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Recurrence Relation When Solving 2nd Order Homogeneous Linear ODEs by Frobenius Method

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Abstract

The Frobenius method, also known as the Extended Power Series method comes into play when solving second-order homogeneous linear ODEs having variable coefficients, about a singular point. The solution is expressed in terms of infinite power series. This straightforward article is merely aimed to lucidly arrive at the recurrence relation/formula by taking proper heed of summation limits and simply manipulating them, which is probably absent in almost all research articles and books. The methodology discussed is followed by a few conspicuous and relevant observations.

Keywords: ODE • Recurrence relation • Singular point • Power series • Homogeneous • Frobenius method

Introduction

The 2nd order homogeneous linear Ordinary Differential Equations (ODEs) find their applications in several fields of discipline, such as thermodynamics, theory of vibrations, electrical engineering, medicine, etc. [1]. The Frobenius method becomes handy when the ODE is quite complex and consists of variable coefficients. Second-order linear ODEs such as Legendre's differential equation and Bessel's differential equation are solved using the Frobenius method, wherein a formal power series solution is obtained.

The Frobenius method is a generalization of the usual power series method. The power series method is applicable when the functions P (x) and Q (x) in the equivalent normalized ODE are analytic x. Therefore, singularity ceases and x0 becomes the ordinary point of the ODE. If one wants to seek an infinite power series solution about point x_0 valid in some deleted interval $0 < |x - x_0| < R$, R being the radius of convergence (R > 0), and x_0 as the regular singular point, then the solution y (x), according to stated theorem [2], takes the form,

$$y(x) = |x - x_0|^m \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{k=0}^{\infty} a_k (x - x_0)^{m+k}$$

where, ak ≠ 0, m = real or complex definite constant, which can be determined from the indicial equation obtained by equating the lowest power of x in the substituted equation to 0.

This short article is centered around how the lower summation limit and exponent of x alter when the assumed solution, y(x) taken about any regular singular point, x_0 is subjected to mathematical operations such as differentiation, multiplication, etc., and how the recurrence relation is obtained by straightforward manipulation of those limits. This is ignored in, principally textbooks [2-12] and open works of literature [13-24]. The recurrence formula, thus obtained is employed to figure out the coefficients (a_k) of the powers of x in the final series solution. The methodology is discussed in the subsequent section, followed by useful observations. Few examples have been taken from renowned books for illustration purposes. It is worthwhile to note that not the complete solution is presented, rather important steps up to the recurrence relation are jotted down.

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Methodology

Let the solution y (x) about $x_0 = 0$ be

$$y(x) = \sum_{k=0}^{\infty} a_k (x)^{m+k}$$

Now, if y (x) is differentiated w.r.t. x, the lower limit of summation will increase by the number of times y (x) is differentiated. Also, the power of x will reduce by the number of times y (x) is differentiated. If y (x) differentiated twice, then

$$y''(x) = \sum_{k=2}^{\infty} (m+k)(m+k-1) a_k (x)^{m+k-2}$$

When *y* (*x*) is multiplied by x^n, where $n \in \mathbf{N}$, the lower limit of summation will decrease by the amount *n*, and the power of *x* will increase by *n*. Suppose *y* (*x*) is multiplied with x^2, then

$$x^2 y(x) = \sum_{k=-2}^{\infty} a_k (x)^{m+k+2}$$

The lower limit of summation reached a value less than 0 i.e., k < 0. In fact, it's not a matter of concern, this is done just to arrive at the recurrence relation, which will be apparent in the forthcoming examples. The following Table 1 summarizes the same. The ODEs normally involve differentiation and multiplication, thus, integration and division can be neglected. Some examples to illustrate the process follows. The solution, y (x) is assumed about $x_0 = 0$ which will be the regular singular point in these examples.

Problem #1 – 3 x y" + 2 y' + y = 0 (1)

Solution. Let

$$y(x) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Here, *P* (*x*) and *Q* (*x*) are not analytic at *x* = 0; but *x P* (*x*) and x^2 *Q* (*x*) are analytic. Thus, $x_0 = 0$ is the regular singular point, and there exist two nontrivial linearly independent solutions of the given ODE. Depending upon the nature of the roots (m_1 and m_2) of the indicial equation associated with x_0, and the difference between the two roots ($m_1 - m_2$), the two linearly independent solutions can be found.

