

Research Article

Phase Spaces and Deformation Theory*

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Abstract We have previously introduced the notion of non-commutative phase space (algebra) associated to any associative algebra, defined over a field. The purpose of the present paper is to prove that this construction is useful in non-commutative deformation theory for the construction of the versal family of finite families of modules. In particular, we obtain a much better understanding of the obstruction calculus, that is, of the Massey products.

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1 Introduction

In [8], we sketched a physical “toy model,” where the space-time of classical physics became a section of a universal fiber space \tilde{E} , defined on the moduli space $\underline{H} = \text{Simp}(H)$ of the physical systems we chose to consider (in this case, the systems composed of an observer and an observed, both sitting in a Euclidean 3-space). This moduli space was called the *time-space*. Time, in this mathematical model, was defined to be a metric ρ on the time-space, measuring all possible infinitesimal changes of the *state* of the objects in the family we are studying. This gave us a model of relativity theory, in which the set of all (relative) velocities turned out to be a projective space. Dynamics was introduced into this picture, via the general construction, for any associative algebra A , of a *phase space* $\text{Ph}(A)$. This is a universal pair of a homomorphism of algebras, $\iota : A \rightarrow \text{Ph}(A)$, and a derivation, $d : A \rightarrow \text{Ph}(A)$, such that for any homomorphism of A into a k -algebra R , the derivations of A in R are induced by unique homomorphisms $\text{Ph}(A) \rightarrow R$, composed with d . Iterating this $\text{Ph}(-)$ -construction, we obtained a limit morphism $\iota(n) : \text{Ph}^n(A) \rightarrow \text{Ph}^\infty(A)$ with image $\text{Ph}^{(n)}(A)$, and a universal derivation $\delta \in \text{Der}_k(\text{Ph}^\infty(A), \text{Ph}^\infty(A))$, the *Dirac-derivation*. A general *dynamical structure of order n* is now a two-sided δ -ideal σ in $\text{Ph}^\infty(A)$ inducing a surjective homomorphism $\text{Ph}^{(n-1)}(A) \rightarrow \text{Ph}^\infty(A)/\sigma =: A(\sigma)$.

In [8] and later in [10], we have shown that, associated to any such *time space* H with a fixed dynamical structure $H(\sigma)$, there is a kind of “Quantum field theory”. In particular, we have stressed the point that, if H is the affine ring of a moduli space of the objects we want to study, the ring $\text{Ph}^\infty(H)$ is the complete ring of observables, containing the parameters not only of the iso-classes of the objects in question, but also of all dynamical parameters. The choice made by fixing the dynamical structure σ , and reducing to the k -algebra $H(\sigma)$, would classically correspond to the introduction of a parsimony principle (e.g. to the choice of some Lagrangian).

The purpose of this paper is to study this phase-space construction in greater detail. There is a natural descending filtration of two-sided ideals, $\{\mathcal{F}_n\}_{0 \leq n}$ of $\text{Ph}^\infty(A)$. The corresponding quotients $\text{Ph}^n(A)/\mathcal{F}_n$ are finite dimensional vector spaces, and considered as affine varieties; these are our non-commutative Jet-spaces.

We will first see, in Section 2, that we may extend the usual prolongation-projection procedure of Elie Cartan to this non-commutative setting, and obtain a framework for the study of general systems of (non-commutative) PDEs; see also [9].

In Section 3, we present a short introduction to non-commutative deformations of modules, and the generalized Massey products, as exposed in [4, 5].

Then, in Section 4, the main part of the paper follows: the construction for finitely generated associative algebras A of the versal family of the non-commutative deformation functor of any finite family of finitely dimensional A -modules, based on the phase-space of a *resolution* of the k -algebra A .

Notice that our $\text{Ph}^\infty(A)$ is a non-commutative analogue of the notion of higher differentials treated in many texts (see [1] and the more recent paper [2]).

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2 Phase spaces and the Dirac derivation

Given a k -algebra A , denote by $A/k - \underline{\text{alg}}$ the category where the objects are homomorphisms of k -algebras $\kappa : A \rightarrow R$, and the morphisms $\psi : \kappa \rightarrow \kappa'$ are commutative diagrams:

$$\begin{array}{ccc} & A & \\ \kappa \swarrow & & \searrow \kappa' \\ R & \xrightarrow{\psi} & R' \end{array}$$

and consider the functor

$$\text{Der}_k(A, -) : A/k - \underline{\text{alg}} \longrightarrow \underline{\text{Sets}}$$

defined by $\text{Der}_k(A, \kappa) := \text{Der}_k(A, R)$. It is representable by a k -algebra-morphism, $\iota : A \rightarrow \text{Ph}(A)$ with a *universal family* given by a universal derivation $d : A \rightarrow \text{Ph}(A)$. It is easy to construct $\text{Ph}(A)$. In fact, let $\pi : F \rightarrow A$ be a surjective homomorphism of algebras, with $F = k\langle t_1, t_2, \dots, t_r \rangle$, freely generated by the t_i s, and put $I = \ker \pi$. Let,

$$\text{Ph}(A) = k\langle t_1, t_2, \dots, t_r, dt_1, dt_2, \dots, dt_r \rangle / (I, dI),$$

where dt_i is a formal variable. Clearly there is a homomorphism $i'_0 : F \rightarrow \text{Ph}(A)$ and a derivation $d' : F \rightarrow \text{Ph}(A)$, defined by putting $d'(t_i) = \text{cl}(dt_i)$, the equivalence class of dt_i . Since i'_0 and d' both kill the ideal I , they define a homomorphism $i_0 : A \rightarrow \text{Ph}(A)$ and a derivation $d : A \rightarrow \text{Ph}(A)$. To see that i_0 and d have the universal property, let $\kappa : A \rightarrow R$ be an object of the category $A/k - \underline{\text{alg}}$. Any derivation $\xi : A \rightarrow R$ defines a derivation $\xi' : F \rightarrow R$, mapping t_i to $\xi'(t_i)$. Let $\rho_{\xi'} : k\langle t_1, t_2, \dots, t_r, dt_1, dt_2, \dots, dt_r \rangle \rightarrow R$ be the homomorphism defined by

$$\rho_{\xi'}(t_i) = \kappa(\pi(t_i)), \quad \rho_{\xi'}(dt_i) = \xi(\pi(t_i)),$$

where $\rho_{\xi'}$ sends I and dI to zero, and so defines a homomorphism $\rho_{\xi} : \text{Ph}(A) \rightarrow R$, such that the composition with $d : A \rightarrow \text{Ph}(A)$ is ξ . The unicity is a consequence of the fact that the images of i_0 and d generate $\text{Ph}(A)$ as k -algebra.

Clearly $\text{Ph}(-)$ is a covariant functor on $k - \underline{\text{alg}}$, and we have the identities,

$$\begin{aligned} d_* : \text{Der}_k(A, A) &= \text{Mor}_A(\text{Ph}(A), A), \\ d^* : \text{Der}_k(A, \text{Ph}(A)) &= \text{End}_A(\text{Ph}(A)), \end{aligned}$$

with the last one associating d to the identity endomorphism of $\text{Ph}(A)$. In particular, we see that i_0 has a cosection, $\sigma_0 : \text{Ph}(A) \rightarrow A$, corresponding to the trivial (zero) derivation of A .

Let now V be a right A -module, with the structure morphism $\rho(V) =: \rho : A \rightarrow \text{End}_k(V)$. We obtain a universal derivation:

$$u(V) =: u : A \longrightarrow \text{Hom}_k(V, V \otimes_A \text{Ph}(A)),$$

defined by $u(a)(v) = v \otimes d(a)$. Using the long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}_A(V, V \otimes_A \text{Ph}(A)) &\longrightarrow \text{Hom}_k(V, V \otimes_A \text{Ph}(A)) \\ \xrightarrow{\iota} \text{Der}_k(A, \text{Hom}_k(V, V \otimes_A \text{Ph}(A))) &\xrightarrow{\kappa} \text{Ext}_A^1(V, V \otimes_A \text{Ph}(A)) \longrightarrow 0, \end{aligned}$$

we obtain the non-commutative Kodaira-Spencer class

$$c(V) := \kappa(u(V)) \in \text{Ext}_A^1(V, V \otimes_A \text{Ph}(A)),$$

inducing the Kodaira-Spencer morphism

$$g : \Theta_A := \text{Der}_k(A, A) \longrightarrow \text{Ext}_A^1(V, V)$$

via the identity d_* . If $c(V) = 0$, then the exact sequence above proves that there exist a $\nabla \in \text{Hom}_k(V, V \otimes_A \text{Ph}(A))$ such that $u = \iota(\nabla)$. This is just another way of proving that $c(V)$ is the obstruction for the existence of a connection,

$$\nabla : \text{Der}_k(A, A) \longrightarrow \text{Hom}_k(V, V).$$

It is well known, I think, that in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting

$$\text{ch}^i(V) := 1/i! c^i(V) \in \text{Ext}_A^i(V, V \otimes_A \text{Ph}(A))$$

and that if $c(V) = 0$, the curvature $R(\nabla)$ of the connection ∇ induces a curvature class in a generalized Lie-algebra cohomology:

$$R_\nabla \in H^2(k, A; \Theta_A, \text{End}_A(V)).$$

Any $\text{Ph}(A)$ -module W , given by its structure map,

$$\rho(W)^1 =: \rho^1 : \text{Ph}(A) \longrightarrow \text{End}_k(W),$$

corresponds bijectively to an induced A -module structure $\rho : A \rightarrow \text{End}_k(W)$, together with a derivation $\delta_\rho \in \text{Der}_k(A, \text{End}_k(W))$, defining an element $[\delta_\rho] \in \text{Ext}_A^1(W, W)$. Fixing this last element, we find that the set of $\text{Ph}(A)$ -module structures on the A -module W is in one-to-one correspondence with $\text{End}_k(W)/\text{End}_A(W)$. Conversely, starting with an A -module V and an element $\delta \in \text{Der}_k(A, \text{End}_k(V))$, we obtain a $\text{Ph}(A)$ -module V_δ . It is then easy to see that the kernel of the natural map

$$\text{Ext}_{\text{Ph}(A)}^1(V_\delta, V_\delta) \longrightarrow \text{Ext}_A^1(V, V)$$

induced by the linear map

$$\text{Der}_k(\text{Ph}(A), \text{End}_k(V_\delta)) \longrightarrow \text{Der}_k(A, \text{End}_k(V))$$

is the quotient

$$\text{Der}_A(\text{Ph}(A), \text{End}_k(V_\delta)) / \text{End}_k(V)$$

and the image is a subspace $[\delta_\rho]^\perp \subseteq \text{Ext}_A^1(V, V)$, which is rather easy to compute; see examples below.

