Journal of
Applied & Computational Mathematics
ISSN: 2168-9679

Bakery, J Appl Computat Math 2015, 4:6
DOI: 10.4172/2168-9679.1000270

Research Article | Open Access

Oscillation of a Nonlinear Difference Equation of Power Type

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Abstract

We give in this work the sufficient conditions on the positive solutions of the difference equation $x_{n+1} = \alpha x_n^8 + \frac{x_{n-1}}{x_n}$, n=0,1,…., where α ≥ 0 and s > 0 with arbitrary positive initials x_{-1}; x_0 to be bounded and the equilibrium point to be globally asymptotically stable. Finally we present the condition for which every positive solution converges to a prime two periodic solution. We have given a non-oscillatory positive solution which converges to the equilibrium point.

Keywords: Difference equation; Boundedness; Global asymptotic stability; Oscillation; Period two solution

Introduction

In this work we study the positive solutions of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}) = \alpha x_n^8 + \frac{x_{n-1}}{x_n}, n = 0,1,...., \quad (1)$$

where f is a continuously differentiable function, α ≥ 0 and s > 0 with arbitrary positive initials x_{-1}, x_0. Now we recall some basic definitions and results which will be used in the sequel.

Remark 3.1: A point $\bar{x} \in \mathbb{R}$ is an equilibrium point of equation (1), if and only if,

$$g(\bar{x}) = \bar{x} - 1 - \alpha \bar{x}^8 = 0 \quad (2)$$

If we replace x_n and x_{n-1} in (1) by the variables u and v respectively, then we have

$$z = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{s\alpha\bar{x} - 1}{\bar{x}},$$

$$h = \frac{\partial f}{\partial u}(\bar{x}, \bar{x}) = \frac{1}{\bar{x}}.$$

The linearized equation related with equation (1) about the equilibrium point $\bar{x}$ is

$y_{n+1} = zy_n + hy_{n-1}$, n = 0, 1, 2,…..

Its characteristic equation is

$$\lambda^2 - z\lambda - h = 0 \quad (3)$$

Definition 3.2: An equilibrium [1] point x of the difference equation (1) is called locally stable if for every ϵ>0 there exists δ > 0 such that, if x_{-1}, $x_0 \in (0;\infty)$ with $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$ then $|x_n - \bar{x}| < \varepsilon$ for all n≥-1.

Definition 3.3: An equilibrium [2] point $\bar{x}$ of the difference equation (1) is called a global attractor if for every $x_{-1}, x_0 \in (0;\infty)$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

Definition 3.4: An equilibrium [3] point $\bar{x}$ of the equation (1) is called globally asymp-totically stable if it is locally stable and a global attractor.

Definition 3.5: An equilibrium [3] point $\bar{x}$ of the difference equation (1) is called unstable if it is not locally stable.

Definition 3.6: A sequence [4] $(x_n)_n^\infty = -1$ is said to be periodic with period p if $x_{n+p} = x_n$ for all n ≥ -1.

Definition 3.7: A sequence [4] $(x_n)_n^\infty = -1$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Theorem 3.8: (i) $\bar{x}$ is locally asymptotically stable, if and only if [5,6],

$|z| < 1 - h < 2$.

(ii) $\bar{x}$ is unstable and called a saddle point, if and only if,

$z^2 + 4h > 0$ and $|z| > |1 - h|$.

(iii) $\bar{x}$ is called a non-hyperbolic point, if and only if,

$|z| = |1 - h|$

or

h=-1 and $|z| \le 2$.

For s = 0, the difference equation (1) will reduce to

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n} \quad (4)$$

Amleh et al. [7] gave the following results:

(1) If α > 1, the equilibrium point $\bar{x} = \alpha + 1$ of (4) is globally asymptotically stable.

(2) If α = 1, then every positive solution converges to a solution of prime period-two.

(3) Every positive solution is bounded, if and only if, α ≥ 1.

(4) The equilibrium point is an unstable saddle point, if α ϵ (0,1).

As of late, there has been incredible enthusiasm for examining nonlinear and rational difference equations, see, for instance the

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Received October 27, 2015; **Accepted** November 09, 2015; **Published** November 13, 2015

Citation: Bakery AA (2015) Oscillation of a Nonlinear Difference Equation of Power Type. J Appl Computat Math 4: 270. doi:10.4172/2168-9679.1000270

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references therein [8-16].

