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Particle Motion and Scattering in Finslerian-Kropina Schwarzschild Metric

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Abstract

In theory of relativity, Finsler geometry is a without motion there is no position, in universe. In this paper, we study the motion of photon and massless particles for Schwarzschild metric in Finsler space time in the case of Kropina-Finsler space with constant flag curvature. Then, we analyzed qualitatively of potential energy of particles at various distances near Schwarzschild black hole and the results are compared with in the Riemannain geometry in particular Lorenz case. Also, described the scattering of particles near Schwarzschild black hole in Finsler space time by finding capture cross section of absorption by the black hole.

Keywords: Schwarzschild metric; Einstein field equations; Black hole; Finsler space; Berwald space

Introduction

The Einstein's general relativity is one of the most successful gravity theories. It coincides with the experiments at the highest achievable precision such as the perihelion precession of mercury, bending of light, radar echo delay, gravitational red-shift and so on. However, Einstein's gravity theory may exist some problems in understanding gravity correctly at all scales. It cannot accurately describe the accelerated expansion of universe on large scales, and it is very difficult to establish a complete theory of quantum gravity at the micro-scale. This leads to various modifications of gravity theory, including Superstring theory [1], scalar-tensor theories [2], Lovelock gravity [3], HoravaLifshitz gravity [4], $f(R)$ gravity [5], etc.

It is widely known that space-time of Einstein's general relativity is described by a Riemannian geometry. Since there exist some problems in general relativity, scientists have proposed to modify gravity theory by improving differential geometry. As the generalization of Riemannian geometry, Finsler geometry [6], [7] is the most general geometry, in which the line element is dependent on not only space-time coordinates but also tangent vectors. Namely $ds^2=f(x^\alpha,dx^\beta$; $ds^2=f(x^\alpha,\mu dx^\beta)=\mu^2 f(x^\alpha,dx^\beta)$, and the metric can also be given by $ds^2=G_{\alpha\beta}dx^\alpha dx^\beta$, where noting that $G_{\alpha\beta}$ is related to dx^α. This geometry can reduce to a Riemannian geometry when $f(x^\alpha,dx^\beta)$ becomes a quadratic form in the dx^β. The applications of the Finsler geometry to general relativity and black hole physics have attracted a great deal of attention.

In general relativity, gravitational force is due to distortion of matter with space-time. General relativity relates curvature of space-time to energy and momentum of matter and radiation. The classical tests of general theory of relativity are the perihelion precession of mercury's orbit, the deflection of light by the sun and gravitational redshift of light. Schwarzschild derived vacuum solutions for Einstein field equations under symmetry assumptions. The Schwarzschild metric describes gravitational field outside a spherical body (of mass M, say) with zero electric charge and zero angular momentum. It has singularities at r=0 and r=2M [8]. But r=2M is a coordinate singularity which could be removed by transforming the metric to retarded or advanced time coordinate.

Finsler geometry provides metric generalization to Riemannian geometry which includes Lorentz metric as a special case. Riemannian geometry defines inner product structure over tangent bundle whereas Finsler geometry generalizes metric geometry by defining a general length functional for curves on the manifold [9]. By imposing few requirements on Finsler structure, one could model a geometric space-time which provides physical effects that are useful and suitable for physics [10]. Finsler geometry is used in various fields of research such as medical imaging, optics, relativity and cosmology [11]. Finsler space times are generalizations of lorentzian metric manifolds which preserve causality.

Li et al. [9] studied exact solution of vacuum field equations in finsler space-time and described it to be same as vanishing ricci scalar which implies that the geodesic rays are parallel to each other. Silagadze [12] proposed finslerian extension of the Schwarzschild metric. Chang [13,14] argued that finsler geometry, in principle, can address the rotational curves of galaxies and the acceleration of the universe without introducing dark matter or dark energy. Minguzzi [15] studied singularity theorems, raychaudhuri equation and its consequences for chronality in the context of Finsler space times. Recently, Fuster [16] presented finslerian version of the well-known pp-waves which generalizes the very special relativity (VSR) line element. Pfeifer [17] gave precise geometric definition of observers and their measurements, and showed that two different observers are related by a transformation composed out of a certain parallel transport and a Lorentz transformation. In [14], Javaloyes described few links between finsler Geometry and the geometry of space times.

