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Polygonal Figurate Number Sequences

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Abstract

This paper proves the log-concavity of polygonal and centered polygonal figurate number sequences and derives two recurrence formulas for these sequences. The log-concavity property is established by examining the second-order difference between consecutive terms, showing its non-negativity. The derived recurrence relations offer a practical method to compute subsequent terms based on previous ones. The proofs for both the log-concavity property and the recurrence formulas are provided, enhancing our understanding of these sequences and their mathematical properties.

Keywords: Figurate numbers • Log-concavity • m-gonal • Number sequences

Introduction

Figurate numbers, as well as a majority of classes of special numbers, have long and rich history. They were introduced in the Pythagorean school as an attempt to connect geometry and arithmetic [1]. A figurate number is a number that can be represented by a regular and discrete geometric pattern of equally spaced points [2]. It may be, say, a polygonal, polyhedral or polytopic number if the arrangement forms a regular polygon, a regular polyhedron or a regular polytope, respectively. In particular, polygonal numbers generalize numbers which can be arranged as a triangle (triangular numbers), or a square (square numbers), or in general as an m-gon for any integer $m \geq 3$ [3].

Beyond classical polygonal numbers, there exists a multitude of other numbers that can be formed in the plane through points (or balls). Among them, the centered polygonal numbers emerge as a significant and distinct class of these numbers. The centered polygonal numbers (or, sometimes, polygonal numbers of the second order) form a class of figurate numbers, in which layers of polygons are drawn centered about a point. Each centered polygonal number is formed by a central dot, surrounded by polygonal layers with a constant number of sides. Each side of a polygonal layer contains one dot more than any side of the previous layer, so starting from the second polygonal layer each layer of a centered m-gonal number contains m more points than the previous layer [1].

Some scholars studied the log-concavity or log-convexity of different numbers sequences such as Fibonacci and hyper-Fibonacci numbers, Lucas and hyper-Lucas numbers, Bell numbers, hyper-Pell numbers, Motzkin numbers, Fine numbers, Franel numbers of order 3 and 4, Ap ́ery numbers, large Schröder numbers, central Delannoy numbers, Catalan-Larcombe–French numbers sequences, and so on [4-12].

The author has reviewed the existing literature on the log-concavity and log-convexity of number sequences. While there is a wealth of research available, it is worth noting that no previous studies have specifically investigated the log-concavity or log-convexity of polygonal (or m-gonal) figurate number sequences. In light of this gap in the literature, the objective of this work is to thoroughly investigate the log-concavity behavior exhibited by polygonal figurate number sequences. By focusing on this particular type of number sequence, the research aims to shed light on a previously unexplored property of these sequences.

In a prior work [13], the author made an attempt to establish the proof of log-concavity for centered polygonal numbers. However, the present work aims to provide a more straightforward and concise proof for the log-concavity of centered polygonal numbers.

Additionally, this work goes beyond the investigation of log-concavity by introducing two new recurrence formulas that are closely associated with polygonal and centered polygonal figurate number sequences. These recurrences not only enhance our understanding of the sequences themselves but also contribute to the broader field of mathematical research.

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By delving into the log-concavity behaviors of figurate number sequences and introducing new recurrences, this research not only addresses an existing gap in the literature but also expands our knowledge and insights in the field of combinatorics. The findings of this study have the potential to uncover new patterns and relationships within these sequences and may pave the way for further investigations and applications in various mathematical contexts.

The paper's organization is as follows: Section 2 presents the definitions and mathematical formulations of figurate numbers. In section 3, we explore the log-concavity of sequences of figurate numbers. Section 4 discusses recurrence relations associated with polygonal and centered polygonal figurate numbers, and finally, section 4 is about the conclusions.

Discussion

Basic concepts and notations

Some properties of figurate numbers are given. In this paper we continue discussing the properties of m-gonal figurate numbers. Now we recall some definitions involved in this paper.

Definition 1. Let $\{s_n\}_n \geq 0$ be a sequence of positive numbers. If for all $j \geq 1$, $s^2_j \geq s_{j-1}s_{j+1}$, the sequence $\{s_n\}_n \geq 0$ is called log-concave.

Definition 2. Let $\{s_n\}_n \geq 0$ be a sequence of positive numbers. If for all $j \geq 1$, $s^2_j \leq s_{j-1}s_{j+1}$, the sequence $\{s_n\}_n \geq 0$ is called log-convex. In case of equality, $s^2_j = s_{j-1}s_{j+1}$, $j \geq 1$, we call the sequence $\{s_n\}_n \geq 0$ geometric or log-straight.

