \leq \frac{3}{2} B_2 < 1.47588$ and $B_4 \leq \frac{4}{3} B_4 < 1.311894$ One easily checks that $\phi_6 < 11.62$ for $B_5 B_3 B_2 B_1 < 1.795 \times 1.47588 \times 1.311894$, $B_1 < 2.939477$ and $B_2 \geq 2.018$. Hence we have a contradiction.

Proof of Theorem 1 for n=12

Here we have $w_{12}=13, B_5 \leq \gamma_9 < 2.5217871$ Using (2.5), we have $w_{12}=0.3376 < B_{12} < 2.2254706 = m_{12}$, and using (2.3) we have $B_2 B_{11} \leq \frac{1}{x_1}$ i.e. $B_2 \leq \frac{\gamma_{11}}{x_2} < 2.2254706$.

The inequality (11; 1) gives 11.62(B_5) + 11.62 > B_12 > 13. But this is not true for 0.4165 < B_5 < 2.17 So we must have either B_12 < 0.4165 or B_12 > 2.17.

0.3376 < B_{12} < 0.4165

Claim (i) $B_{11} < 0.459$

Suppose $B_{11} \geq 0.459$ then $B_5 \geq \frac{3}{4} B_2 > 0.34425$ and 2B_{11} > B_{12}, so (10; 2) holds, i.e. $\phi_5 = 4B_1 + 4B_2 + 4B_3 - 2B_5B_2 - \sqrt{B_5^2 B_2 B_3 B_4} > 13$ Left side is a decreasing function of $B_1$, so we can replace $B_1$ by $B_1$ to find that $\phi_5 < 13$ for 0.34425 < B_{12} < 0.4165, a contradiction. Hence we have $B_{11} < 0.459$.

Claim (ii) $B_{10} \leq 0.5432$

Suppose $B_{10} \geq 0.5432$ From Lemma 3, $B_5 B_3 \geq \frac{1}{2} B_3$ and $B_{10} \leq \frac{3}{2} B_{12}$. Therefore $\frac{B_1}{B_2 B_3 B_4} \leq \frac{1}{2} (0.5432) \leq \frac{1}{2} B_2 B_3$. So $B_5 B_3 B_2 B_1$. the inequality (9; 3) holds, i.e. $9 \left( \frac{1}{B_2 B_3 B_4} \right)^2 + 4B_1 - \frac{B_5}{B_2 B_3} > 13$ One easily checks that it is not true noting that left side is a decreasing function of $B_1$. Hence we must have $B_{10} < 0.5432$.

Claim (iii) $B_9 < 0.6655$

Suppose $B_9 \geq 0.6655$ then $\frac{B_1}{B_2 B_3 B_4 B_5} > 2$. So the inequality (8; 4) holds. This gives $\phi_8(x) = 4B_1 + 2B_3 > 13$ where $x = B_2 B_3$. The function $\phi_8(x)$ has its maximum value at $x = \left( \frac{1}{B_2} \right)^2$ so $\phi_8(x) < \phi_8 \left( \frac{1}{B_2} \right)^2 < 13$ for 0.6655 < B_5 < 1.795. Applying AM-GM to the inequality (1, 2, 4, 4, 1) we get $\phi_4 = B_1 + 4B_2 + 2B_3 > 2B_1 + 4B_2 + B_3 - \sqrt{B_5 B_2 B_3 B_4 B_5} > 13$. We find that left side is a decreasing function of $B_2$ and $B_3$. So we
can replace B2 by 2.0299, B8 by "B4 and B12 by "B5. Then it turns
a decreasing function of "B5, so can replace B4 by 1.646 to find that
\( \phi_{1,13} < 1.646 \) for each of B1, B2, B12, and B3:19517, a contradiction. Hence we have B1<1.646.

**Claim (viii) B1<2.4273**

Suppose B1 ≥ 2.4273. Considering following two cases:

Case (i) B1>B3

Here B1>each of B2, B11, as B1 > 1.137 > each of B7,..., B12.
Also B2B9B12<2.2254706×1.9517×1.646<7.15. So \( \frac{B_2}{B_1 B_3} > 2 \) Hence the inequality (4, 8") holds. This gives 4B2 = \( \frac{B_1}{B_3} + 9(B_2 B_4 B_5) \)

left side is an increasing function of B4 B5 B6, and decreasing function of B2. So we can replace B2 B3 B4 by 7,5.11 and B1 by 2.4273 to get a contradiction.