Preliminaries

Finsler geometry uses generalization of the notion of distance between two neighbouring points x^i, x^i+dx^i which is given by a function $F(x^i,dx^i)$ with quasimetric structure and minkowski norm[16].

$$ds=F(x^i,dx^i). \tag{2.1}$$

Quasimetric: A metric d which satisfies the following three axioms is called quasimetric.

I. Positivity: $d(x,y)$=0.

II. Positive Definiteness: $d(x,y)$=0 if and only if $x=y$.

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III. Triangle Inequality: $d(x,z)=d(x,y)+d(y,z)$. We can see that quasimetric need not satisfy symmetry.

Minkowski Norm: A function F: $TM \to 0,\infty)$ is a minkowski norm if F is smooth,

Homogeneous of degree 1 and is of symmetric bilinear form [15]. The finite distance between two points on a given curve $x^i=x^i(s)$ between t_1 and t_2 is given by the length functional [18]:

$$s=\int_{t_1}^{t_2} F\left(x^i,\frac{dx^i(t)}{dt}\right)dt. \tag{2.2}$$

For s to be independent of parameter t, F must satisfy homogeneity of degree one i.e., $F(x^i,kdx^i)=kF(x^i,dx^i)$.

Consider an n-dimensional smooth manifold M. Let T_xM be tangent space at the point $x \in M$. The set of all tangent spaces $TM = \{T_xM : x \in M\}$ is called tangent bundle on the manifold M.

We denote $(x,y) \in TM$ where $x \in M$ and $y \in T_xM$. Let M be an n-dimensional c^∞ manifold. The tangent space at $x \in M$ is denoted by T_xM. The tangent bundle of M is defined by $TM=\cup_{x\in M}T_xM$. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. The natural projection π: $TM \to M$ is given by $\pi(x,y) = x$. The dual space of T_xM is called the cotangent space at x and is denoted by T_x^*M . The union $T^*M = U_{x\in M}T_x^*M$ is the cotangent bundle of M.

Definition 4.1. *A Finsler structure of M is a function F : TM → [0,∞) with the following properties:*

1. *Regularity: F is c^∞ on the entire slit tangent bundle TM\{0}.*

2. *Positive homogeneity: $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$. 3. Strong Convexity: The $n \times n$ Hessian matrix*

$$\left(g_{ij}\right)=\left(\frac{1}{2}\partial_i\partial_i F^2\right)=\left(\frac{1}{2}\frac{\partial}{\partial_{y^i}}\frac{\partial}{\partial_{y^i}}\left(F^2\right)\right)$$

with i,j=1,2,3,....,n is positive definite at every point of TM\{0}.

Note that the metric tensor g_{ij} is symmetric under exchange of indices.

Noticed that; For pseudo-finsler spaces, metric tensor g_{ij} need not obey positive definiteness [19].

Because of homogeneity, we get $F^2(x^k,y^k)=g_{ij}(x^k,y^k)y^iy^j$. Hence $ds^2=g_{ij}(x^k,y^k)y^iy^j$. The space is riemannian if g_{ij} does not depend on y^k i.e., it depends only on position x^k but not on velocity y^k.

The geodesic equation in finsler manifold is given by

$$\frac{d^2x^i}{ds^2}+2G^i=0. \tag{2.3}$$

where s is parameterizing the geodesics curve.

The geodesic spray coefficients G^i are given [15] by

$$G^i(x,y)=\frac{1}{4}g^{il}(x,y)\left\{2\frac{\partial g_{il}}{\partial x^k}(x,y)-\frac{\partial g_{jk}}{\partial x^i}(x,y)\right\}y^jy^k.$$

G is globally defined vector field on TM. The notion of Riemannian curvature for Riemann metrics can be extended to Finsler metrics. For a non-zero vector $y \in T_xM\backslash\{0\}$ the Riemannian curvature R_y: $T_xM \to T_xM$ is linear map defined by

$$R_y(u)=R^i_k(y)u^k\frac{\partial}{\partial x^{i,}}u=u^i\frac{\partial}{\partial x^{i,}}$$

where

$$R^i_k(y)=2\frac{\partial G^i}{\partial x^k}-\frac{\partial^2 G^i}{\partial x^j\partial y^k}y^j+2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The trace of Riemann curvature R_y is scalar function Ric on TM defined by

$Ric(y)=tr(R_y)$,

which is called the Ricci curvature of (M,F).

