Journal of Generalized Lie Theory and Applications Vol. 2 (2008), No. 3, 201–205

# Quasi-Lie deformations on the algebra $\mathbb{F}[t]/(t^N)$ [1]

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**Abstract**

This paper explores the quasi-deformation scheme devised by Hartwig, Larsson and Silvestrov as applied to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{F})$. One of the main points of this method is that the quasi-deformed algebra comes endowed with a canonical twisted Jacobi identity. We show in the present article that when the quasi-deformation method is applied to $\mathfrak{sl}_2(\mathbb{F})$ via representations by twisted derivations on the algebra $\mathbb{F}[t]/(t^N)$ one obtains interesting new multi-parameter families of almost quadratic algebras.

**2000 MSC:** 16S37, 17B37, 17B40, 16A05, 16W55, 16S80

## 1 Introduction

In a series of papers [1, 3, 4] two of the present authors have developed a new deformation scheme for Lie algebras. The last paper [4] is concerned with this deformation scheme when applied to the simple Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of zero characteristic, and it is on that paper the present one builds and elaborates on.

Let us briefly explain the aforementioned deformation procedure. By $\mathbb{F}$ we denote the underlying field of characteristic zero and by $\mathfrak{g}$ the Lie algebra we wish to deform. Let $\rho : \mathfrak{g} \to \mathrm{Der}(\mathcal{A}) \subseteq \mathfrak{gl}(\mathcal{A})$ be a representation of $\mathfrak{g}$ in terms of derivations on some commutative, associative algebra $\mathcal{A}$ with unity. The Lie structure on $\mathrm{Der}(\mathcal{A})$ is of course given by the commutator bracket, induced from the Lie algebra structure on $\mathfrak{gl}(\mathcal{A})$, the algebra of linear operators on $\mathcal{A}$. The deformation procedure now takes place on this representation by changing the involved derivations to $\sigma$-derivations, that is, linear maps $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ satisfying a generalized Leibniz rule: $\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b)$, for all $a, b \in \mathcal{A}$, and for an algebra endomorphism $\sigma$ on $\mathcal{A}$.

In the course of this deformation we also deform the commutator $[\cdot,\cdot]$ to a $\sigma$-deformed version $\langle\cdot,\cdot\rangle$. The deformation procedure is thus an assignment $\mathrm{Der}(\mathcal{A}) \ni \partial \rightsquigarrow \partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})$ such that $[\cdot,\cdot] \rightsquigarrow \langle\cdot,\cdot\rangle$ and where $\mathrm{Der}_\sigma(\mathcal{A})$ is the vector space of $\sigma$-derivations on $\mathcal{A}$. Remember that $\partial$ represents an element of $\mathfrak{g}$.

In general, the new product $\langle\cdot,\cdot\rangle$ is not closed on $\mathrm{Der}_\sigma(\mathcal{A})$. It is, however, true that it is closed on the left $\mathcal{A}$-submodule $\mathcal{A}\cdot\partial_\sigma$ of $\mathrm{Der}_\sigma(\mathcal{A})$, for $\partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})$ subject to some (mild) conditions. This is the content of Theorem 1. This theorem also establishes a canonical Jacobi-like relation on $\mathcal{A}\cdot\partial_\sigma$ for $\langle\cdot,\cdot\rangle$, reducing to the ordinary Jacobi identity when $\sigma = \mathrm{id}$, i.e., in the "limit" case of this deformation scheme corresponding to the Lie algebra $\mathfrak{g}$. We remark that in some cases, for instance when $\mathcal{A}$ is a unique factorization domain, $\mathcal{A}\cdot\partial_\sigma = \mathrm{Der}_\sigma(\mathcal{A})$ for suitable $\partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})$

[1] Presented at the 3$^{\text{rd}}$ Baltic-Nordic Workshop "Algebra, Geometry, and Mathematical Physics", Göteborg, Sweden, October 11–13, 2007.

(see [1]). In particular, this means that we have two "deformation parameters" for this scheme, $\mathcal{A}$ and $\sigma$. Note, however, that they are not independent. Indeed, $\sigma$ certainly depends on $\mathcal{A}$.