Definition 3. Let $\{s_n\}_n \geq 0$ be a sequence of positive numbers. The sequence $\{s_n\}_n \geq 0$ is log-concave (log-convex) if and only if its quotient sequence $\{(s_{n+1})/s_n\}_n \geq 0$ is non-increasing (non-decreasing).

Log-concavity and log-convexity are important properties of combinatorial sequences and they play a crucial role in many fields for instance economics, probability, mathematical biology, quantum physics and white noise theory [14-20].

We consider the sets of points forming some geometrical figures on the plane. Starting from a point, one can add two points to it so that the three points form an equilateral triangle. By adding three further points, one can form a six-point equilateral triangle. This can be repeated, see Figure 1a for an example and for further details. The numbers thus obtained are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, . . . Sloane's, and are called triangular numbers. For n ≥ 1, the n th triangular number is given by the formula

$S_n=n(n+1)/2$ (1)

By adding to a point three, five, seven, etc. points and arranging them in the form of a square, one obtain the numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . . (Sloane's A000290), which are called square numbers, see Figure 1b for an example. For n ≥ 1, the n^{th} square number is given by the formula

$S_n=n^2$ (2)

By adding to a point four, seven, ten, etc. points and arranging them in the form of a regular pentagon, one obtain the numbers 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, . . .(Sloane's A000326), which are called pentagonal numbers, see Figure 1c for an example. For n ≥ 1, the nth pentagonal number is given by the formula

$S_n=n(3n-1)/2$ (3)

Following a similar approach, one can construct hexagonal, heptagonal, octagonal, nonagonal, decagonal numbers, . . . , m-gonal numbers if the arrangement forms a regular m-gon [1]. Form ≥ 3, the n^{th} term m-gonal number denoted by $S_n(m)$ is the sum of the first n elements of the arithmetic progression

1, 1 + (m−2), 1 + 2(m−2), 1 + 3(m−2), ... (4)

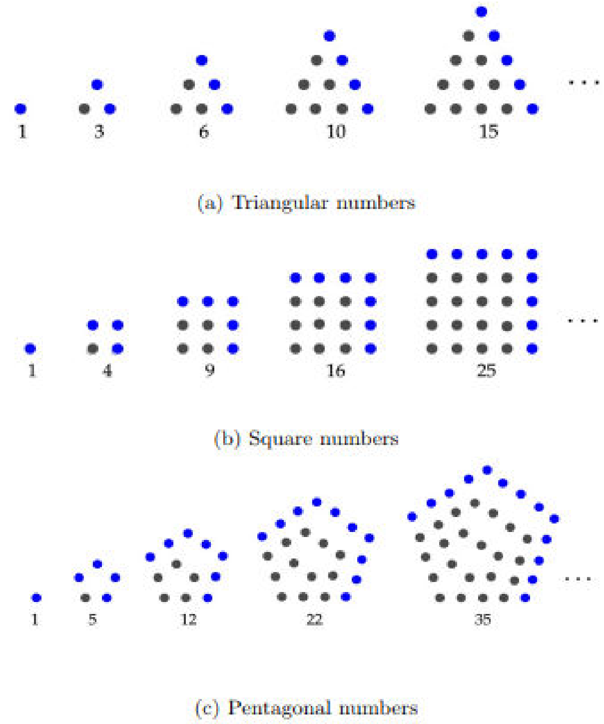


Figure 1. Polygonal numbers.

The following lemma is important for understanding the m-gonal figurate number sequences. It presents a formula that allows us to calculate the n^{th} term of the sequence using the values of n and m

Lemma 4. For all m ≥ 3 and n ≥ 1, the n^{th} term of m-gonal figurate number is given by

$S_n(m)=n/2[(m-2)\,n-m+4]$ (5)

Proof. To prove (5), it suffices to find the sum of the first n elements of (4). Hence the first n elements of the arithmetic progression given in (4) is:

$1, 1 + (m - 2), 1 + 2(m-2), 1 + 3(m - 2), \ldots, 1 + (n - 1)(m-2), \forall m \geq 3.$

Since the sum of the first n elements of an arithmetic progression $s_1, s_2, s_3, \ldots, s_n$ is equal to $n/2\, [s_1+s_n]$, it follows that

$S_n(m) = n/2\, [s_1+s_n]$

$= n/2\, [1+(1 + (n-1)(m-2))]$

$n/2\, [2+(m-2)n-m+2]$

$= n/2\, [(m-2)n-m+4]$ or

$S_n(m)=(m-2/2)\, [n^2-n] +n$

This completes the proof.

