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Physical Interpretation of Noncommutative Algebraic Varieties

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Abstract

The theory of algebraic varieties gives an algebraic interpretation of differential geometry, thus of our physical world. To treat, among other physical properties, the theory of entanglement, we need to generalize the space parametrizing the objects of physics. We do this by introducing noncommutative varieties.

Keywords: Noncommutative algebra; Differential geometry; Matrix polynomial algebras

Introduction

Differential geometry is simplified by applying algebraic geometry. Then all holomorphic functions are interpreted as polynomials, and completed to power series. We have used infinitesimal methods, or rather deformation theory of modules, to construct varieties as moduli of the closed points. Eriksen [1] and Laudal [2] for a precise treatment. Then a generalization to deformation theory over commutative rings to noncommutative, have made the definition of noncommutative varieties possible.

In this short text, we show how our results can be stated without mentioning deformation theory, and we show one way of interpreting noncommutative varieties of matrix polynomial algebras.

Finally, we show how our notion of entanglement gives a kind of smoothing of a singularity. In articles to come, we will show how these varieties can (and must) be constructed using deformation theory.

Matrix Polynomial Algebras

Definition 1

Let $M \in M_r$ be a matrix with natural numbers m_{ij} as entries, and let R be a commutative ring. We then let R[M] be the R-algebra generated by the matrices (paths) $t_{ij}(l_{ij})$, $1 \le l_{ij} \le m_{ij}$, $1 \le i, j \le r$. Alternatively $R[M] = \mathbb{T}_{R^r}(V_M)$ the tensor algebra over R^r of the module generated by the $t_{ij}(l_{ij})$ given by matrix multiplication.

Example 1. We let r=2 and M=$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Then

$$R[M] = \begin{pmatrix} R[t_{11}(1), t_{11}(2)] & \langle t_{12} \rangle \\ 0 & R[t_{22}(1), t_{22}(2)] \end{pmatrix},$$

the $R^2 = R \cdot \mathrm{id}(2)$-algebra generated by the matrix generators.

We should notice that in general algebraic theory, there is no reason why the algebras on the diagonal, that is $R_{ii} = e_i R e_i$, should be commutative. This is the reason that we use the square brackets in the notation. The general, not necessarily commutative, polynomial algebra over R with M variables should then be named $R\langle M \rangle$.

For the rest of this text, we will let R=k, an algebraically closed field of characteristic 0. And of course, because we are giving the physical interpretation, we think of k as the field of complex numbers.

The Non Commutative Affine Space

We put $\mathbb{A}^M = \prod_{1 \le i \le r} \mathbb{A}^{m_{ii}}$

For a matrix polynomial, that is an element $f = (f_{ij}) \in k[M]$, and a point $P = (p_1, \ldots, p_r) \in k^r$, we let $f(P) = (f_{11}(p_1), \ldots, f_{rr}(p_r))$. Then as usual, we let the complements of the algebras sets be the topology of $\mathbb{A}^M$.

Definition 2

Let $\mathfrak{a} \subseteq k[M]$ be a two-sided ideal. Then the set $Z(\mathfrak{a}) = \{P \in \mathbb{A}^M \mid f(P) = 0 \text{ for all } f \in \mathfrak{a}\}$ is called an algebraic set, or a Zariski closed set.

The resulting topology is obviously the product topology of the Zariski topology, as the entries off the diagonal doesn't influence the state. As we will see, these variables are present to make the dynamics in the space. Each of the com- mutative affine spaces in the product corresponds to the ordinary Euclidean space.

As in the commutative situation, we let the coordinate ring of an algebraic set $Z(\mathfrak{a})$ be $A[M]/\mathfrak{a}$ (we assume that algebraic sets are reduced). The ring of regular functions is the most problematic one to define in the noncommutative case. This is where the dynamics, or entanglement, comes into the picture. We need to translate the deformation theory to the concept of a semi-local ring.

Lemma 1: The simple one-dimensional k[M]-modules corresponds to the maximal ideals $\mathfrak{m}_{ii} \subseteq k[M]_{ii}$. Thus there is a one-to-one correspondence between point in A^M and tuples of maximal ideals with one-dimensional quotient.