The Finslerian Ricci tensor is given as:

$$Ric_{ij}=\frac{1}{2}\partial_{y^i}\partial_{y^j}\left(F^2Ric\right),$$

It reduces to Ricci tensor when Finsler structure F is Riemannian.

Berwald spaces: If geodesic spray coefficients for a Finsler space are quadratic in y, then such spaces are called Berwald spaces [17].

Berwald spaces satisfy $Ric_{ij}=0$.

Particle Motion in Schwarzschildmetric in Finsler-Kropina Space Structure

The Finsler structure in Schwarzschild metric is given by [15];

$$F^2=A(r)dt^2-B(r)dr^2-r^2F^2(\theta,\varphi,d\theta,d\varphi), \tag{3.1}$$

where $A(r)=1-\frac{2GM}{\lambda r}$, $B(r)=(\lambda-\frac{2GM}{\lambda r})-1$.

The modified schwarzschild metric radius is given as [15]; $r_S=\frac{2GM}{\lambda r}$.

We consider the Kropina-Finsler space with constant positive flag curvature $\lambda=2$, which is given by;

$$\overline{F}=\frac{\left(1-\epsilon^2\sin^2\theta\right)d\theta^2+\sin^2\theta d\phi^2}{\epsilon\sin^2\theta d\phi^2-\epsilon^3\sin^3\theta d\phi}, \tag{3.2}$$

where $0 \le \epsilon < 1$.

We know that the exterior metric of vaccum field equations is given by

$$ds^2=A(r)dt^2-B(r)dr^2-r^2\left(\frac{\left(1-\epsilon^2\sin^2\theta\right)d\theta^2+\sin^2\theta d\phi^2}{\epsilon\sin^2\theta d\phi^2-\epsilon^3\sin^3\theta d\phi}\right)^2. \tag{3.3}$$

Here, we use advanced time coordinates (v,r,θ,φ) as t breaks down near the horizon and also radially infalling particle moves along line of advanced time.

So the Schwarzschild metric in (v,r,θ,φ) coordinates is given as

$$ds^2=V(r)dr^2-2dvdr-r^2\left(\frac{\left(1-\epsilon^2\sin^2\theta\right)d\theta^2+\sin^2\theta d\phi^2}{\epsilon\sin^2\theta d\phi^2-\epsilon^3\sin^3\theta d\phi}\right)^2. \tag{3.4}$$

Here, $V(r)=1-\frac{2M}{r}$.

Consider an action

$$I=\int g_{ij}x^ix^jds.$$

$$I=\int\left(V(r)v^2-2vr-r^2\left(\frac{\left(1-\epsilon^2\sin^2\theta\right)d\theta^2+\sin^2\theta d\phi^2}{\epsilon\sin^2\theta d\phi^2-\epsilon^3\sin^3\theta d\phi}\right)^2\right). \tag{3.5}$$

In above equation indicates $\frac{d}{ds}, s$ represents the propogation along worldline of the particle. Observed that equation (3.5) corresponds to geodesic equation, i.e., $\delta I=0$. consider $g_{ij}\dot{x}^i\dot{x}^j=k$. Here, $k=0$ corresponds to null propagation of light and $k=1$

Corresponds to time like geodesics

Consider θ equation of motion of the (3.5) as:

$$2r^2\left(\frac{\left(1-\epsilon^2\sin^2\theta\right)d\theta^2+\sin^2\theta d\phi^2}{\epsilon\sin^2\theta d\phi^2-\epsilon^3\sin^3\theta d\phi}\right)^2 * \frac{D(\theta,\phi)terms}{\left(\epsilon\sin^2\theta d\phi^2-\epsilon^3\sin^3\theta d\phi\right)^2}, \quad (3.6)$$

where

$$D(\theta,\phi)terms = 2\epsilon\sin^2\theta\theta^2\theta - 2\epsilon^2\sin^3\theta\cos\theta\theta - 2\epsilon^3\sin^4\theta\theta^2\theta + 2\epsilon\sin^4\theta\cos\theta\theta d\phi^2$$
$$-2\epsilon^3\sin^2\theta\theta\theta d\phi + 2\epsilon^4\sin^3\theta\cos\theta d\phi + 2\epsilon^5\sin^4\theta\theta\theta d\phi - 2\epsilon^3\sin^4\theta\cos\theta d\phi^3$$
$$-2\sin\theta\cos\theta\theta^2 + \epsilon\sin^2\theta\theta^2\theta + 2\epsilon^3\sin^2\theta\cos\theta\theta^2 d\phi - 2\epsilon^2\sin^3\theta\cos\theta\theta^2 - \epsilon^3\sin^4\theta\theta^2\theta$$
$$+3\epsilon^5\sin^4\theta\cos\theta d\phi + 2\sin^3\theta\cos\theta d\phi^2 + \epsilon\sin^4\theta\theta d\phi^2 - 3\epsilon^3\sin^4\theta\cos\theta(d\phi)^3.$$