Remark 1. Defining *time* as a metric on the moduli space, $\text{Simp}(A)$, of simple A -modules, in line with the philosophy of [8], noticing that $\text{Ext}_A^1(V, V)$ is the tangent space of $\text{Simp}(A)$ at the point corresponding to V , we see that the non-commutative space $\text{Ph}(A)$ also parametrizes the set of *generalized momenta*, that is, the set of pairs of a point $V \in \text{Simp}(A)$, and a tangent vector at that point.

Example 2. (i) Let $A = k[t]$, then obviously, $\text{Ph}(A) = k\langle t, dt \rangle$ and d is given by $d(t) = dt$, such that for $f \in k[t]$, we find $d(f) = J_t(f)$ with the notations of [7], that is, the non-commutative derivation of f with respect to t . One should also compare this with the non-commutative Taylor formula of loc.cit. If $V \simeq k^2$ is an A -module, defined by the matrix $X \in M_2(k)$, and $\delta \in \text{Der}_k(A, \text{End}_k(V))$ is defined in terms of the matrix $Y \in M_2(k)$, then the $\text{Ph}(A)$ -module V_δ is the $k\langle t, dt \rangle$ -module defined by the action of the two matrices $X, Y \in M_2(k)$, and we find

$$\begin{aligned} e_V^1 &:= \dim_k \text{Ext}_A^1(V, V) = \dim_k \text{End}_A(V) = \dim_k \{Z \in M_2(k) \mid [X, Z] = 0\}, \\ e_{V_\delta}^1 &:= \dim_k \text{Ext}_{\text{Ph}(A)}^1(V_\delta, V_\delta) = 8 - 4 + \dim \{Z \in M_2(k) \mid [X, Z] = [Y, Z] = 0\}. \end{aligned}$$

We have the following inequalities:

$$2 \leq e_V^1 \leq 4 \leq e_{V_\delta}^1 \leq 8.$$

(ii) Let $A = k[t_1, t_2]$, then we find

$$\text{Ph}(A) = k\langle t_1, t_2, dt_1, dt_2 \rangle / ([t_1, t_2], [dt_1, t_2] + [t_1, dt_2]).$$

In particular, we have a surjective homomorphism

$$\text{Ph}(A) \longrightarrow k\langle t_1, t_2, dt_1, dt_2 \rangle / ([t_1, t_2], [dt_1, dt_2], [t_i, dt_i] - 1),$$

with the right-hand side algebra being the Weyl algebra. This homomorphism exists in all dimensions. We also have a surjective homomorphism,

$$\text{Ph}(A) \longrightarrow k[t_1, t_2, \xi_1, \xi_2],$$

that is, onto the affine algebra of the classical phase-space.

The phase-space construction may, of course, be iterated. Given the k -algebra A , we may form the sequence $\{\text{Ph}^n(A)\}_{0 \leq n}$, defined inductively by

$$\text{Ph}^0(A) = A, \quad \text{Ph}^1(A) = \text{Ph}(A), \dots, \text{Ph}^{n+1}(A) := \text{Ph}(\text{Ph}^n(A)).$$

Let $i_0^n : \text{Ph}^n(A) \rightarrow \text{Ph}^{n+1}(A)$ be the canonical imbedding, and let $d_n : \text{Ph}^n(A) \rightarrow \text{Ph}^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $\text{Ph}^n(A) \rightarrow \text{Ph}^{n+2}(A)$, there exists by universality a homomorphism $i_1^{n+1} : \text{Ph}^{n+1}(A) \rightarrow \text{Ph}^{n+2}(A)$, such that

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

Notice that we compose functions and functors from left to right. Clearly, we may continue this process constructing new homomorphisms

$$\{i_j^n : \text{Ph}^n(A) \rightarrow \text{Ph}^{n+1}(A)\}_{0 \leq j \leq n}$$

with the property

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

Notice also that we have the “bi-gone” $i_0^0 i_0^1 = i_0^1 i_1^0$ and the “hexagone”

$$i_0^1 i_0^2 = i_0^2 i_1^1, \quad i_1^1 i_0^2 = i_0^2 i_1^2, \quad i_1^1 i_1^2 = i_1^2 i_2^1$$

and, in general,

$$i_p^{n,n+1} i_q^{n+1} = i_{q-1}^n i_p^{n+1} \quad (p < q), \quad i_p^{n,n+1} = i_p^n i_{p+1}^{n+1}, \quad i_p^{n,n+1} = i_q^n i_{p+1}^{n+1} \quad (q < p)$$

which is all easily proved by composing with i_0^{n-1} and d_{n-1} . Thus, the $\text{Ph}^*(A)$ is a semi-cosimplicial algebra with a cosection onto A . Therefore, for any object

$$\kappa : A \rightarrow R \in A/k\text{-alg}$$

the semi-cosimplicial algebra above induces a semi-simplicial k -vector space, $\text{Der}_k(\text{Ph}^*(A), R)$, and one should be interested in its homology.

The system of k -algebras and homomorphisms of k -algebras $\{\text{Ph}^n(A), i_j^n\}_{n, 0 \leq j \leq n}$ has an inductive (direct) limit, $\text{Ph}^\infty(A)$, together with homomorphisms $i_n : \text{Ph}^n(A) \rightarrow \text{Ph}^\infty(A)$ satisfying

$$i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \dots, n.$$

Moreover, the family of derivations $\{d_n\}_{0 \leq n}$ define a unique derivation $\delta : \text{Ph}^\infty(A) \rightarrow \text{Ph}^\infty(A)$, such that $i_n \circ \delta = d_n \circ i_{n+1}$. Put

$$\text{Ph}^{(n)}(A) := \text{im } i_n \subseteq \text{Ph}^\infty(A).$$

The k -algebra $\text{Ph}^\infty(A)$ has a descending filtration of two-sided ideals, with $\{\mathcal{F}_n\}_{0 \leq n}$ given inductively by

$$\mathcal{F}_1 = \text{Ph}^\infty(A) \cdot \text{im}(\delta) \cdot \text{Ph}^\infty(A),$$

$$\delta \mathcal{F}_n \subseteq \mathcal{F}_{n+1}, \quad \mathcal{F}_{n_1} \mathcal{F}_{n_2} \cdot \mathcal{F}_{n_r} \subseteq \mathcal{F}_n, \quad n_1 + \dots + n_r = n,$$

such that the derivation δ induces derivations $\delta_n : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1}$. Using the canonical homomorphism $i_n : \text{Ph}^n(A) \rightarrow \text{Ph}^\infty(A)$, we pull the filtration $\{\mathcal{F}_p\}_{0 \leq p}$ back to $\text{Ph}^n(A)$, not bothering to change the notation.

Definition 3. Let $\mathcal{D}(A) := \varprojlim_{n \geq 1} \text{Ph}^\infty(A)/\mathcal{F}_n$ be the completion of $\text{Ph}^\infty(A)$ in the topology given by the filtration $\{\mathcal{F}_n\}_{0 \leq n}$. The k -algebra $\text{Ph}^\infty(A)$ will be referred to as the k -algebra of higher differentials, and $\mathcal{D}(A)$ will be called the k -algebra of formalized higher differentials. Put

$$\mathcal{D}_n := \mathcal{D}_n(A) := \text{Ph}^\infty(A)/\mathcal{F}_{n+1}.$$

Clearly, δ defines a derivation on $\mathcal{D}(A)$, and an isomorphism of k -algebras

$$\epsilon := \exp(\delta) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$$

and, in particular, an algebra homomorphism

$$\tilde{\eta} := \exp(\delta) : A \longrightarrow \mathcal{D}(A),$$

inducing the algebra homomorphisms

$$\tilde{\eta}_n : A \longrightarrow \mathcal{D}_n(A)$$

which, by killing, in the right-hand side algebra, the image of the maximal ideal, $\mathfrak{m}(\underline{t})$, of A corresponding to a point $\underline{t} \in \text{Simp}_1(A)$, induces a homomorphism of k -algebras

$$\tilde{\eta}_n(\underline{t}) : A \longrightarrow \mathcal{D}_n(A)(\underline{t}) := \mathcal{D}_n/(\mathcal{D}_n\mathfrak{m}(\underline{t})\mathcal{D}_n)$$

and an injective homomorphism

$$\tilde{\eta}(\underline{t}) : A \longrightarrow \varinjlim_{n \geq 1} \mathcal{D}_n(A)(\underline{t});$$

see [8]. More generally, let A be a finitely generated k -algebra and let $\rho : A \rightarrow \text{End}_k(V)$ be an n -dimensional representation (e.g. a point of $\text{Simp}_n(A)$) corresponding to a two-sided ideal $\mathfrak{m} = \ker \rho$ of A . Then $\tilde{\eta}$ induces a homomorphism

$$\tilde{\eta}(\mathfrak{m}) : A \longrightarrow \mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$$

and we will be interested in the image; see Section 4.