Throughout the article we denote by! the class

$\omega := \{(x_n) : x_n \in \mathbb{R}^+ \; and\; x_n \; is\; a\; solution\; of\; (1)\}$,

and l_∞ denote the class of bounded sequences of real numbers.

Global Behavior of Solutions and Boundedness

Firstly we determine the classification of the equilibrium points for equation (1) and its uniqueness. By applying theorem (3.8) in the special difference equation (1), we obtain the following result.

Lemma 4.1: (i) x is locally asymptotically stable, if and only if,

$$\frac{2-\bar{x}}{s(\bar{x})^s} < \alpha < \frac{(\bar{x})^{1-s}}{s}.$$

(ii) $\bar{x}$ is unstable, if and only if,

$$\alpha > \frac{(\bar{x})^{1-s}}{s} \; or\; \alpha < \frac{2-\bar{x}}{s(\bar{x})^s}.$$

(iii) $\bar{x}$ is non-hyperbolic point, if and only if,

$$\alpha = \frac{(\bar{x})^{1-s}}{s} \; or\; \alpha = \frac{2-\bar{x}}{s(\bar{x})^s}.$$

Proof: The proof is easy, so omitted.

Lemma 4.2: (1) If s = 1 and α ϵ (0,1), then there exist an unique equilibrium point $\bar{x} = \frac{1}{1-\alpha} > 1$ of equation (1).

(2) If s ϵ (0,1), then there exist an unique equilibrium point $\bar{x} > 1$ of equation (1).

(3) For s > 1, if

(i) $0 < \alpha < \frac{1}{s}\left(\frac{s-1}{s}\right)^{(s-1)}$, then (1) has two equilibrium points $\bar{x} > 1$.

(ii) $\alpha = \frac{1}{s}\left(\frac{s-1}{s}\right)^{(s-1)}$, then there exist an unique equilibrium point $\bar{x} > 1$ of equation (1).

(iii) $\alpha > \frac{1}{s}\left(\frac{s-1}{s}\right)^{(s-1)}$, then there is no equilibrium point of equation (1).

Proof: (1) For s = 1 and α ϵ (0,1), then from the definition of the function g, we get $g(\bar{x}) = \bar{x} - \alpha\bar{x} - 1 = 0$, which gives $\bar{x} = \frac{1}{1-\alpha} > 1$.

(2) Let s ϵ (0; 1), from the definition of the function g, we have $g(1) = -\alpha < 0, \lim_{x\to\infty} g(x) = \infty$. The function g is decreasing on $\left[0, (s\alpha)^{\frac{1}{1-s}}\right]$ and increasing on $\left[(s\alpha)^{\frac{1}{1-s}}, \infty\right]$ hence g has a unique root $\bar{x} > 1$.

(3) For s > 1, since g(0) = -1 and g(1) = -α, we have also g'(x) = 0 if and only if $x = \left(\frac{1}{s\alpha}\right)^{\frac{1}{s-1}}$ which is a maximum point of g(x). Since

$$g\left(\left(\frac{1}{s\alpha}\right)^{\frac{1}{s-1}}\right) = \left(\frac{1}{s\alpha}\right)^{\frac{1}{s-1}}\left(\frac{s-1}{s}\right) - 1.$$

We have three cases, the first one is

$$g\left(\left(\frac{1}{s\alpha}\right)^{\frac{1}{s-1}}\right) > 0 \Leftrightarrow 0 < \alpha < \frac{1}{s}\left(\frac{s-1}{s}\right)^{s-1}.$$

Hence there is two equilibrium points $\bar{x} > 1$.

The second case is

$$g\left(\left(\frac{1}{s\alpha}\right)^{\frac{1}{s-1}}\right) = 0 \Leftrightarrow \alpha = \frac{1}{s}\left(\frac{s-1}{s}\right)^{s-1}.$$

Hence there is one equilibrium point $\bar{x} > 1$.

The third case is

$$g\left(\left(\frac{1}{s\alpha}\right)^{\frac{1}{s-1}}\right) < 0 \Leftrightarrow \alpha < \frac{1}{s}\left(\frac{s-1}{s}\right)^{s-1}.$$

Hence there is no equilibrium point. This ends the proof.

Secondly we give the sufficient condition for the positive solutions of equation (1) to be bounded and its equilibrium point to be global asymptotically stable.