Now,

$$y'(x)=\sum_{k=1}^{\infty}(m+k)a_k(x)^{m+k-1}$$

$$y''(x)=\sum_{k=2}^{\infty}(m+k)(m+k-1)a_k(x)^{m+k-2}$$

Plugging y, y', and y'' into (1), we obtain

$$3x\sum_{k=2}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}+2\sum_{k=1}^{\infty}(m+k)a_k x^{m+k-1}+\sum_{k=0}^{\infty}a_k x^{m+k}=0$$

$$\sum_{k=1}^{\infty}3(m+k)(m+k-1)a_k x^{m+k-1}+\sum_{k=1}^{\infty}2(m+k)a_k x^{m+k-1}+\sum_{k=0}^{\infty}a_k x^{m+k}=0$$

The lower limit of summation and the power of x in the 1st summation above is changed as per Table 1. Upon combining the coefficients of x^{m+k-1} (lowest power) in the above equation and setting $k = 0$, one can obtain the indicial equation, which can be solved for roots, m_1, and m_2. In the 3rd summation, we set $k = k - 1$; in order to make the exponent of x the same in all the terms.

$$\sum_{k=1}^{\infty}3(m+k)(m+k-1)a_k x^{m+k-1}+\sum_{k=1}^{\infty}2(m+k)a_k x^{m+k-1}+\sum_{k=1}^{\infty}a_{k-1} x^{m+k-1}=0$$

Now, all the terms in the above equation have the same summation limits and exponent of x. Thus, we can rewrite it as

$$\sum_{k=1}^{\infty}\left\{3(m+k)(m+k-1)a_k+2(m+k)a_k+a_{k-1}\right\}x^{m+k-1}=0$$

The equation above is much simplified and we can obtain the recurrence formula by equating the coefficient of x^{m+k-1} to 0. Therefore,

$$3(m+k)(m+k-1)a_k+2(m+k)a_k+a_{k-1}=0$$

$$a_k=\frac{-a_{k-1}}{(m+k)(3m+3k-1)};k\ge 1\,\forall\, k\in\mathbf{N}$$

$$\text{When k}=1, a_1=\frac{-a_0}{(m+1)(3m+2)}$$

$$\text{When k}=2,\ a_2=\frac{-a_1}{(m+2)(3m+5)}=\frac{a_0}{(m+1)(m+2)(3m+2)(3m+5)}$$

and so on.

One can proceed by substituting the m values so found, and obtaining the final solution, y (x) in terms of two linearly independent solutions, y_1 (x) and y_2 (x), based upon the roots of the indicial equation.

Problem #2 – $2x^2y'' + x y' + (x^2 - 3) y = 0$ (2)

Solution. Here, P (x) and Q (x) are not analytic at $x = 0$; but $x\,P$ (x) and $x^2\,Q$ (x) are analytic. Thus, $x_0 = 0$ is the regular singular point, and there exist two nontrivial linearly independent solutions of the given ODE.

Replacing y, y', and y'' found in the previous problem into (2), we obtain

$$2x^2\sum_{k=2}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}+x\sum_{k=1}^{\infty}(m+k)a_k x^{m+k-1}+\left(x^2-3\right)\sum_{k=0}^{\infty}a_k x^{m+k}=0$$

$$\sum_{k=0}^{\infty}2(m+k)(m+k-1)a_k x^{m+k}+\sum_{k=0}^{\infty}(m+k)a_k x^{m+k}+\sum_{k=-2}^{\infty}a_k x^{m+k+2}-\sum_{k=0}^{\infty}3a_k x^{m+k}=0$$

The lower limits of summations and the powers of x in all the summations above are changed as per Table 1. Upon combining the coefficients of x^{m+k} (lowest power) in the above equation and setting k = 0, one can obtain the indicial equation, which can be solved for roots, m_1, and m_2. In the 3rd summation, we set $k = k - 2$; in order to make the exponent of x the same in all the terms.

$$\sum_{k=0}^{\infty}2(m+k)(m+k-1)a_k x^{m+k}+\sum_{k=0}^{\infty}(m+k)a_k x^{m+k}+\sum_{k=0}^{\infty}a_{k-2} x^{m+k}-\sum_{k=0}^{\infty}3a_k x^{m+k}=0$$

Now, all the terms in the above equation have the same summation limits and exponent of x. Thus, we can rewrite it as

$$\sum_{k=0}^{\infty}\left\{2(m+k)(m+k-1)a_k+(m+k)a_k+a_{k-2}-3a_k\right\}x^{m+k}=0$$

We can obtain the recurrence formula by equating the coefficient of x^{m+k} to 0. Therefore,

$$2(m+k)(m+k-1)a_k+(m+k)a_k+a_{k-2}-3a_k=0$$

$$a_k=\frac{-a_{k-2}}{(m+k)(2m+2k-1)-3};k\ge 2\,\forall\, k\in\mathbf{N}$$

$$\text{When k}=2, a_2=\frac{-a_0}{(m+2)(2m+3)-3}$$

$$\text{When k}=3,\ a_3=\frac{-a_1}{(m+3)(2m+5)-3}$$

$$\text{When k}=4,\ a_4=\frac{-a_2}{(m+4)(2m+7)-3}=\frac{a_0}{\{(m+4)(2m+7)-3\}\{(m+2)(2m+3)-3\}}$$

and so on. One can proceed by substituting the m values so found, and obtaining the final solution, y (x) in terms of two linearly independent solutions, y_1 (x) and y_2 (x), based upon the roots of the indicial equation. Now, the following problem, in fact, is a special case with $m = 0$, i.e. no singular point exists, and the problem can be solved by the ordinary power series method. Hence, the Frobenius method generalizes the power series method.