Assume initially the particle is in $\theta=\pi/2$ plane. Then we get $(r^2\dot{\theta}^2)=0$. Hence, if the particle is initially in equatorial plane i.e., $\theta=\pi/2$, then one can always choose a coordinate system such that particle is in the same plane.

So we put $\theta = \pi/2$ the metric (3.4) becomes;

$$ds^2 = Vr^2 - 2vr - r^2\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)\phi^2. \quad (3.7)$$

We consider

$r^2\dot{\varphi}$ *=constant, say Land*

$V(r)\dot{v}^2 - \dot{r} = -E,$

where L is angular momentum per unit mass and E corresponds to energy of the particle. Let us consider radial equation of motion

$$g_{ij}x^i x^j = k \quad (3.8)$$

$$V(r)v^2 - 2vr - r^2\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)\phi^2 = k \quad (3.9)$$

Substituting $v=\frac{-E+r}{V(r)}$ and $\phi=\frac{L}{r^2}$ in (3.8) second equation, we get;

$$V(r)\left(\frac{r-E}{V(r)}\right)^2 - 2r\left(\frac{r-E}{V(r)}\right) - r^2\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)^2\left(\frac{L}{r^2}\right)^2 = k, \quad (3.10)$$

$$\frac{E^2}{V(r)} - \frac{r^2}{V(r)} - \frac{L^2}{r^2}\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)^2 = k. \quad (3.11)$$

Since

$$r = \frac{2M}{u},$$
$$r = \frac{-2M}{u^2}u\frac{-2M}{u^2}\frac{du}{d\phi}\frac{l}{r^2},$$
$$-\frac{l}{2M}\frac{du}{d\phi}.$$
$$V(r) = 1-u.$$

Therfore, second equation in (3.10) becomes

$$\left(E^2 - \frac{L^2}{4M^2}\left(\frac{du}{d\phi}\right)^2\right)\frac{1}{1-u} + \frac{L^2}{4M^2}u^2\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)^2 = k, \quad (3.12)$$

$$\frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \left(\frac{u^3}{2} - \frac{u^4}{2}\right)\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)^2 - \frac{2m^2uk}{L^2} = \frac{2M}{L^2}\left(E^2 - k\right). \quad (3.13)$$

Now, consider the particle motion (of unit mass) to be one dimensional with potential $P(x)$.

The total energy is given by the following energy equation as;

$$\mathrm{TE} = \frac{1}{2}x^2 + P(x). \quad (3.14)$$

Discussion of motion of photon and massive particle near Schwarzschild black hole at $k=0$ and $k=1$.

- Motion of photon near Schwarzschild black hole at $k=0$.

The potential is maximum at $u=\frac{2}{3}$ and $r=\frac{2M}{u}=3M$.

Hence, there is a circular null geodesic at $3M$.

The potential energy in case of photon is;

$$P(u) = \left(\frac{u^3}{2} - \frac{u^4}{2}\right)\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)^2.$$

Now, we have the comparison if potential energy in this case with Schwarzschild metric in Lorentz metric $(\epsilon=0)$.

- Motion of photon near Schwarzschild black hole at $k=1$. The potential energy in case of photon is (Figure 1);

$$P(u) = \left(\frac{u^3}{2} - \frac{u^4}{2}\right)\left(\frac{1-\epsilon^2}{1-\epsilon^3}\right)^2 - \frac{2m^2u}{L}.$$

Scattering of Particle in Schwarzschild Metric in Finsler-Kropina Setting

From Schwarzschild black hole in Finsler space-time. Let b be the impact parameter which measures the distance of the particle off the axis of center of black hole.

Let's look at scattering of massless and massive particle near the black hole $r=2M$.