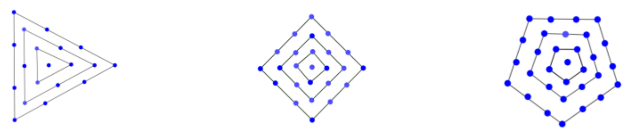
Beyond classical polygonal numbers, there are many other numbers, which can be constructed in the plane from points (or balls). The centered polygonal numbers form the next important class of such numbers.

Centered polygonal numbers, also known as polygonal numbers of the second order, belong to a category of figurate numbers that involve the construction of layers of polygons centered around a point. Each centered polygonal number consists of a central dot, which is then surrounded by successive layers of polygons with a fixed number of sides. In each layer, the number of dots on each side of the polygon increases by one compared to the previous layer. This pattern applies starting from the second polygonal layer, where each layer of a centered m-gonal number contains m additional points compared to the previous layer.

To illustrate this concept, consider the centered triangular number, which represents a triangle with a dot positioned at the center and additional dots arranged in successive triangular layers around it. The accompanying Figure 2a demonstrates the progression of constructing centered triangular numbers by adding a new layer of dots in the shape of a triangle around the previous figure at each step. The first few centered triangular numbers are 1, 4, 10, 19, 31, 46, 64, 85, 109, 136, . . . Sloane's A005448.

A centered square number is consisting of a central dot with four dots around it, and then additional dots in the gaps between adjacent dots, see Figure 2b for an example. The first few centered square numbers are 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, . . . Sloane's A001844.

A centered pentagonal number represents a pentagon with a dot in the center and all other dots surrounding the center in successive pentagonal layers, see Figure 2c for an example. The first few centered pentagonal numbers are 1, 6, 16, 31, 51, 76, 106, 141, 181, 226, . . . Sloane's A005891.



(a) Centered triangular numbers (b) Centered square numbers (c) Centered pentagonal numbers

Figure 2. Centered polygonal numbers.

Following this procedure, we can construct centered hexagonal numbers, centered heptagonal numbers, centered octagonal numbers, centered nonagonal numbers, centered decagonal numbers, etc.

In algebraic terms, the n^{th} centered m-gonal number, denoted as $C_n(m)$, can be derived by summing the first n elements of the sequence 1, m, 2 m, 3 m, Thus, according to the definition, we have:

$$C_n(m) = 1 + m + 2m + 3m + \cdots + (n-1)m \qquad (6)$$

By definition, the above formula leads to a recurrence relation for centered m-gonal numbers:

$$C_{n+1}(m) = C_n(m) + nm,\ C_1(m)=1 \qquad (7)$$

Furthermore, since the sum of the series $m+2m+3m+\cdots+(n-1)m$ can be expressed as $m\,(1+2+3+\cdots+n-1)=(m(n-1)n)/2$, we can derive a general formula for the n^{th} centered m-gonal number:

$$C_n(m) = 1+(m(n-1)n)/2 \qquad (8)$$

This formula allows us to directly calculate the n^{th} centered m-gonal number without having to sum the individual terms of the sequence.

Log-concavity of polygonal numbers

In this section, our objective is to establish the log-concavity of polygonal and centered polygonal figurate numbers. We present a proof showing the log-concavity of these number sequences according to the provided definition. This result contributes to our understanding of the mathematical characteristics of polygonal figurate numbers and their potential applications in various fields.

Theorem 5. For all $m \geq 3$, the sequence $\{S_n(m)\}_{n\geq 1}$ of m-gonal figurate numbers is log-concave. Proof. To prove that the sequence $\{S_n(m)\}_{n\geq 1}$ of m-gonal figurate numbers is log-concave for all $m \geq 3$, we need to show that the second-order differences are non-negative. In other words, we need to prove that for all $m \geq 3$ and for all $n \geq 1$, we have:

$S^2_n(m) \geq S_{n-1}(m)S_{n+1}(m)$ or $S^2_n(m) - S_{n-1}(m)S_{n+1}(m) \geq 0$

We denote $\ell=(m-2)/2$. Computing $S^2_n(m)$, $S_{n-1}(m)$ and $S_{n+1}(m)$, it follows from (5) that

$S_n(m)=n(\ell n-\ell+1)$, $S_{n-1}(m)=(n-1)(\ell(n-1)-\ell+1)$, and

$S_{n+1}(m) = (n+1)(\ell(n+1)-\ell+1)$.