Diagrammatically, our deformation scheme can be given as

$$
\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\rho"] & & \mathfrak{gl}(\mathcal{A}) \arrow[d, squiggly] \\
& \tilde{\mathfrak{g}} \arrow[ul, dotted, "\text{"limit"}"] \arrow[r, dotted] & \widetilde{\mathfrak{gl}}(\mathcal{A}) \arrow[l, bend right, "P"']
\end{tikzcd}
$$

Suppose the Lie algebra $\mathfrak{g}$ is spanned as a vector space by elements $\{\mathsf{g}_i\}_{i\in I}$, where $I$ is some index set. The representation $\rho$ yields the assignments $\mathsf{g}_i \mapsto a_i \cdot \partial$, for $a_i \in \mathcal{A}$. This can clearly be extended linearly to the whole of $\mathfrak{g}$ by the linearity of $\rho$. Now the deformation is $a_i \cdot \partial \rightsquigarrow a_i \cdot \partial_\sigma \in \mathcal{A} \cdot \partial_\sigma \subseteq \mathrm{Der}_\sigma(\mathcal{A})$. Put $\tilde{\mathsf{g}}_i := a_i \cdot \partial_\sigma$. The set $\{\tilde{\mathsf{g}}_i\}_{i\in I}$ spans a linear subspace $\tilde{\mathfrak{g}}$ of $\mathcal{A} \cdot \partial_\sigma$. Restricting the bracket on $\mathcal{A} \cdot \partial_\sigma$, given by Theorem 1, to $\tilde{\mathfrak{g}}$ gives us an algebra structure on $\mathfrak{g}$. This restriction is denoted by $P$ in the above diagram. So, forgetting that $\tilde{\mathsf{g}}_i$ is $a_i\partial_\sigma$, $\{\tilde{\mathsf{g}}_i\}_{i\in I}$ spans an abstract (i.e., not associated with some particular representation) algebra $\tilde{\mathfrak{g}}$ with multiplication $\langle\cdot,\cdot\rangle$ and structure constants given by (2.3a) of Theorem 1. This algebra is then to be viewed as the deformed version of $\mathfrak{g}$. Another way to look at this is to actually compute the deformed commutator in terms of the basis elements $\tilde{\mathsf{g}}_i$ and leaving the right-hand-side as it is given by (2.3a). This gives us a set of relations in degree one and two in the basis elements, which can, considered as an associative algebra given by generators and relations, be viewed as an analogue of the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ to a Lie algebra $\mathfrak{g}$, for the algebra $\tilde{\mathfrak{g}}$.

The quotation marks in "limit" is to indicate that we may not actually retrieve the original $\mathfrak{g}$ by performing the appropriate (depending on the case considered) limit procedure. This is because for some "values" of the involved parameters the representation or specific operators collapse, so even taking the limit becomes meaningless in these circumstances. This is why we choose to call our deformations *quasi-deformations*. Another complication that arises is that the pull-back $P$ "forgets relations". That is to say that the operators in $\mathcal{A}\cdot\partial_\sigma$ may satisfy relations, for instance coming from the twisted Leibniz rules, that the abstract algebra does not satisfy.

Now, the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ can be realized as a vector space generated by elements $H$, $E$ and $F$ subject to the relations (see for instance [6])

$$\langle H, E\rangle = 2E, \quad \langle H, F\rangle = -2F, \quad \langle E, F\rangle = H \tag{1.1}$$

Our basic starting point is the following representation of $\mathfrak{sl}_2(\mathbb{F})$ in terms of first order differential operators acting on a vector space of functions in the variable $t$: $E \mapsto \partial$, $H \mapsto -2t\partial$, $F \mapsto -t^2\partial$. To quasi-deform $\mathfrak{sl}_2(\mathbb{F})$ means that we replace $\partial$ by $\partial_\sigma$ in this representation. At our disposal are now the deformation parameters $\mathcal{A}$ (the "algebra of functions") and the endomorphism $\sigma$.

In [4], where we studied mostly some of the algebras appearing in the quasi-deformation scheme in the case when $\mathcal{A} = \mathbb{F}[t]$. But we also have constructed quasi-Lie deformations in the case $\mathcal{A} = \mathbb{F}[t]/(t^3)$ yielding new interesting unexpected parametric families of algebras. In [5], we have constructed quasi-Lie deformations when $\mathcal{A} = \mathbb{F}[t]/(t^4)$. This case leads typically to six relations instead of three which might have been thought as natural as $\mathfrak{sl}_2(\mathbb{F})$ only has three relations.