The k -algebras $\text{Ph}^n(A)$ are our generalized jet spaces. In fact, any homomorphism of A -algebras

$$P_n : \text{Ph}^n(A) \longrightarrow A$$

composed with

$$\delta^n : A \longrightarrow \text{Ph}^n(A)$$

is a usual differential operator of order $\leq n$ on A . Notice also the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{d^{n-1}} & \text{Ph}^{n-1}(A) & \xrightarrow{d} & \text{Ph}^n(A) & \xrightarrow{d} & \text{Ph}^{n+1}(A) & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \mathcal{F}_{p-1} & \xrightarrow{d} & \mathcal{F}_p & \xrightarrow{d} & \mathcal{F}_{p+1} & \xrightarrow{d} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{F}_{p-1}/\mathcal{F}_p & \xrightarrow{d} & \mathcal{F}_p/\mathcal{F}_{p+1} & \xrightarrow{d} & \mathcal{F}_{p+1}/\mathcal{F}_{p+2} & \xrightarrow{d} & \dots \end{array}$$

Here the upper vertical morphisms are injective, with the lower line being the sequence of *symbols*.

It is easy to see that the differential operators form an associative k -algebra, $\text{Diff}(A)$. In fact, assume two differential operators

$$P_m : \text{Ph}^m(A) \longrightarrow A, \quad P_n : \text{Ph}^n(A) \longrightarrow A,$$

and consider the functorially defined diagram

$$\begin{array}{ccc} A & \xrightarrow{d^m} & \text{Ph}^m(A) & \xrightarrow{d^n} & \text{Ph}^{m+n}(A) \\ & & \downarrow P_m & & \downarrow \text{Ph}^n(P_m) \\ & & A & \xrightarrow{d^n} & \text{Ph}^n(A) \\ & & & & \downarrow P_n \\ & & & & A, \end{array}$$

then the product is defined by the composition

$$P_m P_n = \text{Ph}^{(n)}(P_m) \circ P_n.$$

Let now V be, as above, a right A -module, with structure morphism $\rho(V) : A \rightarrow \text{End}_k(V)$. Consider the linear map

$$\iota_n := id \otimes (i_1 \circ \cdots \circ i_n) : V \otimes_A \text{Ph} A \longrightarrow V \otimes_A \text{Ph}^{n+1}, \quad n \geq 0.$$

Assume that the non-commutative Kodaira-Spencer class, defined above,

$$c(V) := \kappa(u(V)) \in \text{Ext}_A^1(V, V \otimes_A \text{Ph}(A)),$$

vanishes. Then, as we know, there exist a connection, that is, a linear map

$$\nabla_0 \in \text{Hom}_k(V, V \otimes_A \text{Ph}(A))$$

such that $u(V) = \iota(\nabla_0)$. It is also easy to see that this connection induces higher-order connections, that is, k -linear maps,

$$\nabla(n) \in \text{Hom}_k(V \otimes_A \text{Ph}^n(A), V \otimes_A \text{Ph}^{n+1}(A)), \quad n \geq 0,$$

defined by

$$\nabla(n)(v \otimes f) = \iota_n(\nabla_0(v))i_0(f) + v \otimes d_n(f).$$

In fact, we just have to prove that $\nabla(n)$ is well defined, that is, we have to prove that

$$\nabla(n)(va \otimes f) = \nabla(n)(v \otimes af), \quad \forall a \in A, f \in \text{Ph}^n(A).$$

Noticing that

$$\iota_n(v \otimes d_0(a)) = v \otimes d_n(a),$$

where we have put $a := i_0 \circ \cdots \circ i_0(a)$, we find

$$\begin{aligned} \nabla(n)(va \otimes f) &= \iota_n(\nabla_0(va))i_0(f) + va \otimes d_n(f) \\ &= \iota_n(\nabla_0(v)i_0(a) + v \otimes da)i_0(f) + v \otimes ad_n f \\ &= \iota_n(\nabla_0(v))i_0(af) + v \otimes d_n a i_0(f) + v \otimes ad_n f \\ &= \nabla(n)(v \otimes af). \end{aligned}$$

These higher-order connections will induce a diagram

$$\begin{array}{ccccccc} V \otimes_A \text{Ph}^{n-1}(A) & \xrightarrow{\nabla(n-1)} & V \otimes_A \text{Ph}^n(A) & \xrightarrow{\nabla(n)} & V \otimes_A \text{Ph}^{n+1}(A) & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ V \otimes_A \mathcal{F}_{p-1} & \xrightarrow{\nabla(n-1)} & V \otimes_A \mathcal{F}_p & \xrightarrow{\nabla(n)} & V \otimes_A \mathcal{F}_{p+1} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ V \otimes_A \mathcal{F}_{p-1}/\mathcal{F}_p & \xrightarrow{\nabla(n-1)} & V \otimes_A \mathcal{F}_p/\mathcal{F}_{p+1} & \xrightarrow{\nabla(n)} & V \otimes_A \mathcal{F}_{p+1}/\mathcal{F}_{p+2} & \longrightarrow & \cdots, \end{array}$$

where the lower line is the sequence of symbols. Notice that

$$\nabla^n \in \text{Hom}_k(V, V \otimes \text{Ph}^n(A)),$$

as given above, by definition has the property that for all $a \in A$ and all $v \in V$ we have

$$\nabla^n(va) = \nabla^n(v)a + \nabla^{n-1}(v)da + \cdots + v \otimes d^n(a).$$

Assume, in particular, that V and the A -module W are free of ranks p and q , respectively. Let $\{P_{i,j}\}_{i=1,\dots,p, j=1,\dots,q}$ be a family of A -homomorphisms $\text{Ph}^n(A) \rightarrow A$, defining a *generalized differential operator*

$$\mathfrak{D} := \begin{pmatrix} P_{1,1} & \cdots & P_{1,p} \\ \cdots & & \\ P_{q,1} & \cdots & P_{q,p} \end{pmatrix} \circ \nabla^n : V \rightarrow W.$$

The solution space of \mathfrak{D} is by definition $\mathbf{S}(\mathfrak{D}) := \ker \mathfrak{D}$. There are natural generalizations of this set-up, which we will, hopefully, return to in a later paper, extending the classical prolongation-projection method of Elie Cartan to this non-commutative setting. See Example 4 for the commutative analogue.

In [8], we introduced the notion of a *dynamical structure* for a k -algebra A , as a two-sided δ -stable ideal $\sigma \subset \text{Ph}^\infty(A)$, or equivalently as the corresponding quotient $A(\sigma)$ of the δ -algebra $\text{Ph}^\infty(A)$. Any such $A(\sigma)$ will be given in terms of a sequence of ideals, $\sigma_n \subset \text{Ph}^n(A)$ ($n \geq 0$), with the property that $d(\sigma_n) \subset \sigma_{n+1}$. The *solution space* of such a system, should be considered as the non-commutative scheme parametrized by $A(\sigma)$, that is, as the geometric system of all simple representations of $A(\sigma)$; see [6].

This is, in a sense, dual to the classical theory of PDEs, as we will show by considering the following example, leaving the general situation to the hypothetical paper referred to above.

Example 4 (see [9]). (i) Let $A = k[t_1, t_2, \dots, t_n]$, and consider the situation corresponding to a *free particle* (see [8]) that is, where we have obtained $A(\sigma)$ by killing $d^2 t_i$, for every $i = 1, 2, \dots, n$, then the commutativization $A(\sigma)_k^{\text{com}}$ of $A(\sigma)_k := \text{Ph}(A)/\mathcal{F}_{k+1}$ is a free A -module generated by the basis

$$\{dt_{i_1} dt_{i_2} \cdots dt_{i_r}\}_{i_1 \leq i_2 \leq \dots \leq i_r, r \leq k}.$$

Put $|\underline{i}| = r$ if $\underline{i} = \{i_1, i_2, \dots, i_r\}$. The dual basis $\{p_{\underline{i}}\}_{i_1 \leq i_2 \leq \dots \leq i_r, r \leq k}$ may be identified with a basis $D_{\underline{i}}$ of the A -module of all (classical higher-order) differential operators of order less or equal to k . In fact, consider the composition

$$\tilde{\eta} : A \rightarrow \mathcal{D}(A) \rightarrow A(\sigma)_k,$$

then, for $f \in A$ we have

$$p_{\underline{i}}(\tilde{\eta}(f)) = \frac{1}{\mu_1! \mu_2! \cdots \mu_s!} D_{j_1}^{\mu_1} D_{j_2}^{\mu_2} \cdots D_{j_r}^{\mu_r}(f),$$

where we assume

$$j_1 = i_1 = i_2 = \cdots = i_{\mu_1} < j_2 = i_{\mu_1+1} = i_{\mu_1+2} = \cdots = i_{\mu_1+\mu_2} < \cdots < i_{\mu_1+\dots+\mu_s} = j_r$$

and where $D_{i_p}^{\mu_p}$ is μ_p th-order derivation with respect to t_{i_p} . If $\{i_1, i_2, \dots, i_r\} = \emptyset$, we let $D_{\underline{i}}$ to be the identity operator on A .

