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Editorial

Preface to the Special Issue on Deformation Theory and Applications

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The purpose of this issue is to present some contributions that develop deformations of algebraic structures and applications to physics and their interrelations. It shows how basic techniques of deformation theory work in various standard situations.

Deformation is one of the oldest techniques used by mathematicians and physicists. The first instances of the so-called deformation theory were given by Kodaira and Spencer for complex structures and by Gerstenhaber for associative algebras. Abstract deformation theory and deformation functors in algebraic geometry were inspired and developed by works of André, Deligne, Goldman, Grothendick, Illusie, Laudal, Lichtenbaum, Milson, Quillen, Schlessinger, and Stasheff. Among concrete deformation theory developed by Gerstenhaber for associative algebras and later with Schack for bialgebras, the Lie algebras case was studied by Nijenhuis and Richardson and then by Fialowski and her collaborators in a more general framework.

Deformation theory is the study of a family in the neighborhood of a given element. Intuitively, a deformation of a mathematical object is a family of the same kind of objects depending on some parameters. The main and popular tool is the power series ring or more generally any commutative algebras. By standard facts of deformation theory, the infinitesimal deformations of an algebra of a given type are parametrized by a second cohomology of the algebra. More generally, it is stated that deformations are controlled by a suitable cohomology. Deformations help to construct new objects starting from a given object and to infer some of its properties. They can also be useful for classification problems.

A modern approach, essentially due to Quillen, Deligne, Drinfel'd, and Kontsevich, is that, in characteristic zero, every deformation problem is controlled by a differential graded Lie algebra, via solutions of Maurer-Cartan equation modulo gauge equivalence.

Some mathematical formulations of quantization are based on the algebra of observables and consist in replacing the classical algebra of observables (typically complex-valued smooth functions on a Poisson manifold) by a noncommutative one constructed by means of an algebraic formal deformations of the classical algebra. In 1997, Kontsevich solved a longstanding problem in mathematical physics, that is every Poisson manifold admits formal quantization which is canonical up to a certain equivalence.

Deformation theory has been applied as a useful tool in the study of many other mathematical structures in Lie theory, quantum groups, operads, and so on. Even today it plays an important role in many developments of contemporary mathematics, especially in representation theory.

As we mentioned above, the fact that any Poisson manifold can be quantized was proved by Kontsevich. However, finding exact formulas for specific cases of Poisson brackets is an interesting separate problem. There are several well-known examples of such explicit formulas. One of the first was the Moyal product quantizing the standard symplectic structure on R^{2n}. Another example is the standard quantization of the Kirillov-Kostant-Souriau bracket on the dual space g^* to a Lie algebra g (Gutt, 1983). Relations between this quantization and the Yang-Baxter equation were shown by Gekhtman and Stolin (1994). Despite the fact that the formula for the standard quantization of the Kirillov-Kostant-Souriau bracket was known already for a long time, the problem of finding explicit formulas for equivariant quantization of its symplectic leaves (i.e., coadjoint orbits on g^*) was open. Recently this problem was solved in important cases using the relationship with the dynamical Yang-Baxter equation and the Shapovalov form on Verma modules. It turns out that these explicit quantization formulas help to solve the known Kostant problem about description of the algebra of the locally-finite endomorphisms of the highest weight irreducible modules in some important cases.

The papers in the volume cover a number of topics related to the deformation theory. The theory of generalized Massey product given by Laudal is applied to compute open subset of the moduli of graded R-modules in Siqveland's paper. A generalized Burnside theorem in a noncommutative deformation theory is stated in Eriksen's paper. In Laudal's paper a noncommutative phase space associated to any algebra is introduced and studied; a connection to physics is shown and it is proved that it is useful in noncommutative deformation theory for the construction of the versal family of finite families of modules. In a same spirit, a geometry of noncommutative algebra is provided in Siqveland's paper. An illustration of algebraic formal deformation theory is given in Elhamdadi, and Makhlouf's paper by introducing an algebraic cohomology and a deformation theory of alternative algebras. Fialowski, Penkava, and Phillipson study deformations and describe the moduli space of 3-dimensional complex associative algebras. Kamiya and Shibukawa's paper provides a construction and a characterization of a dynamical Yang-Baxter map which is a set-theoretical solution of a version of the quantum dynamical Yang-Baxter equation, by means of homogenous pre-systems. Coll, Giaquinto, and Magnant compute the index of seaweed Lie algebras from its associated meander graph. The relevance of Frobenius Lie algebras (Lie algebras of index equal to zero) to deformations and quantum groups theory stems from their relation to the classical Yang-Baxter equation.

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