Theorem 4.3: (i) If s = 1 and α ≥ 1, then $\omega \cap l_\infty = \theta$.

(ii) If s > 1 and α = 1, then $\omega \backslash l_\infty \neq \theta$.

(iii) If s > 1 and α > 1, then $\omega \backslash l_\infty \neq \theta$.

(iv) If s > 1 and α ϵ (0,1), then $\omega \backslash l_\infty \neq \theta$.

Proof: (i) Let (x_n) be a bounded solution of the difference equation $x_{n+1} = \alpha x_n + \frac{x_{n-1}}{x_n}$.

Since $x_{n+1} \geq \alpha x_n \geq x_n$ for each n ≥ -1. Hence x_n is convergent, which gives a x_n contradiction. So all solutions of equation (1) are unbounded.

(ii) For s > 1 and α = 1, we have

$$\sup_n\left\{x_n : x_{n+1} = x_n^8 + \frac{x_{n-1}}{x_n}\right\} \geq \sup_n\left\{x_n : x_{n+1} = 0 + \frac{x_{n-1}}{x_n}\right\} = \infty.$$

(iii) For s > 1, α > 1 and by using condition (ii), we obtain

$$\sup_n\left\{x_n : x_{n+1} = \alpha x_n^8 + \frac{x_{n-1}}{x_n}\right\} \geq \sup_n\left\{x_n : x_{n+1} = x_n^8 + \frac{x_{n-1}}{x_n}\right\} = \infty.$$

(iv) For s > 1 and 0 < α < 1, we have

$$\sup_n\left\{x_n : x_{n+1} = \alpha x_n^8 + \frac{x_{n-1}}{x_n}\right\} \geq \sup_n\left\{x_n : x_{n+1} = 0 + \frac{x_{n-1}}{x_n}\right\} = \infty.$$

Theorem 4.4: If s = 1, then $\omega = l_\infty$.

Proof: We have two cases:

(i) Let s = 1 and α ϵ (1/2;1), then by using equation (1) we get

$$x_{n+1} = \alpha x_n + \frac{x_{n-1}}{x_n} \Rightarrow x_{n+1} > \alpha x_n \Rightarrow \frac{x_n}{x_{n+1}} < \frac{1}{\alpha} < 2 \Rightarrow \frac{x_{n-1}}{x_n} < 2 \Rightarrow x_{n+1} < \alpha x_n + 2,$$

for all n ≥ -1, hence for odd indices we obtain

$$x_{2n+1} < \alpha x_{2n} + \frac{1}{\alpha} < \alpha\left(\alpha x_{2n-1} + \frac{1}{\alpha}\right) = \alpha^2 x_{2n-1} + \frac{1}{\alpha} + 1$$

$$< \alpha^2\left(\alpha x_{2n-2} + \frac{1}{\alpha}\right) + 1 + \frac{1}{\alpha} = \alpha^3 x_{2n-2} + \alpha + 1 + \frac{1}{\alpha}$$

$$< \alpha^3\left(\alpha x_{2n-3} + \frac{1}{\alpha}\right) + \alpha + 1 + \frac{1}{\alpha} = \alpha^4 x_{2n-2} + \alpha^2 + \alpha^2 + 1 + \frac{1}{\alpha}$$

By induction, we deduce

$$x_{2n+1} < x^{2n+2}x-1+\alpha^{2n}+\alpha^{2n-1}+...+\alpha+\frac{1}{\alpha}+<\frac{1}{\alpha(1-\alpha)}.$$

And similarly, we get for even indices

$$x_{2n} < x^{2n}x_0+x^{2n-2}+\alpha^{2n-3}+...+\alpha+\frac{1}{\alpha}+1<\frac{1}{\alpha(1-\alpha)}.$$

This completes the proof.

Lemma 4.5: Given equation (1), for all x_{-1} and $x_0 \epsilon (0,\infty)$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $x_n \geq 1$.

Proof: Let x_{-1} and $x_0 \epsilon (0,\infty)$ and there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ with $x_n < 1$. By using equation (1), we have

$$x_{n+1}=\alpha x_n^8+\frac{x_{n-1}}{x_n}<1 \Rightarrow \frac{x_{n-1}}{x_n}<1 \Rightarrow x_{n_0-1}<x_{n_0}<1$$

Hence there exists $n_0-1 \in \mathbb{N}$ for every $n \geq n_0 -1$, then $x_n < 1$, so continuing in the same manner we get $x_{-1} < 1$. This gives a contradiction.