Now, let us consider $S^2_n(m)-S_{n-1}(m)S_{n+1}(m)$. We get

$$S^2_n(m)-S_{n-1}(m)S_{n+1}(m) \qquad (9)$$

$$= n^2(\ell n-\ell+1)^2-(n-1)(\ell(n-1)-\ell+1)(n+1)(\ell(n+1)-\ell+1) \qquad (10)$$

$$= n^2(\ell n-\ell+1)^2-(n^2-1)((\ell n-\ell+1)^2-\ell^2) \qquad (11)$$

$$= n^2\ell^2+(\ell n-\ell+1)^2-\ell^2 \qquad (12)$$

$$\geq 0, \qquad (13)$$

where the last inequality follows by $n \geq 1$.

Theorem 6. For all $m \geq 3$ and $n \geq 2$, the sequence $\{C_n(m)\}$ of centered polygonal numbers is log-concave.

Proof. We aim to prove that the sequence of centered m-gonal figurate numbers, denoted as $C_n(m)$, is log-concave for all $m \geq 3$. To establish this, we need to prove that the second-order differences of the sequence are non-negative. In other words, we must show that for any $m \geq 3$ and $n \geq 1$, the inequality holds:

$$C^2_n(m) \geq C_{n-1}(m)C_{n+1}(m).$$

Alternatively, we can express this inequality as:

$$C^2_n(m) - C_{n-1}(m)C_{n+1}(m) \geq 0.$$

To facilitate our proof, we introduce the variable η, defined as $\eta:=(n-1)n/2$. By employing equation (8), we can deduce the following expressions:

$C_n(m)=1+(m(n-1)n)/2=1+mn,$

$C_{n-1}(m)=1+(m(n-1-1)(n-1))/2=1+(m(n-1)n-2(n-1))/2=1+m\eta-mn+m,$

and $C_{n+1}(m)=1+(m(n+1-1)(n+1))/2=1+(m(n-1)n+2n)/2=1+m\eta+mn.$

We proceed to evaluate the inequality:

$$C^2_n(m) \geq C_{n-1}(m)C_{n+1}(m)=(1+m\eta)^2-(1+m\eta-mn+m)(1+m\eta+mn) \quad (14)$$

$$=(1+m\eta)^2-((1+m\eta)(1+m\eta+mn)+(-mn+m)(1+m\eta+mn)) \quad (15)$$

$$=(1+m\eta)^2-((1+m\eta)^2+(1+m\eta)mn+(-mn+m)(1+m\eta)+(-mn+m)mn) \quad (16)$$

$$=-mn(1+n\eta)-(1+m\eta)(-mn)+m(1+m\eta)+(-mn+m)mn \quad (17)$$

$$=m^2n^2-m^2n-m^2\eta-m \quad (18)$$

$$=m^2(n^2-n-\eta)-m \quad (19)$$

$$=m^2(2\eta-\eta)-m \quad (20)$$

$$=m^2\eta-m \quad (21)$$

$$=m(m\eta-1) \quad (22)$$

$$\geq 0, \quad (23)$$

where the last inequality follows by $m \geq 3$.

Therefore, we established that the second-order differences of the sequence $C_n(m)$ are non-negative, showing that the sequence is log-concave for all $m \geq 3$. and $n \geq 2$.

Recurrence formulas for polygonal numbers

In the study of polygonal figurate number sequences, recurrence formulas play a crucial role. These formulas provide a systematic way of generating the terms of a sequence based on its previous terms. By utilizing recurrence formulas, mathematicians can efficiently compute and explore the properties of these sequences, revealing intriguing patterns and relationships.

Theorem 7. For all integers $m \geq 3$ and $n \geq 3$, the following recurrence formulas for the sequence $\{S_n(m)\}$ of m-gonal number sequences hold:

$$S_n(m)=R(n)S_{n-1}(m)+T(n)S_{n-2}(m) \quad (24)$$

where $R(n)=(m+2(n-2)(m-2))/(1+(n-2)(m-2))$ and $T(n)=(-m-1+(n-2)(m-2))/(1+(n-2)(m-2))$, the initial values are $S_1(m)=1$, $S_2(m)=m$, and the recurrence of its quotient sequence is given by

$$x_{n-1}=R(n)+T(n)/x_{n-2} \quad (25)$$

with the initial condition $x_1=m$.