Case (ii) B1 ≤ B3

Using (5.1) we have B1 < B3<1.1589 and so B3 < 4B3 < 1.5452

Therefore \( \frac{B_1}{B_3 B_4} > 2 \) as B2>2.0299 and B1>1.9517. Also from Claim (iv), B1>1.9517>each of B2,..., B12. Hence the inequality (1; 4; 7") holds. This gives B4 + 4B2 = \( \frac{B_1}{B_3} + 7(B_2 B_3 B_4 + B_5 B_6 B_7)^{1/3} > 13 \); left side is an increasing function of B2 B3 B4 and B1, and a decreasing function of B6. One can check that inequality is not true for B1 B2 B3<1.9517×1.5452×1.1589, B1,B2<2.512781 and for B2>2.0299. Hence we must have B1<2.4273:

**Claim (ix) B5<1.396**

Suppose B5 ≥ 1.396. From (5.1), B2B3B4<0.925 and B5B6B7<0.104, so we have \( \frac{B_1}{B_3 B_4} > 2 \) and \( \frac{B_1}{B_3 B_4} > 1 \). Applying AMGM to the inequality (1; 2; 1; 1; 4) we get B2 + 4B4 = 2B1 + 4B2 + 4B2 + 4B2 + 4B2 + 4B2 + 4B2

\( \left( B_2 + B_3 + B_4 + B_5 + B_6 + B_7 \right)^{1/3} > 13 \) We find that left side is a decreasing function of B2 and B9. So we replace B2 by 2.0299 and B1 by B1. Now it becomes a decreasing function of B1 and an increasing function of B1, so replacing B1 by 1.546 and B2 by 2.4273, we find that above inequality is not true for 1.522<B1<1.9517 and 1.426<B2<1.646, giving thereby a contradiction. Hence we must have B1<1.396:

**Claim (x) B3>1.7855**

Suppose B3 ≤ 1.7855. We have B4>1.426>each of B5B6,..., B2, hence the inequality (1; 2; 9") holds. It gives \( \phi_{1,13} = B_2 + 4B_4 = 2B_1 + \frac{1}{B_1 B_3} + 9B_2 B_5 B_6 B_7 + 13 \) It is easy to check that left side of above inequality is a decreasing function of B2 and an increasing function of B1 and B3. So replacing B2 by 2.4273, B1 by 1.7855 and B3 by 2.0299 we get -15.13; a contradiction. Hence we have B3>1.7855.

**Claim (xi) B2 > 2.0733**

Suppose B2 ≤ 2.0733. We have B3>1.7855>each of B5B6,..., B2, hence the inequality (2; 10") holds. It gives \( \phi_{1,13} = 4B_2 + 2B_1 + 10.3B_2 B_3 B_4 B_5 B_6 B_7 + 13 \) The left side is a decreasing function of B1 and an increasing function of B2, so replacing B1 by 2.17 and B2 by 2.0733 we get \( \phi_{1,13} < 1 \) a contradiction.

**Claim (xii) B7<0.92 and B5<1.38**

Suppose B7 ≥ 0.92. Here we have B7 B6<2.67 and B7 B8B9<0.295, so \( \frac{B_2}{B_3 B_4} > 2 \) and \( \frac{B_1}{B_3 B_4} > 2 \). Also \( \frac{B_1}{B_3 B_4} > 2 \); \( \frac{B_1}{B_3 B_4} > 2 \) Also \( \frac{B_1}{B_3 B_4} > 2 \). Applying AM-GM to the inequality (2,4,4,2) we get

\( \phi_1 = 4B_2 + 2B_1 + 10.3B_2 B_3 B_4 B_5 B_6 B_7 + 13 \) We find that left side is a decreasing function of B1 and B7. So we can replace B1 by 2.17 and B7 by B1. Then left side becomes a decreasing function of B1, and an increasing function of B7, so we can replace B1 by 2.646 and B7 by 2.2254706 to see that \( \phi_{1,13} < 1.7855 < B_1<1.9517 \) and 0.3376<B5<0.4156, a contradiction. Hence B7<0.92. Further B5<1.38.