Theorem 4.6: If $s \epsilon (0,1)$, then $\omega = l_\infty$.

Proof: We have three cases:

(i) For $\alpha \epsilon (0,1)$ and $s \epsilon (0,1)$, since from lemma (4.5), there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $x_n \geq 1$. By using equation (1) and theorem (4.4), we obtain

$$\sup_{n\geq n_0}\left\{x_n : x_{n+1}=\alpha x_n^8+\frac{x_{n-1}}{x_n}\right\} \geq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=\alpha x_n+\frac{x_{n-1}}{x_n}\right\}<\infty.$$

And since $(x_n)_{n=-1}^{n_o-1}$ is bounded, so the proof follows.

(ii) For $\alpha>1$ and $s \epsilon (0,1)$, since from lemma (4.5), there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $x_n \geq 1$. By using equation (1) and theorem (4.4), we have

$$\sup_{n\geq n_0}\left\{x_n : x_{n+1}=\alpha x_n^8+\frac{x_{n-1}}{x_n}\right\} \leq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=\alpha x_n+\frac{x_{n-1}}{x_n}\right\}$$

$$\leq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=\alpha x_n+\alpha^2\frac{x_{n-1}}{x_n}\right\}=\sup_{n\geq n_0}\left\{x_n : \frac{1}{\alpha^2}x_{n+1}=\frac{1}{\alpha}x_n+\frac{x_{n-1}}{x_n}\right\}$$

$$\leq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=\frac{1}{\alpha}x_n+\frac{x_{n-1}}{x_n}\right\}<\infty$$

And since $(x_n)_{n=-1}^{n_o-1}$ is bounded, so the proof follows.

(iii) For $\alpha = 1$ and $s \epsilon (0,1)$, since from lemma (4.5), there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $x_n \geq 1$. By using equation (1) and theorem (4.4), we have

$$\sup_{n\geq n_0}\left\{x_n : x_{n+1}=x_n^8+\frac{x_{n-1}}{x_n}\right\} \leq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=x_n+\frac{x_{n-1}}{x_n}\right\} \leq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=2x_n+\frac{4x_{n-1}}{x_n}\right\}$$

$$\leq \sup_{n\geq n_0}\left\{x_n : \frac{1}{4}x_{n+1}=\frac{1}{2}x_n+\frac{x_{n-1}}{x_n}\right\} \leq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=\frac{1}{2}x_n+\frac{x_{n-1}}{x_n}\right\}<\infty$$

And since $(x_n)_{n=-1}^{n_o-1}$ is bounded, so the proof ends.

Theorem 4.7: For $s > 1$; the unique positive equilibrium x of equation (1) is not locally asymptotically stable.

Proof: Since $\overline{x}$ of equation (1) is locally asymptotically stable, if and only if,

$$\frac{2-\overline{x}}{s(\overline{x})^s}<\alpha<\frac{(\overline{x})^{1-s}}{s}$$

For $s > 1$, $\overline{x}=\left(\frac{1}{s\alpha}\right)^{\frac{1}{s-1}}$ and $\alpha=\frac{1}{s}\left(\frac{s-1}{s}\right)^{s-1}$, the previous inequality gives a contradiction.

This ends the proof.

Theorem 4.8: For $s = 1$, the unique positive equilibrium $\overline{x}$ of equation (1) is locally asymptotically stable, if and only if, $\alpha\epsilon\left(\frac{1}{2}, 1\right)$.

Proof: For $s = 1$, we have $\overline{x}=\frac{1}{1-\alpha}$, then $\overline{x}$ is locally asymptotically stable, if and only if, $\alpha\epsilon\left(\frac{1}{2}, 1\right)$.

Theorem 4.9: For $s = 1$ and $\alpha\epsilon\left(\frac{1}{3}, 1\right)$, the unique positive equilibrium $\overline{x}$ of equation (1) is global attractor.

Proof: Since (x_n) is bounded, let (x_n) be a divergent sequence with $x_{-1} = x_0 = 1$, then without loss of generality there exists two subsequence (x_{2n}) and (x_{2n+1}) with $\lim_{n\to\infty} x_{2n}=\inf x_n=1$ and $\lim_{n\to\infty} x_{2n+1}=\sup x_n$, then from equation (1) we have

$$x_{2n+1}=\alpha x_{2n}+\frac{x_{2n-1}}{x_{2n}}, for\, all\, n\geq 0.$$

By taking the limit as $n\to\infty$, we obtain $\alpha = 0$, this is a contradiction which completes the proof.