Now, consider the commutativization of $A(\sigma)$, as a k -linear space, and for every $k \geq 1$,

$$\mathcal{E}_k := A(\sigma)_k^{\text{com}}$$

as a family of affine spaces fibered over $\text{Simp}_1(A)$,

$$\pi_k : \mathcal{E}_k \rightarrow \text{Spec}(A).$$

This family is defined by the homomorphism of k -algebras

$$A \rightarrow \mathcal{O}(\mathcal{E}_k) := A[p_{\underline{i}}], \quad |\underline{i}| \leq k.$$

Let $P_q(t, p_{\underline{i}}) \in \mathcal{O}(\mathcal{E}_{k_q})$, $q = 1, \dots, d$, then the system of equations

$$P_q = 0, \quad q = 1, \dots, d$$

is a system of partial differential equations (an SPDE, for short). Suppose there is a *solution*, that is, an $f \in A$, such that

$$P_q(D_{\underline{i}}(f)) = 0, \quad q = 1, \dots, d,$$

then, for every j , we must have

$$D_j(P_q(D_{\underline{i}}(f))) = 0$$

which amounts to extending the SPDE by, including together with $P_q \in \mathcal{O}(\mathcal{E}_{k_q})$, the polynomials

$$D_j P_q := \frac{\partial P_q}{\partial t_j} + \sum_{\underline{i}} \frac{\partial P_q}{\partial p_{\underline{i}}} p_{\underline{i}+j} \in \mathcal{O}(\mathcal{E}_{k_q+1}),$$

where it should be clear how to interpret the indices. Let us denote by \mathcal{P} the extended family of polynomials,

$$\{D_{j_1} \cdots D_{j_l} P_q \mid P_q \in \mathcal{O}(\mathcal{E}_{k_q})\}_{j_l, q \geq 0}$$

and let $\mathfrak{p}_m \subset \mathcal{O}(\mathcal{E}_m)$ be the ideal, generated by the polynomials in \mathcal{P} , contained in $\mathcal{O}(\mathcal{E}_m)$. Denote by $S_m := S_m \mathcal{P} \subset \mathcal{E}_m$ the corresponding subvariety. Clearly, the canonical map $\mathcal{E}_{m+1} \rightarrow \mathcal{E}_m$ induced by the trivial derivation of $\text{Ph}^m(A)$ has a canonical restriction $pl_l : S_{m+1} \rightarrow S_m$. Denote also by $\pi_k : S_k \rightarrow \text{Spec}(A)$ the restriction of the morphism $\pi_k : \mathcal{E}_m \rightarrow \text{Spec}(A)$, defined above, to S_k . Classically, the system is called regular if all π_k are fiber bundles, so smooth, for all $k \geq 1$. Now, for any closed point of $\text{Spec}(A)$, that is, for any point $\underline{t} \in \text{Simp}_1(A)$, consider the sequence of fibers over \underline{t} , and the corresponding sequence of maps $pl_1(\underline{t}) : S_{m+1}(\underline{t}) \rightarrow S_m(\underline{t})$. An element $\tilde{f} \in \varprojlim_m S_m(\underline{t})$ corresponds exactly to an element $\hat{f} \in \hat{A}_{\underline{t}}$, for which,

$$P_q(D_{\underline{i}}(\hat{f})) = 0, \quad q = 1, \dots, d,$$

that is, to a formal solution of the SPDE. Thus, the projective limit of schemes $\mathcal{SP}(\underline{t}) := \varprojlim_m S_m(\underline{t})$ is the space of formal solutions of the SPDE at $\underline{t} \in \text{Simp}_1(A)$.

A fundamental problem in the classical theory of PDE is then the following.

Find necessary and sufficient conditions on the SPDE $\{P_q\}_{q=1, \dots, d}$ for $\mathcal{SP}(\underline{t})$ to be non-empty, and find, based on $\{P_l\}_l$, its structure. In particular, compute its dimension $\sigma(\underline{t})$.

We will not, here, venture into this vast theory, but just add one remark. The solution space is in fact a family, with parameter-space $\text{Simp}_1(A)$. Given any point $\underline{t} \in \text{Simp}_1(A)$, the (formal) scheme, $\mathcal{SP}(\underline{t})$, of formal solutions may have deformations. We might want to compute the formal moduli \underline{H} , and relate the given family to the corresponding mini-versal family.

The tangent space of \underline{H} is given as

$$A^1(k, \mathcal{O}(\mathcal{SP}(\underline{t})), \mathcal{O}(\mathcal{SP}(\underline{t}))) = \text{Hom}_{\mathcal{O}(\mathcal{E})(\underline{t})}(\mathfrak{p}(\underline{t}), \mathcal{O}(\mathcal{SP}(\underline{t}))) / \text{Der};$$

see [3]. A tangent at the point \underline{t} of $\text{Simp}_1(A)$ is the value at \underline{t} of a linear combination of the fundamental vector fields, the derivations $\{D_j\}$ of A . The map between the tangent space of the given family and the tangent space of H is then easily seen to be the following:

$$\eta : T_{\text{Simp}_1(A), \underline{t}} \longrightarrow \text{Hom}_{\mathcal{O}(\mathcal{E})(\underline{t})}(\mathfrak{p}(\underline{t}), \mathcal{O}(\mathcal{SP}(\underline{t}))) / \text{Der},$$

where $\eta(D_j)$ is the class of the map, associating a $P \in \mathfrak{p}$ to the class at \underline{t} of $D_j(P)$. The image of the tangent at \underline{t} of $\text{Simp}_1(A)$, corresponding to D_j , in the tangent space of H , is zero if this map is a derivation. Now, this is exactly what we have arranged, together with any $P \in \mathfrak{p}$, and also including

$$D_j P := \frac{\partial P}{\partial t_j} + \sum_{\underline{i}} \frac{\partial P}{\partial p_{\underline{i}}} p_{\underline{i}+j}$$

in the ideal \mathfrak{p} . Thus, the map η is trivial, and the given pro-family is formally constant, as one probably should have suspected! Moreover, it is easy to see that if $pl_1 : S_{k+1}(\underline{t}) \rightarrow S_k(\underline{t})$ has a local section, then $\pi_k : S_k \rightarrow \text{Spec}(A)$ is formally constant at $\underline{t} \in \text{Spec}(A)$. The basic problem is to find computable conditions under which the constancy of π_k implies the surjectivity of pl_1 , and thereby the non-triviality of $\mathcal{SP}(\underline{t})$.

We will, hopefully, come back to these questions in a later paper.

(ii) Let $A = k[t]/(t^2)$, then

$$\text{Ph}(A) = k\langle t, dt \rangle / (t^2, tdt + dtt),$$

$$\text{Ph}^{(2)}(A) = k\langle t, dt, d^2t \rangle / (t^2, tdt + dtt, td^2t + 2dt^2 + d^2tt),$$

$$\text{Ph}^\infty(A) = k\langle t, dt, \dots, d^n t, \dots \rangle / (t^2, tdt + dtt, \dots, td^n t + ndtd^{n-1}t + \dots + d^n tt, \dots),$$

and it is easy to see that $\eta(t) = \sum_n 1/n!d^n(t)$ is non-zero in $\mathcal{D}/(\mathcal{D}(t)\mathcal{D})$, and, of course, $\eta(t)^2 = 0$. In particular, there is a homomorphism onto

$$\mathcal{D}/(\mathcal{D}(t)\mathcal{D}) \longrightarrow k[dt]/(dt^2) \simeq A.$$

(iii) Let now $A = k[x, y]/(x^3 - y^2)$, compute \mathcal{D} , and see that $dy^2 = 0$ in $\mathcal{D}/(\mathcal{D}(x, y)\mathcal{D})$, so that there are no natural surjective homomorphisms $\mathcal{D}/(\mathcal{D}(x, y)\mathcal{D}) \rightarrow A$. The map $\tilde{\eta} := \exp(\delta) : A \rightarrow \mathcal{D}$ is, however, injective. The difference between examples (i) and (ii) is, of course, due to the fact that in the first case A is graded, and in the second it is not; see Section 4.

3 Non-commutative deformations of families of modules

In [5,6,7], we introduced non-commutative deformations of families of modules of non-commutative k -algebras, and the notion of *swarm* of right modules (or more generally of objects in a k -linear abelian category). Let \underline{a}_r denote the category of r -pointed not necessarily commutative k -algebras R . The objects are the diagrams of k -algebras

$$k^r \xrightarrow{\iota} R \xrightarrow{\pi} k^r$$

such that the composition of ι and π is the identity. Any such r -pointed k -algebra R is isomorphic to a k -algebra of $r \times r$ -matrices $(R_{i,j})$. The radical of R is the bilateral ideal $\text{Rad}(R) := \ker \pi$, such that $R/\text{Rad}(R) \simeq k^r$. The dual k -vector space of $\text{Rad}(R)/\text{Rad}(R)^2$ is called the tangent space of R .

For $r = 1$, there is an obvious inclusion of categories $\underline{l} \subseteq \underline{a}_1$, where \underline{l} , as usual, denotes the category of commutative local Artinian k -algebras with residue field k .