**Claim (xiii) B6<1.097**

Suppose B6 ≥ 1.097. Here we have B3B4B5<4.44 and B7B8B9<0.5, so \( \frac{B_1}{B_3 B_4} > 2 \) and \( \frac{B_1}{B_3 B_4} > 2 \). Also \( \frac{B_1}{B_3 B_4} > 2 \). So we can replace B1 by B1 by \( \phi_{1,13} < 1.7855 < 2.17 < B_7 < 2.4273 < 2.0733 < B_1 < 2.2254706 \). Hence we must have B6<1.097.

**Claim (xiv) B7>B5**

First suppose B5 ≤ B7, then B5B6<1.646×1.9792×1.981 and \( \frac{B_1}{B_3 B_4} > 2 \) Also B5 ≥ B7 > 0.83 if each B1,..., B12 is a decreasing function of B7 and B13. Hence the inequality (2; 2; 6") holds, i.e., \( \phi_1 = 4B_2 + 2B_1 + 10.3B_2 B_3 B_4 B_5 B_6 B_7 + 13 \) Now the left side is a decreasing function of B1 and B7, and the right side is an increasing function of B1, B2, B3, B4, and B5. But one can check that this inequality is not true for B1<2.17, B7<1.7855, B5<2.2254706 and B6B12<1.981, giving thereby a contradiction. Further suppose \( \frac{B_1}{B_3 B_4} > 2 \) then as B5>B6<1.097<1.097<1.7855, B12, the inequality (4; 8") holds. Now working as in Claim (i) of Claim (vii) we get a contradiction for B1>2.17 and B3B4B5<2.2254706×1.9793×7.14934.

**Claim (xv) B3>1.9 and B1<2.4056**

Suppose B3 ≥ 1.9, then for B1<2.2254706×1.9793×7.14934, B1<2.4056. Also \( \frac{B_1}{B_3 B_4} > 2 \) and \( \frac{B_1}{B_3 B_4} > 2 \). Also B7 ≥ B5 if each B7,..., B12 is a decreasing function of B1, B7, B8, B9, B10, and B11. Hence the inequality (2; 4; 6") holds. Now working as in Claim (xiv) we get a contradiction for B1>2.17, B7<2.2254706, B1<1.9 and B7B8B9<2.4056. So B1<1.9. Further if B1 ≥ 2.4056, then \( \frac{B_1}{B_3 B_4} > 2 \) contradicting Claim (xv).

**Claim (xvi) B6<1.58 and B7<2.373**

Suppose B6 ≥ 1.58 then for B5B6<1.38×1.092×1.393, B7B8B9<2.4056, B5<1.393. Also B5 ≥ B7≥ B6>0.74 each of B1,..., B12. Hence the inequality (1; 2; 4; 5")
holds, i.e. $-19 = B_1 + 4B_2 - \frac{2B_1^3}{B_1^2 + 4B_2 - \frac{1}{2B_1} + \frac{B_1}{2B_1} + \frac{5}{B_1}} > 13$

Left side is a decreasing function of $B_2$ and $B_4$.

So we replace $B_2$ by 2.0733 and $B_4$ by 1.58. Then it becomes an increasing function of $B_1$, $B_3$ and $B_5$. So we replace $B_1$ by 2.4056, $B_3$ by 1.9 and $B_5B_6B_7$ by 1.393 to find that $-19<13$, a contradiction. Further if $B_1 \geq 2.373$, then $\frac{B_1}{B_2B_3} > 2$ contradicting Claim (xiv).

Finally, if $B_1 \geq 2.373$, then $\frac{B_1}{B_2B_3} > 2$ contradicting Claim (xiv).

Final Contradiction:

We have $B_1B_2B_3B_4B_11B_12 < 1.899$. Hence the inequality (1; 4; 7*) holds. Now we get contradiction working as in Case (ii) of Claim (viii).