By using theorems (2.8) and (2.9), we get the following result.

Theorem 4.10: For $s = 1$ and $\alpha\in\left(\frac{1}{2},1\right)$, the unique positive equilibrium $\overline{x}$ of equation (1) is global asymptotically stable. This finishes the proof.

Theorem 4.11: For $s\epsilon(0,1)$ and $\frac{2-\overline{x}}{s(\overline{x})^s}<\alpha<\frac{(\overline{x})^{1-s}}{s}$, the unique positive equilibrium $\overline{x}$ of equation (1) is global asymptotically stable.

Proof: By lemma (2.1), $\overline{x}$ is locally asymptotically stable. So, we have to show that for all $\{x_n\}_{n=-1}^{\infty}\in\omega$, there exists an unique positive equilibrium $\overline{x}$ with $\lim_{n\to\infty}x_n=\overline{x}$. Let $\{x_n\}_{n=-1}^{\infty}\in\omega$. By theorem (2.6), $\{x_n\}_{n=-1}^{\infty}\in l_\infty$. Thus, we obtain $0 < l = \lim\inf x_n$, $L = \lim\sup x_n < \infty$.

Hence from equation (1), we get

$$L\leq\alpha L^s+\frac{L}{l}, l\leq\alpha l^s+\frac{l}{L} \tag{5}$$

We have to show that $L = l$, otherwise $L > l$. From equation (5), we obtain

$$(lL)^{(1-s)}\geq\alpha lL\frac{L^{(1-s)}-l^{(1-s)}}{L-l}. \tag{6}$$

By using the mean value theorem for the function $f(x) = x^{(1-s)}$ in (l,L), we nd a

constant $c \epsilon (l,L)$ and from inequality (6), we obtain

$$(lL)^{(1-s)}\geq\alpha lL(1-s)c^{-s}. \tag{7}$$

From lemma (2.5) and for $\alpha \geq 1$, we have

$$\sup_{n\geq n_0}\left\{x_n : x_{n+1}=\alpha x_n^8+\frac{x_{n-1}}{x_n}\right\} \geq \sup_{n\geq n_0}\left\{x_n : x_{n+1}=1+\frac{x_{n-1}}{x_n}\right\}, \tag{8}$$

which gives $L > l \geq 2$, therefore inequality (7) will be

$2^{-3s} \geq \alpha(1-s)$. (9)

Equation (9) with the values α = 1 and s = 1/3 gives a contradiction. Thus, we find L = l.

Periodicity of the Solutions of $x_{n+1} = \alpha x_n^s + \left(\frac{x_{n-1}}{x_n}\right)$

Firstly we study the convergence of the positive solution of equation (1) when s = 1 to a prime two periodic solution.

Theorem 5.1: If $\alpha < \frac{1}{3}$, then equation (1) has a solution of prime period two.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a periodic solution of period two, we have

$$x_{-1} = \alpha x_0 + \frac{x-1}{x_0} = x_1,\ x_0 = \alpha x_{-1} + \frac{x_0}{x_{-1}} = x_2, \quad (10)$$

then (x_{-1}, x_0) is a solution of the system

$$x = \alpha y + \frac{x}{y} \; and \; y = \alpha x + \frac{y}{x}. \quad (11)$$

Let $\{x_n\}_{n=-1}^{\infty} \in \omega$. It is obvious that if equation (10) holds, hence $\{x_n\}_{n=-1}^{\infty}$ has a solution of period two. The two equalities (11) is correspondent to

$$F(x) = 1 - \frac{\alpha^2 x}{(x-1)} - \frac{(x-1)}{\alpha x^2} = 0.$$

Since $\lim_{x\to 1^+} F(x) = -\infty$ and $F(\bar{x}) = F\left(\frac{1}{1-\alpha}\right) = 0$. To find another root (x = a) of F(x), we must have $F'\left(\frac{1}{1-\alpha}\right) < 0$. By simple calculations, we obtain

$$F'\left(\frac{1}{1-\alpha}\right) < 0 \Leftrightarrow 3\alpha^2 - 4\alpha + 1 > 0 \Leftrightarrow \alpha > 1 \, or \, \alpha < \frac{1}{3}.$$

From theorem (4.3), we omit the condition α > 1. Hence by taking x_{-1} = a and $x_0 = \frac{\alpha a^2}{a-1}$, we get a prime 2-periodic solution. This completes the proof.