Fix a (not necessarily commutative) associative k -algebra A and consider a right A -module M . The ordinary deformation functor $\text{Def}_M : \underline{l} \rightarrow \underline{\text{Sets}}$ is then defined. Assuming $\text{Ext}_A^i(M, M)$ has a finite k -dimension for $i = 1, 2$, it is well known (see [12] or [5]) that Def_M has a pro-representing hull H , the *formal moduli* of M . Moreover, the tangent space of H is isomorphic to $\text{Ext}_A^1(M, M)$, and H can be computed in terms of $\text{Ext}_A^i(M, M)$, $i = 1, 2$, and their *matrix Massey products*; see [5].

In the general case, consider a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right A -modules. Assume that $\dim_k \text{Ext}_A^1(V_i, V_j) < \infty$. Any such family of A -modules will be called a *swarm*. We will define a deformation functor $\text{Def}_{\mathcal{V}} : \underline{a}_r \rightarrow \underline{\text{Sets}}$ generalizing the functor Def_M above. Given an object $\pi : R = (R_{i,j}) \rightarrow k^r$ of \underline{a}_r , consider the k -vector space and the left R -module $(R_{i,j} \otimes_k V_j)$. It is easy to see that

$$\text{End}_R((R_{i,j} \otimes_k V_j)) \simeq (R_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)).$$

Clearly, π defines a k -linear and left R -linear map

$$\pi(R) : (R_{i,j} \otimes_k V_j) \longrightarrow \bigoplus_{i=1}^r V_i,$$

inducing a homomorphism of R -endomorphism rings,

$$\tilde{\pi}(R) : (R_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)) \longrightarrow \bigoplus_{i=1}^r \text{End}_k(V_i).$$

The right A -module structure on the V_i s is defined by a homomorphism of k -algebras:

$$\eta_0 : A \longrightarrow \bigoplus_{i=1}^r \text{End}_k(V_i) \subset (\text{Hom}_k(V_i, V_j)) =: \text{End}_k(V).$$

Notice that this homomorphism also provides each $\text{Hom}_k(V_i, V_j)$ with an A -bimodule structure. Let $\text{Def}_{\mathcal{V}}(R) \in \underline{\text{Sets}}$ be the set of isoclasses of homomorphisms of k -algebras,

$$\eta' : A \longrightarrow (R_{i,j} \otimes_k \text{Hom}_k(V_i, V_j))$$

such that $\tilde{\pi}(R) \circ \eta' = \eta_0$, where the equivalence relation is defined by inner automorphisms in the k -algebra $(R_{i,j} \otimes_k \text{Hom}_k(V_i, V_j))$ inducing the identity on $\bigoplus_{i=1}^r \text{End}_k(V_i)$. One easily proves that $\text{Def}_{\mathcal{V}}$ has the same properties as the ordinary deformation functor and we may prove the following theorem (see [5]).

Theorem 5. *The functor $\text{Def}_{\mathcal{V}}$ has a pro-representable hull, that is, an object H of the category of pro-objects $\hat{\underline{a}}_r$ of \underline{a}_r , together with a versal family*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} \text{Def}_{\mathcal{V}}(H/\mathfrak{m}^n),$$

where $\mathfrak{m} = \text{Rad}(H)$, such that the corresponding morphism of functors on \underline{a}_r

$$\kappa : \text{Mor}(H, -) \longrightarrow \text{Def}_{\mathcal{V}},$$

defined for $\phi \in \text{Mor}(H, R)$ by $\kappa(\phi) = R \otimes_{\phi} \tilde{V}$, is smooth and an isomorphism on the tangent level. Moreover, H is uniquely determined by a set of matrix Massey products defined on subspaces

$$D(n) \subseteq \text{Ext}^1(V_i, V_{j_1}) \otimes \cdots \otimes \text{Ext}^1(V_{j_{n-1}}, V_k)$$

with values in $\text{Ext}^2(V_i, V_k)$.

The right action of A on \tilde{V} defines a homomorphism of k -algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := \text{End}_H(\tilde{V}) = (H_{i,j} \otimes \text{Hom}_k(V_i, V_j))$$

and the k -algebra $O(\mathcal{V})$ acts on the family of A -modules $\mathcal{V} = \{V_i\}$, extending the action of A . If $\dim_k V_i < \infty$, for all $i = 1, \dots, r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to (A, \mathcal{V}) turns out to be a closure operation.

Moreover, we prove the crucial result.

Theorem 6 (a generalized Burnside theorem). *Let A be a finite dimensional k -algebra, with k being an algebraically closed field. Consider the family $\mathcal{V} = \{V_i\}_{i=1}^r$ of all simple A -modules, then*

$$\eta : A \longrightarrow O(\mathcal{V}) = (H_{i,j} \otimes \text{Hom}_k(V_i, V_j))$$

is an isomorphism.

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum H_0 of the formal moduli H ; see [3]. The tangent space of H_0 is determined by a family of subspaces

$$\text{Ext}_0^1(V_i, V_j) \subseteq \text{Ext}_A^1(V_i, V_j), \quad i \neq j,$$

the elements of which should be called the almost split extensions (sequences) relative to the family \mathcal{V} , and by a subspace

$$T_0(\Delta) \subseteq \prod_i \text{Ext}_A^1(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A -modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$; see [4]. If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some Artinian k -algebra A , we show that the above notion of *almost split sequence* coincides with that of Auslander; see [11].

Using this we consider, in [5, 7], the general problem of classification of iterated extensions of a family of modules $\mathcal{V} = \{V_i\}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family \mathcal{V} , and extension type given by a directed representation graph Γ . The main result is the following; see [7].

Proposition 7. *Let A be any k -algebra and $\mathcal{V} = \{V_i\}_{i=1}^r$ any swarm of A -modules, such that*

$$\dim_k \text{Ext}_A^1(V_i, V_j) < \infty, \quad \forall i, j = 1, \dots, r.$$

- (i) *Consider an iterated extension E of \mathcal{V} , with representation graph Γ . Then there exists a morphism of k -algebras $\phi : H_{\mathcal{V}} \rightarrow k[\Gamma]$ such that $E \simeq k[\Gamma] \otimes_{\phi} \tilde{V}$ as right A -algebras.*
- (ii) *The set of equivalence classes of iterated extensions of \mathcal{V} with representation graph Γ is a quotient of the set of closed points of the affine algebraic variety $\underline{A}[\Gamma] = \text{Mor}(H_{\mathcal{V}}, k[\Gamma])$.*
- (iii) *There is a versal family $\tilde{V}[\Gamma]$ of A -modules defined on $\underline{A}[\Gamma]$, containing as fibers all the isomorphism classes of iterated extensions of \mathcal{V} with representation graph Γ .*

To any, not necessarily finite, swarm $\underline{c} \subset \underline{\text{mod}}(A)$ of right- A -modules, we have associated two associative k -algebras (see [6, 7]):

$$O(|\underline{c}|, \pi) = \varprojlim_{\mathcal{V} \subset |\underline{c}|} O(\mathcal{V})$$

and a sub-quotient $\mathcal{O}_\pi(\underline{c})$, together with natural k -algebra homomorphisms

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

and $\eta(\underline{c}) : A \rightarrow \mathcal{O}_\pi(\underline{c})$ with the property that the A -module structure on \underline{c} is extended to an \mathcal{O} -module structure in an optimal way. We then defined an *affine non-commutative scheme* of right A -modules to be a swarm \underline{c} of right A -modules, such that $\eta(\underline{c})$ is an isomorphism. In particular, we considered, for finitely generated k -algebras, the swarm $\text{Simp}_{<\infty}^*(A)$ consisting of the finite dimensional simple A -modules, and the *generic point* A , together with all morphisms between them. The fact that this is a swarm, that is for all objects $V_i, V_j \in \text{Simp}_{<\infty}$ we have $\dim_k \text{Ext}_A^1(V_i, V_j) < \infty$, is easily proved. We have in [7] proved the following result (see [7, Proposition 4.1] for the definition of the notion of *geometric k -algebra*)

Proposition 8. *Let A be a geometric k -algebra, then the natural homomorphism*

$$\eta(\text{Simp}^*(A)) : A \longrightarrow \mathcal{O}_\pi(\text{Simp}_{<\infty}^*(A))$$

is an isomorphism, that is, $\text{Simp}_{<\infty}^(A)$ is a scheme for A .*

In particular, $\text{Simp}_{<\infty}^*(k\langle x_1, x_2, \dots, x_d \rangle)$ is a scheme for $k\langle x_1, x_2, \dots, x_d \rangle$. To analyze the local structure of $\text{Simp}_n(A)$, we need the following lemma (see [7, Lemma 3.23]).

Lemma 9. *Let $\mathcal{V} = \{V_i\}_{i=1, \dots, r}$ be a finite subset of $\text{Simp}_{<\infty}(A)$, then the morphism of k -algebras,*

$$A \longrightarrow O(\mathcal{V}) = (H_{i,j} \otimes_k \text{Hom}_k(V_i, V_j))$$

is topologically surjective.