5.2 2:17; B12<2:2254706

Here $B_1 \geq B_2$ > 2.17 Using Lemma 3 and 4, we have

\[
\left( B_1B_2B_3B_4 \right)^{1/6} + B_11 + B_12 > 13
\]

Now the left side is an increasing function of $B_11$ and $B_12$, so replacing $B_11$ by $B_12$ by 2.196 and $B_1$ by 2.17 to arrive at a contradiction. Hence we must have $B_11 \leq 0.445$.

Using Lemmas 3 and 4 we have:

\[
B_2 < \frac{4}{3}B_10 < 0.594, B_2 \geq \frac{3}{2}B_10 < 0.67, B_1 \leq 0.85
\]

\[
B_11 < \frac{3}{2}B_10 < 0.9494, B_11 < \frac{4}{3}B_10 \leq 1.266, B_11 \leq \frac{3}{2}B_10 < 1.4242
\]

\[
B_2 < \frac{2B_10}{\varepsilon} < 1.899, B_11 < \frac{B_11}{\varepsilon} < 2.0255
\]

Claim (iv) B3<_1:62

Suppose $B_1 \geq 1.62$ From (5.2), we have $B_4B5B6<_1:712$ and $B8B9B10<_0.178$, so $\frac{B_1}{B_2B_3} > 2$ and $\frac{B_1}{B_2B_3B_4} > 2$

Applying AM-GM to the inequality (2,4,4,1,1) we get

\[
\phi = 4B_11 - \frac{2B_10}{\varepsilon} + 4B_11 - \frac{2B_10}{\varepsilon} + B_11 + B_10 > 13
\]

We find that left side is a decreasing function of $B_1$, $B_3$ and $B_5$. So we replace $B_1$ by $\varepsilon B_1$, $B_3$ by $\varepsilon B_3$, and $B_5B_6B_7$ by $\varepsilon B_5B_6B_7$. Then it becomes a decreasing function of $B_1$, so replacing $B_1$ by 1.62 we find that $\phi > 21 < 13$, a contradiction. Hence we must have $B_1 < 1.62$.

Claim (v) B12>2:196

Suppose $B_2 \leq 2.196$ From (5.2), we have $B_2B_3B_4 < 4:674$ and $\frac{B_1}{B_2B_3B_4} > 2$

Applying AM-GM to the inequality (2,4,4,1,1) we get

\[
\phi = 4B_11 - \frac{2B_10}{\varepsilon} + 4B_11 - \frac{2B_10}{\varepsilon} + B_11 + B_10 > 13
\]

So we replace $B_1$ by 2.4056, $B_3$ by 1.9 and $B_5B_6B_7$ by 1.393 to find that $-19 < 13$, a contradiction. Further if $B_1 \geq 2.373$, then $\frac{B_1}{B_2B_3} > 2$ contradicting Claim (xiv).

Final Contradiction:

Now we have $B_1B_2B_3B_4B_11B_12 < 1.899$. Hence the inequality (1; 4; 7*) holds. Now we get contradiction working as in Case (ii) of Claim (viii).

5.2 2:17; B12<2:2254706

Here $B_1 \geq B_2$ > 2.17 Using Lemma 3 and 4, we have

\[
\left( B_1B_2B_3B_4 \right)^{1/6} + B_11 + B_12 > 13
\]

Now the left side is an increasing function of $B_11$ and $B_12$, so replacing $B_11$ by $B_12$ by 2.196 and $B_1$ by 2.17 to arrive at a contradiction. Hence we must have $B_11 < 0.445$.

Suppose $B_1 < 0.445$ then $2B_1B_2B_3B_4B_11B_12$ and the inequality (9*; 2; 1) holds, i.e. $\frac{B_2}{B_1B_2B_3B_4B_11B_12} > 0.35916$ and $B_11B_12 > 1.8223$, $B_12$ by 2.2254706, the left side is increasing function of $B_11$ and a decreasing function of $B_12$, so replacing $B_11$ by 2.2254706 and $B_12$ by 0.445 we find that $\phi < 21 < 13$, for $3(4.0445)<B_11<0.4307$, a contradiction.

Therefore we have $B_1 < 0.445$.

References