Secondly we study the convergence of the positive solution of equation (1) when s ϵ (0,1) to a prime two periodic solution.

Theorem 5.2: If s ϵ (0,1), $0 < \alpha \leq \frac{1}{s}$, there exists a positive number ϵ1 such that

$$\varepsilon_1 < \alpha(1+\varepsilon_1)^s, \quad (12)$$

and

$$1 + \varepsilon_1 - \frac{\varepsilon_1}{\alpha(1+\varepsilon_1)^s} - \frac{\alpha^{(s+1)}(1+\varepsilon_1)(s^2+s)}{\varepsilon_1^s} > 0. \quad (13)$$

Then, equation (1) has a solution of prime period two.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a periodic solution of period two, we have

$$x_{-1} = \alpha x_0^s + \frac{x_{-1}}{x_0} = x_1,\ x_0 = \alpha x_{-1}^s + \frac{x_0}{x_{-1}} = x_2, \quad (14)$$

then x_{-1} and x_0 is a solution of the system

$$x = \alpha y^s + \frac{x}{y},\ y = \alpha x^s + \frac{y}{x} \quad (15)$$

The system (15) is correspondent to the equation

$$F(x) = x - \frac{x-1}{\alpha x^s} - \frac{\alpha^{(s+1)} x^{(s^2+s)}}{(x-1)^s} = 0. \quad (16)$$

Thus, we have

$$\lim_{x\to 1^+} F(x) = -\infty \; and \; F(\bar{x}) = 0.$$

More, from inequality (13) and equation (16), we have

$$F(1+\varepsilon_1) > 0.$$

Hence, the equation F(x) = 0 has a root b = 1 + ϵ_0 other than $\bar{x}$ of equation (1), where 0 < ϵ_0 < ϵ_1, for all ϵ ϵ (1,1 + ϵ_1). So, we have

$$y = \frac{\alpha x^{s+1}}{x-1}.$$

Consider the function

F(x) = ϵ - α(1 + ϵ)^s

For $0 < \alpha \leq \frac{1}{s}$, we have f is increasing and from inequality (12), we get f(ϵ_0) < f(ϵ_1) < 0, which gives

$$x_{-1} = 1 + \varepsilon_0 < \frac{\alpha(1+\varepsilon_0)^{s+1}}{\varepsilon_0} = x_0.$$

By taking x_{-1} = b and $x_0 = \frac{\alpha b^{(s+1)}}{b-1}$ then we have a prime 2-periodic solution. This ends the proof.

Thirdly for s > 1, we study the convergence of the positive solution of equation (1) to a prime two periodic solution.

Theorem 5.3: If $s > 1, \alpha \geq \frac{1}{s}$, there exists a positive number ϵ_1 such that

$$\varepsilon_1 > \alpha(1+\varepsilon_1)^s \quad (17)$$

and

$$1 + \varepsilon_1 - \frac{\varepsilon_1}{\alpha(1+\varepsilon_1)^s} - \frac{\alpha^{(s+1)}(1+\varepsilon_1)^{(s^2+s)}}{\varepsilon_1^s} > 0. \quad (18)$$

Then, equation (1) has a solution of prime period two.

Proof: The same previous proof with $\alpha \geq \frac{1}{s}$, we have f is decreasing and from inequality (17), we get

f(ϵ_0) > f(ϵ_1) > 0,

which gives

$$x_{-1} = 1 + \varepsilon_0 > \frac{\alpha(1+\varepsilon_0)^{s+1}}{\varepsilon_0} = x_0.$$

By taking x_{-1} = b and $x_0 = \frac{\alpha b^{(s+1)}}{b-1}$, we have a prime 2-periodic solution. This completes the proof.

Convergence of the Solutions of $x_{n+1} = \alpha x_n^s + \left(\frac{x_{n-1}}{x_n}\right)$

Firstly we study the convergence of every positive solution of equation (1) to a prime two periodic solution. we begin with the following lemma.

Lemma 6.1: Suppose $\{x_n\}_{n=-1}^{\infty} \in \omega$ and β>1. Then, the following conditions are contented.