Definition 3

Let $P = (p_1, \ldots, p_r) \in V = Z(\mathfrak{a}) \subseteq \mathbb{A}^M$. We define the semi- local ring of P in V as $\mathcal{O}_{V,P} = (k[M]/\mathfrak{a})_P$ where the notation R_P for a matrix algebra R means that each R_{ii} is localized in p_i.

Example 2:

$$R = \begin{pmatrix} k[t_{11}(1), t_{11}(2)] & < t_{12}(1), t_{12}(2) > \\ < t_{21}(1), t_{22}(2) > & k[t_{22}(1), t_{22}(2)] \end{pmatrix}_{p_1, p_2}$$

$$\begin{pmatrix} k[t_{11}(1), t_{11}(2)]_{p_1} & < t_{12}(1), t_{12}(2) > \\ < t_{21}(1), t_{22}(2) > & k[t_{22}(1), t_{22}(2)]_{p_2} \end{pmatrix}$$

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Then the above mentioned transformation of the deformation theory is that a regular function is a global function that is regular in every point:

Definition 4

Let $U \subseteq V \subseteq \mathcal{A}^M$ be an open subset of the affine variety V. Then the ring of regular functions over U is $\mathcal{O}_V(U) = \lim_{\substack{\leftarrow \\ P\in U}} \mathcal{O}_{V,P}$

The Dynamics in Noncommutative Affine Varieties

The dynamics in any space is given by the structure of the tangents. This is because the tangent space is the space generated by the tangent directions, which are the directions in which the actual geometry allows dynamics. Thus the tangent space tells what kind of changes in state that are possible. Let us recall the commutative situation, equivalently the differential geometric one: Consider a derivation

$$\delta : A_{\mathfrak{m}} \to A_{\mathfrak{m}} / \mathfrak{m}A_{\mathfrak{m}} \simeq k,$$

satisfying the Leibniz rule $\delta(ab) = a\delta(b) + \delta(a)b$. This induces a k-linear morphism

$$\delta : \mathfrak{m} / \mathfrak{m}^2 \to k,$$

which is an element in the dual space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. Thus in the ordinary (commutative) setting, the directions of change is the basis element in the tangent space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. When we let one point actually be r points, we have to generalize slightly.

Let R be a quotient of a matrix polynomial algebra, and let V_i and V_j be two simple one-dimensional left R-modules. Then Homk (V_i, V_j) is a left-right bi module by $(r\cdot\phi)(v_i) = \phi(r\cdot v_i)$ and $(\phi\cdot r)(v_i) = r\cdot\phi(v_i)$.

Definition 5: The 1-radical of R is defined as $(1,R) = \bigcap_{\dim_k V=1} \ker \rho_V \rho_V$ where ρ_V: R → End_k (V) is the structure morphism of V .

Of course, this indicates that we should really work with the Jacobson radical, taking all simple modules into account. This applied theory is generalized in work to be published in the near future. For the rest of this section, we use the word radical for the 1-radical, and we use the notation m=I(1, R). Then we define the tangent space in this situation as:

Definition 6: The tangent space of the affine variety V in the point $P = (p_1, \ldots, p_r)$ is $T_P(V) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ where $M^\vee = Hom_k(M, k^r)$ for any A(V)-module M .

$S = \begin{pmatrix} k[t_{11}] & kt_{12} \\ kt_{21} & k[t_{22}] \end{pmatrix}$ and consider two general points

$$V_1 = k[t_{11}]/(t_{11} - a), V_2 = k[t_{22}]/(t_{22} - b).$$

First, we compute

$Der_k(S, \mathrm{Hom}_k(V_i, V_j)) / \mathrm{inner}$.

(1, 1): Let δ ∈ Derk (S, End_k (V_1)). Then

$\delta(e_i) = \delta(e_i^2) = 2\delta(e_i) \Rightarrow \delta(e_i) = 0, i = 1,2$.

$\delta(t_{12}) = \delta(t_{12}e_2) = \delta(t_{12})e_2 = 0$,
$\delta(t_{21}) = \delta(e_2 t_{21}) = e_2\delta(t_{21}) = 0$,
$\delta(t_{22}) = \delta(t_{22})e_2 = 0$,

and finally

$\delta(t_{11}) = \alpha$.