Proof. Since the simple modules V_i ($i = 1, \dots, r$) are distinct, there is an obvious surjection

$$\eta_0 : A \longrightarrow \prod_{i=1, \dots, r} \text{End}_k(V_i).$$

Put $\mathfrak{r} = \ker \eta_0$, and consider for $m \geq 2$ the finite dimensional k -algebra, $B := A/\mathfrak{r}^m$. Clearly, $\text{Simp}(B) = \mathcal{V}$ so that by the generalized Burnside theorem (see [5, Theorem 3.4]) we find

$$B \simeq O^B(\mathcal{V}) := (H_{i,j}^B \otimes_k \text{Hom}_k(V_i, V_j)).$$

Consider the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & (H_{i,j}^A \otimes_k \text{Hom}_k(V_i, V_j)) =: O^A(\mathcal{V}) \\ \downarrow & & \downarrow \\ B & \longrightarrow & (H_{i,j}^B \otimes_k \text{Hom}_k(V_i, V_j)) \xrightarrow{\alpha} O^A(\mathcal{V})/\mathfrak{m}^m, \end{array}$$

where all morphisms are natural. In particular α exists since $B = A/\mathfrak{r}^m$ maps into $O^A(\mathcal{V})/\text{Rad}^m$, and therefore induces the morphism α commuting with the rest of the morphisms. Consequently, α has to be surjective, and we have proved the contention. \square

Example 10. As an example of what may occur in rank infinity, we will consider the invariant problem $\mathbf{A}_{\mathbf{C}}^1/\mathbf{C}^*$. Here we are talking about the algebra $A = \mathbf{C}[x](\mathbf{C}^*)$ crossed product of $\mathbf{C}[x]$ with the group \mathbf{C}^* . If $\lambda \in \mathbf{C}^*$, the product in A is given by $x \times \lambda = \lambda \times \lambda^{-1}x$. There are two ‘‘points’’ (i.e. orbits) modeled by the obvious origin $V_0 := A \rightarrow \text{End}_{\mathbf{C}}(\mathbf{C}(0))$, and by $V_1 := A \rightarrow \text{End}_{\mathbf{C}}(\mathbf{C}[x, x^{-1}])$. We may also choose the two points $V_0 := \mathbf{C}(0), V_1 := \mathbf{C}[x]$, in line with the definitions of [6]. Obviously, $\mathbf{C}[x]$ corresponds to the closure of the orbit $\mathbf{C}[x, x^{-1}]$. This choice is the best if we want to make visible the adjacencies in the quotient, and we will therefore treat both cases.

We need to compute

$$\text{Ext}_A^p(V_i, V_j), \quad p = 1, 2, \quad i, j = 1, 2.$$

Now,

$$\text{Ext}_A^1(V_i, V_j) = \text{Der}_{\mathbf{C}}(A, \text{Hom}_{\mathbf{C}}(V_i, V_j)) / \text{Triv}, \quad i, j = 1, 2$$

and since x acts as zero on V_1 , and \mathbf{C}^* acts as identity on V_1 and as a homogenous multiplication on V_0 , we find

$$\text{Der}_k(A, \text{Hom}_k(V_0, V_0)) / \text{Triv} = \text{Der}_k(A, \text{Hom}_k(V_0, V_0)) = \text{Der}_{\mathbf{C}}(A, \mathbf{C}(0)).$$

Any $\delta \in \text{Der}_k(A, \mathbf{C}(0))$ is determined by its values $\delta(x), \delta(\lambda) \in \mathbf{C}(0) \mid \lambda \in \mathbf{C}^*$. Moreover, since in A we have $(\lambda) \times (\lambda^{-1}x) = x \times (\lambda)$, we find

$$\delta(\lambda\mu) = \delta(\lambda) + \delta(\mu), \quad \delta((\lambda) \times (\lambda^{-1}x)) = \delta(x \times (\lambda)).$$

The left-hand side of the last equation is $\delta((\lambda^{-1}x)) = \lambda^{-1}\delta(x)$, and the right-hand side is $\delta(x)$, and since this must hold for all $\lambda \in \mathbf{C}^*$, we must have $\delta(x) = 0$. Moreover, since $\delta(\lambda\mu) = \delta(\lambda) + \delta(\mu)$, it is clear that the continuity of δ implies that δ must be equal to $\alpha \ln(| \cdot |)$, for some $\alpha \in \mathbf{C}$. (To simplify the writing, we will put $\log := \ln(| \cdot |)$.) Therefore,

$$\text{Ext}_A^1(V_0, V_0) = \text{Der}_k(A, \text{Hom}_{\mathbf{C}}(V_0, V_0)) = \mathbf{C}.$$

The cup-product of this class, $\log \cup \log$, sits in $HH^2(A, \mathbf{C}(0)) = \text{Ext}_a^2(V_0, V_0)$, and is given by the 2-cocycle

$$(\lambda, \mu) \longrightarrow \log(\lambda) \times \log(\mu).$$

This is seen to be a boundary, that is, there exists a map $\psi : \mathbf{C}^* \rightarrow \mathbf{C}(0)$, such that for all $\lambda, \mu \in \mathbf{C}^*$ we have

$$\log(\lambda) \times \log(\mu) = \psi(\lambda) - \psi(\lambda\mu) + \psi(\mu).$$

Just put $\psi_{1,1} := \psi_2 = -1/2 \log^2$. Therefore, the cup product is zero, and if we, in general, put

$$\psi_n := \psi_{1,1,\dots,1} = (-)^{n+1} 1/(n!) \log^n, \quad n \geq 1,$$

where n is the number of 1s in the first index, then computing the Massey products of the element $\log \in \text{Ext}_A^1(V_0, V_0)$, we find the n th Massey product

$$[\log, \log, \dots, \log] = \left\{ (\lambda, \mu) \longrightarrow \sum_{p=1, \dots, n-1} \psi_p \psi_{n-p} \right\}$$

and this is easily seen to be the boundary of the 1-cochain

$$\psi_{n+1} = (-)^{n+2} 1/((n+1)!) \log^{n+1}.$$

Therefore, all Massey products are zero. Of course, we have not yet proved that they could be different from zero, that is, we have not computed the *obstruction* group $\text{Ext}_A^2(V_0, V_0)$ and found it non-trivial! Now this is unnecessary.

Now, assume first $V_0 = \mathbf{C}[x, x^{-1}]$, then every

$$\delta \in \text{Ext}_A^1(V_0, V_0) = \text{Der}_{\mathbf{C}}(A, \text{Hom}_{\mathbf{C}}(V_0, V_0)) / \text{Triv}$$

is determined by the values of $\delta(x)$ and $\delta(\lambda)$, $\lambda \in \mathbf{C}^*$. Since $\text{Ext}_{\mathbf{C}[x]}^1(V_0, V_1) = 0$, we may find a trivial derivation such that subtracting from δ we may assume $\delta(x) = 0$. But then the formula

$$\delta(x \times \lambda) = \delta(\lambda \times (\lambda^{-1}x))$$

implies

$$x\delta(\lambda) = \delta(\lambda)(\lambda^{-1}x)$$

from which it follows that

$$\delta(\lambda)(x^p) = (\lambda^{-1}x)^p \delta(\lambda)(1).$$

Now, since $\lambda\mu = \mu\lambda$ in \mathbf{C}^* , we find

$$(\lambda^{-1}\mu x)^p \delta(\lambda)(1)(\mu x) = (\lambda\mu^{-1}x)^p \delta(\lambda)(1)(\lambda x)$$

which should hold for any pair of $\mu, \lambda \in \mathbf{C}^*$, and any p . This obviously implies $\delta = 0$.

This argument shows not only that

$$\text{Ext}_A^1(V_1, V_1) = \text{Der}_{\mathbf{C}}(A, \text{Hom}_{\mathbf{C}}(V_1, V_1)) / \text{Triv} = 0$$

when $V_1 = \mathbf{C}[x, x^{-1}]$, but also when $V_1 = \mathbf{C}[x]$. Finally, we find that the formula above,

$$x\delta(\lambda) = \delta(\lambda)(\lambda^{-1}x),$$

shows that for

$$\delta \in \text{Ext}_A^1(V_1, V_0) = \text{Der}_{\mathbf{C}}(A, \text{Hom}_{\mathbf{C}}(V_1, V_0)) / \text{Triv}$$

we have $\delta(\lambda)(xx^p) = 0$ for all p . Therefore,

$$\text{Ext}_A^1(V_1, V_0) = \text{Der}_{\mathbf{C}}(A, \text{Hom}_{\mathbf{C}}(V_1, V_0)) / \text{Triv} = 0$$

when $V_1 = \mathbf{C}[x, x^{-1}]$. However, when $V_1 = \mathbf{C}[x]$, we find that δ with $\delta(\lambda)(1) \neq 0$ and with $\delta(\lambda)(x^p) = 0$, for $p \geq 1$, survives. These will, as above, give rise to a logarithm of the real part of \mathbf{C}^* . Therefore, in this case $\text{Ext}_A^1(V_1, V_0) = \mathbf{C}$. The miniversal families look like

$$H = \begin{pmatrix} \mathbf{C}[[t]] & 0 \\ 0 & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x, x^{-1}]$, and like

$$H = \begin{pmatrix} \mathbf{C}[[t]] & 0 \\ \langle \mathbf{C} \rangle & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x]$.