(i) $\lim_{x\to\infty} x_{2n} = \beta$, if and only if, $\lim_{n\to\infty} x_{2n+1} = \alpha\beta^{s+1}/(\beta-1)$.

(ii) $\lim_{x\to\infty} x_{2n+1} = \beta$, if and only if, $\lim_{x\to\infty} x_{2n} = \alpha\beta^{s+1}/(\beta-1)$.

Proof: (i) Replace n by 2n + 1 in equation (1), we get

$$x_{2n+1} = \alpha x_{2n}^s + \frac{x_{2n-1}}{x_{2n}} \; for \, all \, n \geq 0.$$

Citation: Bakery AA (2015) Oscillation of a Nonlinear Difference Equation of Power Type. J Appl Computat Math 4: 270. doi:10.4172/2168-9679.1000270

If $\lim_{n\to\infty} x_{2n} = \beta$, then $\lim_{n\to\infty} x_{2n+1} = \alpha\beta^{s+1}/(\beta-1)$ and vice versa.

(ii) Replace n by 2n in equation (1), we have

$$x_{2n} = \alpha x_{2n-1}^{s} + \frac{x_{2n-2}}{x_{2n-1}} \text{ for all } n \geq 0.$$

Let $\lim_{n\to\infty} x_{2n+1} = \beta$, then $\lim_{n\to\infty} x_{2n} = \alpha\beta^{s+1}/(\beta-1)$ and vice versa.

Theorem 6.2: If one of the following

(1) s = 1 and α < 1/3.

(2) s ϵ (0,1), 0 < α ≤ 1/s, inequalities (12) and (13).

is satisfied, then each positive solution of equation (1) converges to a prime two periodic solution.

Proof: Let (x_n) be a positive solution of equation (1), then from theorem (4.4) or (4.6) we have (x_n) is bounded and not convergent. By using Bolzano-Weierstrass theorem, there exists a subsequence convergent let it without loss of generality (x_{2n}), hence from lemma (6.1) also (x_{2n+1}) is convergent and with theorem (5.1) or (5.2) the proof follows.

By the same manner as above we give the following result.

Theorem 6.3: If s > 1, α ≥ 1/s, inequalities (17) and (18) are hold, then there exists a positive solution of equation (1) converges to a prime two periodic solution.

Secondly we study the existence of a non-oscillatory solution of equation (1) and the convergence of every positive solution of equation (1) to the positive equilibrium $\bar{x}$.

Theorem 6.4: There exist a non-oscillatory solution of equation (1).

Proof: By choosing x_{-1} and $x \geq \bar{x}$, we obtain

$$x_1 = \alpha x_0^s + \frac{x-1}{x_0} \geq \alpha\bar{x}^s + 1 = \bar{x},$$

$$x_2 = \alpha x_1^s + \frac{x_0}{x_1} \geq \alpha\bar{x}^s + 1 = \bar{x},$$

By induction, we deduce $x_n \geq \bar{x}$ for all n ≥ 0. If x_{-1} and $x_0 \leq \bar{x}$ we have also $x_n \leq \bar{x}, n \geq 0$.

Theorem 6.5: All non-oscillatory solutions of equation (1) converge to the positive equilibrium $\bar{x}$.

Proof: Let (x_n) be a non-oscillatory solutions of equation (1), then we get

$$x_n \geq \bar{x} \text{ for all } n \geq -1 \Rightarrow x_{n+1} = \alpha x_n^s + \frac{x_{n-1}}{x_n} \geq \alpha\bar{x}^s + 1 \text{ for all } n \geq 0 \Rightarrow \frac{x_{n-1}}{x_n} \geq 1 \text{ for all } n \geq 0$$
$$\Rightarrow x_{n-1} \geq x_n \text{ for all } n \geq 0.$$

Then (x_n) is decreasing and bounded from below, then it is convergent. Also if

$$x_n \leq \bar{x} \text{ for all } n \geq -1 \Rightarrow x_{n+1} = \alpha x_n^s + \frac{x_{n-1}}{x_n} \leq \alpha\bar{x}^s + 1 \text{ for all } n \geq 0 \Rightarrow \frac{x_{n-1}}{x_n} \leq 1 \text{ for all } n \geq 0$$
$$\Rightarrow x_{n-1} \leq x_n \text{ for all } n \geq 0.$$

Then (x_n) is increasing and bounded from above, hence it is convergent. This ends the proof.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this article.

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