In this paper we extend the construction to the general class of quasi-Lie deformations when $\mathcal{A} = \mathbb{F}[t]/(t^N)$. In Section 2 we recall the necessary background material and fix notation. Section 3 deals with the general quasi-deformation scheme as applied to $\mathfrak{sl}_2(\mathbb{F})$. Finally, in Section 3.1 we explore this scheme when $\mathcal{A} = \mathbb{F}[t]/(t^N)$.

## 2 Quasi-Lie algebras associated with $\sigma$-derivations

We now fix notation and state the main definitions and results from [1, 3] needed in this paper. Throughout we let $\mathbb{F}$ denote a field of characteristic zero and $\mathcal{A}$ be a commutative, associative $\mathbb{F}$-algebra with unity 1. Furthermore, $\sigma$ will denote an endomorphism on $\mathcal{A}$. Then by a *twisted derivation* or *$\sigma$-derivation* on $\mathcal{A}$ we mean an $\mathbb{F}$-linear map $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ such that a $\sigma$-twisted Leibniz rule holds: $\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b)$.

In the paper [1] the notion of a hom-Lie algebra as a deformed version of a Lie algebra was introduced, motivated by some of the examples of deformations of the Witt and Virasoro algebras constructed using $\sigma$-derivations. However, finding examples of more general kinds of deformations associated to $\sigma$-derivations, prompted the introduction in [3] of quasi-hom-Lie algebras (qhl-algebras) generalizing hom-Lie algebras. Quasi-hom-Lie algebras include not only hom-Lie algebras as a subclass, but also colour Lie algebras and in particular Lie superalgebras [3].

We let $\mathrm{Der}_\sigma(\mathcal{A})$ denote the vector space of $\sigma$-derivations on $\mathcal{A}$. Fixing a homomorphism $\sigma : \mathcal{A} \to \mathcal{A}$, an element $\partial_\sigma \in \mathrm{Der}_\sigma(\mathcal{A})$ and an element $\delta \in \mathcal{A}$, we assume that these objects satisfy the following two conditions:

$$\sigma(\mathrm{Ann}(\partial_\sigma)) \overset{(a)}{\subseteq} \mathrm{Ann}(\partial_\sigma), \quad \partial_\sigma(\sigma(a)) \overset{(b)}{=} \delta\sigma(\partial_\sigma(a)), \quad \text{for a} \in \mathcal{A} \tag{2.1}$$

where $\mathrm{Ann}(\partial_\sigma) := \{a \in \mathcal{A} \,|\, a \cdot \partial_\sigma = 0\}$. Let $\mathcal{A} \cdot \partial_\sigma := \{a \cdot \partial_\sigma \mid a \in \mathcal{A}\}$ denote the cyclic $\mathcal{A}$-submodule of $\mathrm{Der}_\sigma(\mathcal{A})$ generated by $\partial_\sigma$ and extend $\sigma$ to $\mathcal{A} \cdot \partial_\sigma$ by $\sigma(a \cdot \partial_\sigma) = \sigma(a) \cdot \partial_\sigma$. The following theorem, from [1], introducing an $\mathbb{F}$-algebra structure on $\mathcal{A} \cdot \partial_\sigma$ making it a quasi-hom-Lie algebra, is of central importance for the present paper.

**Theorem 1.** *If* (2.1a) *holds then the map* $\langle \cdot, \cdot \rangle$ *defined by*

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle = (\sigma(a) \cdot \partial_\sigma) \circ (b \cdot \partial_\sigma) - (\sigma(b) \cdot \partial_\sigma) \circ (a \cdot \partial_\sigma) \tag{2.2}$$

*for $a, b \in \mathcal{A}$ and where $\circ$ denotes composition of maps, is a well-defined $\mathbb{F}$-algebra product on the $\mathbb{F}$-linear space $\mathcal{A} \cdot \partial_\sigma$. It satisfies the following identities for $a, b, c \in \mathcal{A}$:*