Problem #3 – $y'' + x\,y' + (x^2 + 2)\,y = 0$ (3)

Solution: Here, P (x) and Q (x) are analytic $\forall\, x$ as they are polynomial functions. Thus, $x_0 = 0$ is the ordinary point, and there exist two nontrivial linearly independent power series solutions of the given ODE. These power series converge in some interval $|x - x_0| < R$ ($R > 0$) about x_0. Substituting y, y', and y'' found in the previous problem into (3), we obtain

$$\sum_{k=2}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}+x\sum_{k=1}^{\infty}(m+k)a_k x^{m+k-1}+\left(x^2+2\right)\sum_{k=0}^{\infty}a_k x^{m+k}=0$$

As $m = 0$, the above equation reduces to

$$\sum_{k=2}^{\infty}(k)(k-1)a_k x^{k-2}+x\sum_{k=1}^{\infty}(k)a_k x^{k-1}+\left(x^2+2\right)\sum_{k=0}^{\infty}a_k x^{k}=0$$

$$\sum_{k=2}^{\infty}(k)(k-1)a_k x^{k-2}+\sum_{k=0}^{\infty}(k)a_k x^{k}+\sum_{k=-2}^{\infty}a_k x^{k+2}+\sum_{k=0}^{\infty}2a_k x^{k}=0$$

The lower limits of summation and the powers of x are altered in the summations 2nd, 3rd, and 4th above as per Table 1.

In the 1st summation, replace k with (k + 2), and k with (k – 2) in the 3rd summation; in order to make the exponent of x the same in all the terms. This yields

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2} x^{k}+\sum_{k=0}^{\infty}(k)a_k x^{k}+\sum_{k=0}^{\infty}a_{k-2} x^{k}+\sum_{k=0}^{\infty}2a_k x^{k}=0$$

Now, all the terms in the above equation have the same summation limits and exponent of x. Thus, we can rewrite it as

$$\sum_{k=0}^{\infty}\left\{(k+2)(k+1)a_{k+2}+k\,a_k+a_{k-2}+2a_k\right\}x^{k}=0$$

We can obtain the recurrence formula by equating the coefficient of x^k to

Table 1. Various mathematical operations and their effect on the two parameters.

Parameters	Differentiation	Multiplication	Integration	Division
Lower limit of summation	Increase	Decrease	Decrease	Increase
Power of *x*	Decrease	Increase	Increase	Decrease

0. Therefore,

$$a_{k+2} = \frac{-a_{k-2} - (k+2)a_k}{(k+2)(k+1)}; k \geq 2 \; \forall k \in \mathbf{N}$$

$$\text{When k} = 2, a_4 = \frac{-a_0 - 4a_2}{12}$$

$$\text{When k} = 3,\; a_5 = \frac{-a_1 - 5a_3}{20}$$

$$\text{When k} = 4,\; a_6 = \frac{-a_2 - 6a_4}{30} = \frac{a_0 + 2a_2}{60}$$

and so on. One can proceed by substituting the coefficients so found, and obtaining the final solution, $y(x)$ in terms of a_0, a_1, a_2, and a_3.

Observations

Few simple noteworthy observations can be made. Only the exponent of *x* has to be taken care of before the equating-coefficient step. The summation limits automatically get adjusted by minute alteration, and the summation can be collected as common from all the terms present in the equation. One does not have to expand the series, or through inspection, arrive at the recurrence relation. It can easily be achieved by slight manipulation of the summation limits, for the sole purpose to match the powers of *x* in the terms, before the equating-coefficient step. The number of coefficients with which the final solution shall be expressed in terms of, can be found out by computing the difference between the smallest and the largest subscripts in the recurrence formula. For instance, in example #3, the number of coefficients which $y(x)$ will be expressed in terms of = |subscript (a_{k+2}) – subscript (a_{k-2}) = $|k + 2 - k + 2|$ = 4. However, the more the difference, the more the number of coefficients $y(x)$ will be expressed by, hence, the solution would become quite complicated. Thereby, sometimes expanding the series and later deriving the recurrence formula could assist in developing a link/relation between the coefficients which $y(x)$ is expressed in terms of, resulting in $y(x)$ being expressed in terms of less number of coefficients, hence making the final solution less complex.

Conclusion

It is discernible how smoothly one can arrive at the recurrence relation and compute the required coefficients of *x* without much painstaking expansion of the summations. However, it is deduced that it could lead to a marginally more intricate solution, since the end result, in certain cases, have to be expressed more in terms of the unknown coefficients.

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