As all inner derivations are zero (easily seen from the computation above, in particular because $\mathrm{ad}_\beta(t_{11}) = a\beta - \beta a = 0$, we find that Der_k (V_1, V_1)/Inner is generated by the derivation sending t_{11} to α, and all other generators to 0.

(1, 2):

For ∈ Der_k (S, Hom_k (V_1, V_2)) things are slightly different. $\delta(e_1) = \delta(e_1^2) = e_1\delta(e_1) + \delta(e_1)e_1 = \delta(e_1)$, that is, the above trick doesn't work quite the same way. However, as $\delta(1) = \delta(e_1 + e_2) = 0$, for every derivation $\delta : S \to Hom_k(V_1, V_2)$, we find $\delta(e_1) = \alpha, \delta(e_2) = -\alpha$

$$\delta(e_1) = \alpha,\ \delta(e_2) = -\alpha,$$
$$\delta(t_{11}) = \delta(t_{11}e_1) = \delta(t_{11})e_1 + t_{11}\delta(e_1) = a\alpha,$$
$$\delta(t_{21}) = \delta(t_{21}e_1) = \delta(t_{21})e_1 = 0,$$
$$\delta(t_{22}) = \delta(e_2 t_{22}) = \delta(e_2)t_{22} = -b\alpha,$$
$$\delta(t_{12}) = \varrho.$$

So a general derivation can be written, the ∗ denoting the dual,

$$\delta = \alpha e_1^* - \alpha e_2^* + a\alpha t_{11}^* - b\alpha t_{22}^* + \varrho t_{12}^*.$$

For the inner derivations, we compute

$ad_\beta(e_1) = \beta e_1 - e_1\beta = -\beta$,

$ad_\beta(e_2) = \beta e_2 - e_2\beta = \beta$,

$ad_\beta(t_{11}) = -\beta a$,

$ad_\beta(t_{22}) = \beta b$,

$\mathrm{ad}_\beta = \gamma e_1^* - \gamma e_2^* + a\gamma t_{11}^* - b\gamma t_{22}^*$, where we have put $\gamma = -\beta$.

So as $ad_\beta(t_{12}) = 0$, and there are no conditions on $\delta(t_{12})$, we Der_k $(V_1, V_2) = kt_{12}^* = kd_{t_{12}}$.

The cases (2, 1) and (2, 2) are exactly similar.

Generalizing the computation in the above example, we have proved the following:

Lemma 2: Let S be a general free r×r matrix polynomial algebra, and let $V_i = V_{ii}(p_{ii})$ be the point p_{ii} in entry i, i. Then the tangent space from V_i to V_j is Der_k $(V_i, V_j) = \oplus_{l=1}^{d_{ij}} k\mathbf{d}_{t_{ij}(l)}$.

Entanglement: Blowing Up a Singularity

For a precise treatment of the deformation theory leading to this examples, see [3]. We consider a coincidently chosen singularity, the E_6-singularity

$$E_6 = k[[x, y]]/(x^4 + y^3)$$

This is a singularity because the tangent space is of dimension d>1 which is too big. So we need to make space for the entanglement, that is the interrelations. Let

$$R = \begin{pmatrix} k[t_{11}] & \langle t_{12}\rangle \\ 0 & k[t_{22}] \end{pmatrix} / (t_{11}^3 t_{12} + t_{12} t_{22}^4)$$

For a point P = (α_1, α_2) in this space, we compute the tangent space in this point by applying Lemma 2.

$$\mathbf{d}(t_{11}^3 t_{12} + t_{12} t_{22}^4) = \mathbf{d}_{t_{ij}}(P) \cdot (\alpha_1^3 + \alpha_2^4) = 0$$

Suggest that the points are pairwise identified, and with an extra tangent from zero to zero. This justifies that we can call this a blowup of the singularity.

Of course, it would be more obvious if we, in the matrix above, localized in origo, then the identification is superfluous. Anyway, the interpretation must be taken by care.

Also, there is a long way to go to prove that the physicist's notion of entanglement has anything to do with this momentum between different points. Our main achievement is to give an example on how extra space for a tangent direction can be given.

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