Scattering of photon

Since $L=r^2\varphi=(r\dot{\varphi})r=cb=b$, where $c=1$ for photon.

The total energy in this case;

$$\mathrm{TE} = \frac{2M^2E^2}{L^2}.$$

For the photon to be absorbed by black hole it should be greater than maximum potential;

$$P_{max} = \frac{2}{27} \Rightarrow \frac{ME^2}{L^2} > \frac{2}{27}$$
$$\Rightarrow \frac{ME^2}{L^2} > \sqrt{\frac{2}{27}}.$$

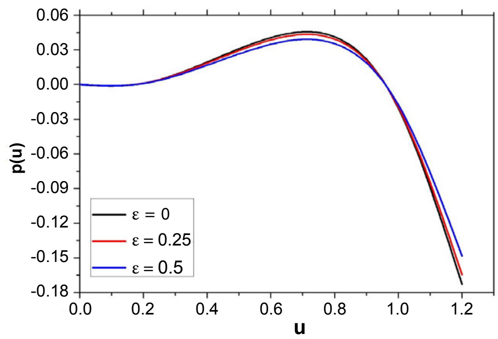

Figure 1: Motion of photon near Schwarzschild black hole.

Therefore,

$$L = r^2\phi = r^2\frac{d\phi}{dt}\frac{dt}{ds} = Eb.$$
$$\Rightarrow \frac{E}{L} = \frac{2}{b}. \tag{4.1}$$

Hence, if $M > \frac{b}{\sqrt{27}}$ the photon gets absorbed

The maximum impact parameter is; $b_{max} = 3\sqrt{3}M$.

The cross section for absorption of photon by black hole is; $\sigma_{abs}=\pi b^2_{max}=27\pi M^2$.

Scattering of massive particle

For relativistic motion moving with velocity v_m,

$$E = \left(1 + \frac{1}{v_m^2} + \ldots\ldots\right).$$
$$\Rightarrow E^2 = 1 + v_m^2 + \ldots\ldots\ldots$$

The total energy of massive particle is;

$$\frac{2m^2}{L^2}\left(E^2 - k\right) = \frac{2m^2}{L^2}\left(1 + v_m^2 + \ldots\ldots\ldots - 1\right) = \frac{2m^2 v_m^2}{L^2}.$$

The total energy of massive particle in this case is;

$$p(u) = \frac{1}{2}\left\{u^2 - u^3\left(\frac{1-\in^2}{1-\in^3}\right)^2 - \frac{4m^2 u}{L^2}\right\}.$$

The boundary conditions required for the particle to be absorbed by the black hole are,

$$p_{max} = 0 \; and \; p'(u) = 0.$$
$$p_{max} = 0 \Rightarrow u^2 - u + \frac{4M^2}{L^2}\left(\frac{1-\in^3}{1-\in^2}\right)^2 = 0. \tag{4.2}$$
$$p'(u) = 0 \Rightarrow 3u^2 - 2u + \frac{4M^2}{L^2}\left(\frac{1-\in^3}{1-\in^2}\right)^2 = 0. \tag{4.3}$$

Solving second and third equation in (4.2), we get $u = 1/2$.

Substitute u=1/2 in (second eqn.,) (4.2) , we have

$$\frac{16M^2}{L^2} = \left(\frac{1-\in^2}{1-\in^3}\right)^2.$$
$$L = v_m b_s.$$
$$\Rightarrow b_c^2 = \frac{16M^2}{L^2}\left(\frac{1-\in^2}{1-\in^3}\right)^2$$

If $b<b_c$, the, massive particle is captured by the black hole.

Therefore, capture cross section for massive particle

$$\sigma_c = \pi b_c^2 = 16\pi\frac{m^2}{v_m^2}\left(\frac{1-\in^3}{1-\in^2}\right)^2,$$
$$\frac{352}{7}\frac{m^2}{v_m^2}\left(\frac{1-\in^3}{1-\in^2}\right)^2.$$

which shows that, σ_c diverges, when $v_m \to 0$.

This is a consequence of attractive nature of gravity.

Discussion

The present work is to the Finsler-Kropina version of motion of photon and scattering of massless particles in Schwarzschild metric. As this we estimate the maximum potential acquired by the particle depend on the Finsler parameter. Finally, we manipulated the capture cross section of the particle by the black hole. Also, observed that the results are agree as compared to Riemannain case only when ∈=0.

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