Proof. By definition, we have $S_{n+1}(m)=S_n(m)+(1+(m-2)n)$ (26)

It follows that $S_{n+2}(m)\ S_{n+1}(m)+(m-1+(m-2)n)$ (27)

Rewriting (26) and (27) for all $n \geq 3$, we have

$$S_{n-1}(m)=S_{n-2}(m)+(1+(m-2)(n-2)) \quad (28)$$

$$S_n(m)=S_{n-1}(m)+(m-1+(m-2)(n-)) \quad (29)$$

Multiplying (28) by $m-1+(m-2)(n-2)$ and (29) by $1+(m-2)(n-2)$, and subtracting as to cancel the non-homogeneous part, one can obtain the homogeneous second-order linear recurrence for $S_n(m)$:

$$S_n(m)=\left[\frac{m+2(n-2)(m-2)}{1+(n-2)(m-2)}\right]S_{n-1}(m)-\left[\frac{m-1+(n-2)(m-2)}{1+(n-2)(m-2)}\right]S_{n-2}(m), \forall n, m \geq 3.$$

By denoting

$R(n)=(m+2(n-2)(m-2))/(1+(n-2)(m-2))$

and

$T(n)=(m-1+(n-2)(m-2))/(1+(n-2)(m-2)$

one can obtain $S_n(m)=R(n)S_{n-1}(m)+T(n)S_{n-2}(m), \forall n, m \geq 3$ (30)

with given initial conditions $S_1(m)=1$ and $S_2(m)=m$. By dividing (30) through by $S_{n-1}(m)$, one can also get the recurrence of its quotient sequence x_{n-1} as

$$x_{n-1}=R(n)+(T(n)/x_{n-2}),\ n \geq 3 \quad (31)$$

with initial condition $x_1=m$

Theorem 8. For all integers $m \geq 3$ and $n \geq 3$, the following recurrence formulas hold for the sequence $\{C_n(m)\}$ of centered polygonal number sequences:

$$C_n(m)=R(n)C_{n-1}(m)+T(n)C_{n-2}(m)$$

where $R(n)=2n-3/n-2$ and $T(n)=n-1/n-2$, the initial values

$C_2(m)=1+m$, and the recurrence of its quotient sequence is given by

$$y_{n-1}=R(n)+(T(n)/y_{n-2})$$

with the initial condition $y_1=1+m$.

Proof. To prove this theorem, we begin by using equation (8) to derive the following equation:

$$C_{n+1}(m)=C_n(m)+mn \quad (32)$$

It follows that

$$C_{n+2}(m)=C_{n+1}(m)+m(n+1) \quad (33)$$

Rewriting (32) and (33) for $n \geq 3$, we have

$C_{n-1}(m)=C_{n-2}(m)+m(n-2)$ (34)

$C_n(m)=C_{n-1}(m)+m(n-1)$ (35)

Next, we multiply equation (34) by m(n−1) and equation (35) by m(n−2), and then subtract the two equations to cancel out the non-homogeneous part. This allows us to obtain the homogeneous second-order linear recurrence for $C_n(m)$:

$$C_n(m) = \left[\frac{2n-3}{n-2}\right]C_{n-1}(m) - \left[\frac{n-1}{n-2}\right]C_{n-2}(m), \forall n, m \geq 3.$$

By denoting

$2n-3/n-2=R(n)$

and $-n-1/n-2=T(n)$,

one can rewrite the recurrence as follows:

$C_n(m)=R(n)C_{n-1}(m)+T(n)C_{n-2}(m), \ \forall n, m \geq 3$ (36)

These recurrence relations hold for given initial conditions $C_1(m)=1$ and $C_2(m)=1+m$. Dividing equation (36) by $C_{n-1}(m)$, we can derive the recurrence relation for the quotient sequence y_{n-1} as:

$y_{n-1}=R(n)+(T(n)/y_{n-2}), \ n \geq 3$ (37)

with the initial condition $y_1=1+m$.

This completes the proof of the theorem.

Conclusion

The paper presented the results on polygonal figurate number sequences, uncovering the remarkable property of log-concavity in both polygonal and centered polygonal numbers. Furthermore, we introduced two recurrence formulas for these figurate number sequences that provide a systematic way to find subsequent numbers in the sequence based on previous terms. These contributions enhance our understanding and provide valuable tools for further exploration and analysis of polygonal figurate numbers.

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