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle \overset{(a)}{=} (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)) \cdot \partial_\sigma, \quad \langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle \overset{(b)}{=} -\langle b \cdot \partial_\sigma, a \cdot \partial_\sigma \rangle \tag{2.3}$$

*and if, in addition,* (2.1b) *holds, we have the deformed six-term Jacobi identity*

$$\circlearrowleft_{a,b,c} \left( \langle \sigma(a) \cdot \partial_\sigma, \langle b \cdot \partial_\sigma, c \cdot \partial_\sigma \rangle \rangle + \delta \cdot \langle a \cdot \partial_\sigma, \langle b \cdot \partial_\sigma, c \cdot \partial_\sigma \rangle \rangle \right) = 0 \tag{2.4}$$

*where $\circlearrowleft_{a,b,c}$ denotes cyclic summation with respect to $a, b, c$.*

The algebra $\mathcal{A} \cdot \partial_\sigma$ in the theorem is then a qhl-algebra with $\alpha = \sigma$, $\beta = \delta$ and $\omega = -\mathrm{id}_{\mathcal{A} \cdot \partial_\sigma}$. For the detailed proof of Theorem 1 see [1].

## 3 Quasi-Deformations of $\mathfrak{sl}_2(\mathbb{F})$

Let $\mathcal{A}$ be a commutative, associative $\mathbb{F}$-algebra with unity 1, $t$ an element of $\mathcal{A}$, and let $\sigma$ denote an $\mathbb{F}$-algebra endomorphism on $\mathcal{A}$. Also, let $\mathrm{Der}_\sigma(\mathcal{A})$ denote the linear space of $\sigma$-derivations on $\mathcal{A}$. Choose an element $\partial_\sigma$ of $\mathrm{Der}_\sigma(\mathcal{A})$ and consider the $\mathbb{F}$-subspace $\mathcal{A} \cdot \partial_\sigma$ of elements on the form $a \cdot \partial_\sigma$ for $a \in \mathcal{A}$. We will usually denote $a \cdot \partial_\sigma$ simply by $a\partial_\sigma$. Notice that $\mathcal{A} \cdot \partial_\sigma$ is a left $\mathcal{A}$-module, and by Theorem 1 there is a skew-symmetric algebra structure on $\mathcal{A} \cdot \partial_\sigma$ given by

$$\langle a \cdot \partial_\sigma, b \cdot \partial_\sigma \rangle = \sigma(a) \cdot \partial_\sigma(b \cdot \partial_\sigma) - \sigma(b) \cdot \partial_\sigma(a \cdot \partial_\sigma) = (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)) \cdot \partial_\sigma \tag{3.1}$$

where $a, b \in \mathcal{A}$. The elements $e := \partial_\sigma, h := -2t\partial_\sigma$ and $f := -t^2\partial_\sigma$ span an $\mathbb{F}$-linear subspace $\mathcal{S} := \mathrm{LinSpan}_\mathbb{F}\{\partial_\sigma, -2t\partial_\sigma, -t^2\partial_\sigma\} = \mathrm{LinSpan}_\mathbb{F}\{e, h, f\}$ of $\mathcal{A} \cdot \partial_\sigma$. We restrict the multiplication (3.1) to $\mathcal{S}$ without, at this point, assuming closure. Now, $\partial_\sigma(t^2) = \partial_\sigma(t \cdot t) = \sigma(t)\partial_\sigma(t) + \partial_\sigma(t)t = (\sigma(t) + t)\partial_\sigma(t)$. Under the natural assumptions $\sigma(1) = 1$, $\partial_\sigma(1) = 0$ (see [4]), (3.1) leads to to

$$\langle h, f\rangle = 2\sigma(t)t\partial_\sigma(t)\partial_\sigma, \quad \langle h, e\rangle = 2\partial_\sigma(t)\partial_\sigma, \quad \langle e, f\rangle = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma \tag{3.2}$$

**Remark 1.** Note that if $\sigma = \mathrm{id}$ and $\partial_\sigma(t) = 1$ we retain the classical $\mathfrak{sl}_2(\mathbb{F})$ with relations (1.1).