4 The infinite phase space construction and Massey products

Let, as above, $\mathcal{V} = \{V_i\}_{i=1, \dots, r}$ be a family of A -modules. To compute the relevant cohomology for the deformation theory, that is, the $\text{Ext}_A^*(V_i, V_j)$, we may use the Leray spectral sequence of [3], together with the formulas

$$\begin{aligned} \text{Ext}_A^n(V_i, V_j) &= HH^{n+1}(k, A; \text{Hom}_k(V_i, V_j)), \\ HH^{n+1}(k, A; W) &= \varprojlim_{\text{Free}/A}^{(n)} \text{Der}_k(-, W), \quad n > 0, \end{aligned}$$

where W is any A -bimodule. Choose a surjective morphism $\mu : F \rightarrow A$ of a free k -algebra F onto A , and put $I = \ker \mu$, then we find that

$$\begin{aligned} HH^3(k, A; W) &= \varprojlim_{\text{Free}/A}^{(2)} \text{Der}_k(-, W) = \text{Hom}_F(I/I^2, \ker \mu) / \text{Der}, \\ \text{Ext}_A^2(V_i, V_j) &= \text{Hom}_F(I/I^2, \text{Hom}_k(V_i, V_j)) / \text{Der}, \end{aligned}$$

where Der is the restriction of the derivations, $\text{Der}_k(F, -)$, to I/I^2 . Moreover, consider a commutative diagram of homomorphisms of algebras, in which $\tilde{\rho}$ is not yet included

$$\begin{array}{ccccc} A & \longleftarrow & F & \longleftarrow & I \\ \rho \downarrow & & \searrow \tilde{\rho} & & \downarrow \rho' \\ S & \longleftarrow & R & & \end{array}$$

and where $J := \ker \pi$ has square 0. The composition map $O : I/I^2 \rightarrow \ker \pi$ induces an element $o \in HH^3(k, A; J)$, independent upon the choice of ρ' . If this (obstruction) element vanishes, then O is the restriction to I of a derivation $\xi : F \rightarrow \ker \pi$. Subtracting this from ρ' , we may assume that $\rho'(I) = 0$, so there exists a lifting $\tilde{\rho}$ of ρ . If there exists a lifting $\tilde{\rho}$, then we may obviously assume that $O = 0$.

Now, let $\{\psi_{i,j}(l) \in \text{Der}_k(A; V_i, V_j)\}_{l=1, \dots, d_{i,j}}$ represent a basis of $\text{Ext}_A^1(V_i, V_j)$, and let $E_{i,j} := \{t_{i,j}(l)\}_{l=1, \dots, d_{i,j}}$ denote the dual basis. Consider, the free matrix k -algebra (quiver) $(T_{i,j}^1)$, generated in slot (i, j) by the (formal) elements of $E_{i,j}$. There is a unique homomorphism

$$\pi : T^1 := (T_{i,j}) \longrightarrow \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \cdot & \cdot & \cdots & 0 \\ 0 & \cdot & \cdots & k \end{pmatrix}.$$

Denote by the same letter the completion of T^1 with respect to the powers of the radical $\text{Rad}(T^1) := \ker \pi$. Then $T^1 \in \hat{\underline{a}}_r$. Consider the k -algebra and the π -induced homomorphism

$$\pi_1 : (T_{i,j}^1 \otimes_k \text{Hom}_k(V_i, V_j)) \longrightarrow (\text{Hom}_k(V_i, V_j)).$$

Clearly, π_1 splits, and it is easy to see that

$$\xi : A \longrightarrow (T_{i,j}^1 \otimes \text{Hom}_k(V_i, V_j))$$

defined by

$$\xi = \sum_{i,j,l} t_{i,j}(l) \psi_{i,j}(l)$$

is a derivation, $\xi : A \rightarrow ((T^1/\text{Rad}^2(T^1))_{i,j} \otimes_k \text{Hom}_k(V_i, V_j))$, therefore inducing a unique homomorphism, $\tilde{\rho}_1$, makes the following diagram commute

$$\begin{array}{ccccc} A & \xrightarrow{id+\delta} & \mathcal{D}_1(A) & \longleftarrow & \mathcal{D}_2(A) \\ \rho \downarrow & \searrow \tilde{\rho}_1 & \downarrow \rho_1 & & \\ (\text{Hom}_k(V_i, V_i)) & \longleftarrow & ((T^1/\text{Rad}^2(T^1))_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)) & & \end{array}$$

Now, we would have liked to extend this diagram, completing it with commuting homomorphisms,

$$\begin{array}{ccc} A & \xrightarrow{\exp(\delta)} & \mathcal{D}_n(A) \\ \tilde{\rho}_1 \downarrow & & \downarrow \rho_n \\ (T^1(1)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)) & \longleftarrow & (H(n)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)), \end{array}$$

where

$$(T^1(n) := T^1/\text{Rad}^{n+1}(T^1)), \quad (H(n) := H/\text{Rad}^{n+1}(H)).$$

However, as will be clear in the next construction, the obvious continuation of this procedure does not work. In fact, the formalized higher differentials $\mathcal{D}(A)$ is not really the natural phase-space to work with for all purposes. In an obvious sense it is too homogenous. We are therefore led to the construction of a kind of *projective resolution* of A . Consider as above a surjective homomorphism, $\mu : F \rightarrow A$, with $F = k\langle x_1, x_2, \dots, x_s \rangle$ a free k -algebra, and $I = \ker \mu$. Obviously $\text{Ph}^{(p)}(F)$, for $p \geq 1$, are also free, and $\text{Ph}^{(p+1)}(F)$ is a free $\text{Ph}^{(p)}(F)$ -algebra. Let $\exp(\delta) : F \rightarrow \mathcal{D}(F)$ be defined as in Section 2 by

$$\exp(\delta) = id + d + 1/2d^2 + \dots$$

and denote by $\eta_p : F \rightarrow \mathcal{D}_p(F)$ the induced homomorphism. Define

$$\mathcal{H}_p := \mathcal{D}_p(F)/(i_0(I), \eta_p(I)).$$

Clearly, $\mathcal{H}_p = \mathcal{D}_p(A)$, $\mathcal{H} = \text{proj lim } \mathcal{H}_p$, for $p = 0, 1$. For $p \geq 2$, there are only natural surjective homomorphisms, $\kappa_p : \mathcal{H}_p \rightarrow \mathcal{D}_p(A)$. By functoriality, the diagram above induces another commutative diagram, which may be completed to the commutative diagram (ρ'_2 and ρ_2 not yet included)

$$\begin{array}{ccccc}
 I & \longrightarrow & \mathcal{D}_1(I) & \longleftarrow & \mathcal{D}_2(I) \\
 \downarrow & & \downarrow & & \downarrow \\
 F & \xrightarrow{id+\delta} & \mathcal{D}_1(F) & \longleftarrow & \mathcal{D}_2(F) \\
 \downarrow \mu & & \downarrow \mu_1 & & \downarrow \mu_2 \\
 A & \xrightarrow{id+\delta} & \mathcal{D}_1(A) & \longleftarrow & \mathcal{H}_2(A) \\
 \downarrow \tilde{\rho}_1 & \nearrow \rho_1 & \downarrow \rho_1 & \nearrow \rho_2 & \downarrow \rho_2 \\
 \mathcal{O}(1) & \longleftarrow & \mathcal{O}'(2), & &
 \end{array}$$

where we, in expectation of later constructions, put

$$\begin{aligned}
 H(1) &= T^1 / \text{Rad}(T^1)^2, & H'(2) &= T^1 / \text{Rad}(T^1)^3, \\
 \mathcal{O}(n) &:= (H(n)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)), & n &\geq 1, \\
 \mathcal{O}'(n) &:= (H'(n)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)), & n &\geq 2.
 \end{aligned}$$

Now the map

$$(id + \delta) \circ \mu_1 \circ \rho_1 : I \longrightarrow \mathcal{O}(1)$$

is zero, and the resulting map $\tilde{\rho}_1 : A \rightarrow \mathcal{O}(1)$ is, as deformation of the family \mathcal{V} , the universal family at the tangent level. Since $\text{Ph}^{(n+1)}(F)$ is a free algebra over $\text{Ph}^{(n)}(F)$, there is lifting ρ'_2 . We want an induced ρ_2 . Consider the composition

$$\mathcal{O}' := \text{exp}(\delta) \circ \rho_2 : F \longrightarrow \mathcal{O}'(2)$$

lifting $\mu \circ \tilde{\rho}_1$. The restriction to I vanishes on I^2 and induces a map

$$\mathcal{O}(2) : I/I^2 \longrightarrow \left(\left(\text{Rad}(T^1)^2 / \text{Rad}(T^1)^3 \right)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j) \right).$$

It is easily seen to be F -linear, both from left and right, and so it induces the obstruction

$$o_2 \in \left(\left(\text{Rad}(T^1)^2 / \text{Rad}(T^1)^3 \right)_{i,j} \otimes_k \text{Ext}_A^2(V_i, V_j) \right)$$

independent upon the choice of extension ρ'_2 . Now

$$\left(\left(\text{Rad}(T^1)^2 / \text{Rad}(T^1)^3 \right)_{i,j} \otimes_k \text{Ext}_A^2(V_i, V_j) \right)$$

may be identified with

$$\text{Hom}_k \left(\left(\text{Ext}_A^2(V_i, V_j)^* \right), \left(\text{Rad}(T^1)^2 / \text{Rad}(H)^3 \right)_{i,j} \right)$$

which is a subspace of

$$\text{Mor}_{\underline{a}_r} \left(k^r \oplus \left(\text{Ext}_A^2(V_i, V_j)^* \right), T^1 / \text{Rad}(T^1)^3 \right).$$

Denote by T^2 the free matrix algebra (quiver), in \underline{a}_r , generated by $\text{Ext}_A^2(V_i, V_j)^*$, just like the construction of T^1 above, such that

$$T^2 / \text{Rad}(T^2)^2 = k^r \oplus \left(\text{Ext}_A^2(V_i, V_j)^* \right).$$

We may now state and prove the main result of this paper.