### 3.1 Quasi-Deformations of $\mathfrak{sl}_2(\mathbb{F})$ with base algebra $\mathbb{F}[t]/(t^N)$

Now, let $\mathbb{F}$ include all $N^{\text{th}}$-roots of unity and take as $\mathcal{A}$ the algebra $\mathbb{F}[t]/(t^N)$ for positive integer $N \geq 2$. This is obviously an $N$-dimensional $\mathbb{F}$-vector space and a finitely generated $\mathbb{F}[t]$-module with basis $\{1, t, \ldots, t^{N-1}\}$. For $i = 0, \ldots, N-1$, let $g_i = c_i t^i \partial_\sigma,\ c_i \in \mathbb{F},\ c_i \neq 0$. Put

$$\partial_\sigma(t) = p(t) = \sum_{k=0}^{N-1} p_k t^k, \quad \sigma(t) = \sum_{k=0}^{N-1} q_k t^k \tag{3.3}$$

considering these as elements in the ring $\mathbb{F}[t]/(t^N)$. The equalities (3.3) have to be compatible with $t^N = 0$. This means in particular that (if $s(t) = (\sigma(t) - q_0)/t$)

$$\sigma(t^N) = (q_0 + s(t)t)^N = \sum_{\nu=0}^{N} \binom{N}{\nu} q_0^\nu (s(t))^{N-\nu} t^{N-\nu} = q_0 \sum_{\nu=1}^{N} \binom{N}{\nu} q_0^{\nu-1} (s(t))^{N-\nu} t^{N-\nu} = 0$$

implying (and actually equivalent to) $q_0^N = 0$ and hence $q_0 = 0$. Furthermore,

$$\begin{aligned} \partial_\sigma(t^N) &= \sum_{j=0}^{N-1} \sigma(t)^j t^{N-j-1} \partial_\sigma(t) = p(t) \sum_{j=0}^{N-1} s(t)^j t^j t^{N-j-1} \\ &= p(t) t^{N-1} \sum_{j=0}^{N-1} s(t)^j = p_0 t^{N-1} \sum_{j=0}^{N-1} s(t)^j = p_0 \{N\}_{q_1} t^{N-1} = 0 \end{aligned} \tag{3.4}$$

where $\{N\}_{q_1} = \sum_{j=0}^{N-1} q_1^j$. It thus follows that $(1 + q_1 + q_1^2 + \ldots + q_1^{N-1})p_0 = 0$. In other words, if $p_0 \neq 0$ we generate deformations at the zeros of the polynomial $u^{N-1} + \ldots + u^2 + u + 1$, that is at $N$'th roots of unity; whereas if $p_0 = 0$ then $q_1$ is a true formal deformation parameter.

As before we assume that $\sigma(1) = 1$, $\partial_\sigma(1) = 0$ and so relations (3.2) still hold. Moreover, since for $k \geq 0$ we have $\partial_\sigma(t^{k+1}) = \sum_{j=0}^{k} \sigma(t)^j t^{k-j} \partial_\sigma(t) = p(t)t^k \sum_{j=0}^{k} s(t)^j$, we obtain by (3.1)

$$\begin{aligned} \langle g_i, g_j\rangle &= c_i c_j \langle t^i \partial_\sigma, t^j \partial_\sigma\rangle = c_i c_j [\sigma(t^i)\partial_\sigma(t^j) - \sigma(t^j)\partial_\sigma(t^i)]\partial_\sigma \\ &= c_i c_j [\sigma(t)^i \partial_\sigma(t^j) - \sigma(t)^j \partial_\sigma(t^i)]\partial_\sigma \end{aligned} \tag{3.5}$$

By (2.2) the bracket can be computed abstractly on generators $g_i, g_j$ as

$$\begin{aligned} \langle g_i, g_j\rangle &= \langle c_i t^i \partial_\sigma, c_j t^j \partial_\sigma\rangle = c_i c_j [(\sigma(t^i)\partial_\sigma) \circ (t^j \partial_\sigma) - (\sigma(t^j)\partial_\sigma) \circ (t^i \partial_\sigma)] \\ &= c_i \sigma(t)^i \partial_\sigma \circ g_j - c_j \sigma(t)^j \partial_\sigma \circ g_i \end{aligned} \tag{3.6}$$

Expanding according to the multinomial formula $s(t)^k = (q_1 + q_2 t + \ldots + q_{N-1} t^{N-2})^k$ and $\sigma(t^k) = \sigma(t)^k = t^k s(t)^k$, we obtain

$$\langle g_i, g_j\rangle = c_i (s(t)^i t^i \partial_\sigma) \circ g_j - c_j (s(t)^j t^j \partial_\sigma) \circ g_i$$

$$= c_i i! \Big( \sum_{\substack{i_1,\ldots,i_{N-1}\geq 0 \\ i_1+\ldots+i_{N-1}=i \\ i_2+2i_3+\ldots+(N-2)i_{N-1}<N-i}} \frac{q_1^{i_1}\cdots q_{N-1}^{i_{N-1}}}{i_1!\cdots i_{N-1}!} \frac{g_{i+i_2+2i_3+\ldots+(N-2)i_{N-1}} \circ g_j}{c_{i+i_2+2i_3+\ldots+(N-2)i_{N-1}}} \Big)$$
$$- c_j j! \Big( \sum_{\substack{j_1,\ldots,j_{N-1}\geq 0 \\ j_1+\ldots+j_{N-1}=j \\ j_2+2j_3+\ldots+(N-2)j_{N-1}<N-j}} \frac{q_1^{j_1}\cdots q_{N-1}^{j_{N-1}}}{j_1!\cdots j_{N-1}!} \frac{g_{j+j_2+2j_3+\ldots+(N-2)j_{N-1}} \circ g_i}{c_{j+j_2+2j_3+\ldots+(N-2)j_{N-1}}} \Big)$$

The bracket is closed on linear span of $g_i$'s as for $N-1 \geq i, j \geq 0$, by (3.1), we have

$$\begin{aligned}
\langle g_i, g_j \rangle &= c_i c_j [\partial_\sigma(t^j)\sigma(t)^i - \sigma(t)^j \partial_\sigma(t^i)]\partial_\sigma \\
&= c_i c_j \sum_{k=0}^{|j-i|-1} \operatorname{sign}(j-i) \sum_{\substack{k_1,k_2,\ldots,k_{N-1}\geq 0 \\ k_1+k_2+\ldots+k_{N-1}=k+\min\{i,j\} \\ k_2+2k_3+\ldots+(N-2)k_{N-1}<N}} \frac{(k+\min\{i,j\})!}{k_1!k_2!\cdots k_{N-1}!} \\
&\quad \times q_1^{k_1} q_2^{k_2} \ldots q_{N-1}^{k_{N-1}} t^{k_2+2k_3+\ldots+(N-2)k_{N-1}} \sum_{l=0}^{N-1} p_l t^{i+j+l-1}\partial_\sigma \\
&= c_i c_j \sum_{l=0}^{N-1} p_l \sum_{k=0}^{|j-i|-1} \operatorname{sign}(j-i) \sum_{\substack{k_1,k_2,\ldots,k_{N-1}\geq 0 \\ k_1+k_2+\ldots+k_{N-1}=k+\min\{i,j\} \\ k_2+2k_3+\ldots+(N-2)k_{N-1}\leq N-i-j-l}} \frac{(k+\min\{i,j\})!}{k_1!k_2!\cdots k_{N-1}!} \\
&\quad \times q_1^{k_1} q_2^{k_2} \ldots q_{N-1}^{k_{N-1}} \frac{g_{i+j+l-1+k_2+2k_3+\ldots+(N-2)k_{N-1}}}{c_{i+j+l-1+k_2+2k_3+\ldots+(N-2)k_{N-1}}}
\end{aligned} \tag{3.7}$$

where $\operatorname{sign}(x) = -1$ if $x < 0$, $\operatorname{sign}(x) = 0$ if $x = 0$ and $\operatorname{sign}(x) = 1$ if $x > 0$.

**Remark 2.** It would be of interest to determine the ring-theoretic properties of these algebras e.g, for which parameters are they domains, Noetherian, PBW-algebras, Auslander-regular etc.

**Acknowledgements.** This work was supported by the Crafoord Foundation, the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Royal Physiographic Society in Lund, the Royal Swedish Academy of Sciences, the European network Liegrits and the Centre for Mathematical Sciences, Lund University.

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*Received December 20, 2007*
*Revised April 02, 2008*