Theorem 11. (i) For any finite family of (finite dimensional) A -modules, $\mathcal{V} := \{V_i\}_{i=1, \dots, r}$, there is a homomorphism $\tilde{\rho}$, making the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{\exp(\delta)} & \mathcal{H} \\ \rho_V \downarrow & & \downarrow \tilde{\rho} \\ \text{End}_k(V) & \longleftarrow & (H_{i,j} \otimes \text{Hom}_k(V_i, V_j)), \end{array}$$

such that the versal family $\tilde{\rho} = \exp(\delta) \circ \rho$.

(ii) Moreover, $H = (H_{i,j})$ may be constructed recursively, as a quotient of $T^1 = (T^1_{i,j})$, by annihilating a series of obstructions, o_n , defining a morphism in \underline{a}_r , $o : T^2 \rightarrow T^1$, such that $H \simeq T^1 \otimes_{T^2} k^r$.

Proof. We have above constructed an obstruction for lifting ρ_1 to a ρ_2 . It is a unique element;

$$o_2 \in \text{Mor}_{\underline{a}_r} \left(k^r \oplus \left(\text{Ext}_A^2(V_i, V_j)^* \right) T^1 / \text{Rad}(T^1)^3 \right).$$

Obviously, the image

$$o_2 \left(\left(\text{Ext}_A^2(V_i, V_j)^* \right) \subset T^1 / \text{Rad}(T^1)^3 \right)$$

generates an ideal of T^1 , contained in $\text{Rad}(T^1)^2$. Call it σ_2 , and put

$$H(2) = T^1 / (\text{Rad}(T^1)^3 + \sigma_2).$$

Then, there is a commutative diagram

$$\begin{array}{ccccccc} I & \xrightarrow{\quad} & \mathcal{D}_1(I) & \xleftrightarrow{\quad} & \mathcal{D}_2(I) & \cdots & \\ \downarrow & & \downarrow & & \downarrow & & \\ F & \xrightarrow{\eta_1} & \mathcal{D}_1(F) & \xleftrightarrow{\quad} & \mathcal{D}_2(F) & & \\ \mu \downarrow & & \downarrow \mu_1 & & \downarrow \mu_2 & & \\ A & \xrightarrow{\eta_1} & \mathcal{D}_1(A) & \xleftrightarrow{\quad} & \mathcal{H}_2(A) & & \\ \tilde{\rho}_1 \downarrow & \nearrow \rho_1 & & \nwarrow \tilde{\rho}_2 & & & \\ \mathcal{O}_1 & \longleftarrow & \mathcal{O}_2 = (H(2)_{i,j} \otimes \text{Hom}_k(V_i, V_j)) & & & & \end{array}$$

In fact, since we have divided out with the obstruction, we know that the morphism

$$O(2) : I/I^2 \longrightarrow \left(\text{Rad}(T^1)^2 / \text{Rad}(T^1)^3 + \sigma_2 \right)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)$$

is the restriction of a derivation

$$\psi'_2 : F \longrightarrow \left(\text{Rad}(T^1)^2 / \text{Rad}(T^1)^3 + \sigma_2 \right)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j).$$

Now change the morphism ρ'_2 , to ρ''_2 mapping $d^2 x_i$ to $\rho'_2(d^2 x_i) - 2\psi_2(x_i)$. It is easily seen that for this new morphism, $\eta_2 \circ \rho''_2$ is zero, restricted to I , proving the existence of $\tilde{\rho}_2$. Recall that $\mathcal{D}_1(A) = \mathcal{H}_1$.

Now $\tilde{\rho}_2$ defines $\tilde{\rho}_2 := \eta_2 \circ \rho_2$. Let σ'_3 be the two-sided ideal in T^1 generated by

$$\text{Rad}(T^1)^4 + \text{Rad}(T^1)\sigma_2 + \sigma_2 \text{Rad}(T^1)$$

and let us put

$$H'(3) := T^1 / \sigma'_3, \quad \mathcal{O}'(3) := (H'(3) \otimes_k \text{Hom}_k(V_i, V_j)).$$

The diagram above induces a commutative diagram, ρ'_3 , constructed as above, but where $\bar{\rho}_3$ is the problem,

$$\begin{array}{ccccc}
 I & \longrightarrow & \mathcal{D}_2(I) & \longleftarrow & \mathcal{D}_3(I) \\
 \downarrow & & \downarrow & & \downarrow \\
 F & \xrightarrow{\eta_2} & \mathcal{D}_2(F) & \longleftarrow & \mathcal{D}_3(F) \\
 \downarrow \mu & & \downarrow \mu_2 & & \downarrow \mu_3 \\
 A & \xrightarrow{\tilde{\eta}_2} & \mathcal{H}_2(A) & \longleftarrow & \mathcal{H}_3 \\
 \downarrow \rho_2 & \swarrow \bar{\rho}_2 & \swarrow \rho'_3 & \swarrow \bar{\rho}_3 & \\
 \mathcal{O}(2) & \xleftarrow{\pi} & \mathcal{O}'(3) & &
 \end{array}$$

Consider now the map $\eta_3 \circ \rho'_3 : I \rightarrow \mathcal{O}'(3)$ ending up in

$$\left((\text{Rad}(T^1)^3 + \sigma_2) / \sigma_3 \right)_{i,j} \otimes_k \text{Hom}_k(V_i, V_j)$$

which clearly is killed by $\text{Rad}(\mathcal{O}'(3))$, and therefore really is a matrix of vector spaces, as an $\mathcal{O}'(3)$ -module. As above, this map is easily seen to be a left and right linear map as F -modules, F acting on $\mathcal{O}(3)$ via $\tilde{\eta}_3 : F \rightarrow \mathcal{D}_3(F)$. Moreover, the induced element

$$\begin{aligned}
 o_3 &\in \left((\text{Rad}(T^1)^3 + \sigma_2) / \sigma'_3 \right)_{i,j} \otimes_k \text{Ext}_A^2(V_i, V_j) \\
 &= \left(\text{Hom}_k(\text{Ext}_A^2(V_i, V_j)^*, (\text{Rad}(T^1)^3 + \sigma_2) / \sigma'_3)_{i,j} \right)
 \end{aligned}$$

is independent on the choice of ρ'_3 . Now, we define $H(3) := H'(3) / \sigma_3$, where σ_3 is defined by the image of o_3 , and define $\kappa : \mathcal{O}'(3) \rightarrow \mathcal{O}(3)$ as above. Since, by functoriality, the morphism

$$\eta_3 \circ \rho'_3 \kappa : I \longrightarrow \mathcal{O}(3)$$

must induce the zero element in the corresponding

$$\left(\text{Ext}_A^1(V_i, V_j) \otimes \left((\text{Rad}(T^1)^3 + \sigma_2) / \sigma'_3 \right)_{i,j} \right) / \text{im } o_3,$$

it must be the restriction of a derivation $\xi : F \rightarrow \mathcal{O}(3)$. Now change ρ'_3 by sending $d^3 x_i$ to $\rho'_3(d^3 x_i) - 3! \psi_3(x_i)$, leaving the other values of the parameters unchanged. Then, a little calculation shows that the new ρ'_3 maps each $\eta_3(f)$, $f \in I$, to zero, inducing a morphism $\bar{\rho}_3 : \mathcal{H}_3 \rightarrow \mathcal{O}(3)$. We now have a new situation, given by a commutative diagram, not yet including ρ_4 ,

$$\begin{array}{ccccc}
 I & \longrightarrow & \mathcal{D}_3(I) & \longleftarrow & \mathcal{D}_4(I) \\
 \downarrow & & \downarrow & & \downarrow \\
 F & \xrightarrow{\tilde{\eta}_3} & \mathcal{D}_3(F) & \longleftarrow & \mathcal{D}_4(F) \\
 \downarrow \mu & & \downarrow \mu_3 & & \downarrow \mu_4 \\
 A & \xrightarrow{\tilde{\eta}_3} & \mathcal{H}_3 & \longleftarrow & \mathcal{H}_4 \\
 \downarrow \bar{\rho}_3 & \swarrow \bar{\rho}_3 & \swarrow \rho'_4 & \swarrow \bar{\rho}_4 & \\
 \mathcal{O}(3) & \xleftarrow{\pi} & \mathcal{O}'(4) & &
 \end{array}$$

and it is clear how to proceed. This proves (i), and the rest is a consequence of the general theorem [3, Theorem 4.2.4]. □

We cannot replace \mathcal{H} by \mathcal{D} . This follows from the trivial Example 4(iii) above. However, if we are in a graded situation, things are nicer.

Corollary 12. *Assume that A is a finitely generated, graded, in degree 1, k -algebra, and assume that \mathcal{V} is a family of graded A -modules. Then there is a corresponding graded formal moduli $(H_{i,j})^{\text{gr}}$, and there is a commutative diagram,*

$$\begin{array}{ccc} A & \xrightarrow{\exp(\delta)} & \mathcal{D} \\ \rho_{\mathcal{V}} \downarrow & & \downarrow \bar{\rho}^{\text{gr}} \\ \text{End}_k(V) & \longleftarrow & (H_{i,j}^{\text{gr}} \otimes \text{Hom}_k(V_i, V_j)), \end{array}$$

such that the graded versal family $\tilde{\rho}^{\text{gr}} = \exp(\delta) \circ \bar{\rho}^{\text{gr}}$.

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