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One Dimensional Heat Equation and its Solution by the Methods of Separation of Variables, Fourier Series and Fourier Transform

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Abstract

The aim of this paper was to study the one-dimensional heat equation and its solution. Firstly, a model of heat equation, which governs the temperature distribution in a body, was derived depending on some physical assumptions. Secondly, the formulae in which we able to obtain its solution were derived by using the Method of separation of variables along with the Fourier series and the method of Fourier transform. Then after, some real-life applications of the equations were discussed through examples. Finally, a numerical simulation of the raised examples was studied by using MATLAB program and the results concluded that the numerical simulations match the analytical solutions as expected.

Keywords: Heat Equation • Separation of Variables • Fourier series • Fourier Transform • MATLAB

Introduction

The heat equation is an important partial differential equation which describe the distribution of heat (or variation in temperature) in a given region over time. The heat equation is a wonderland for mathematical analysis, numerical computations, and experiments. It's also highly practical: engineers have to make sure engines don't melt and computer chips don't overheat. Due to this and other many real-life applications of heat equations, we need to analyze the concepts in detail.

For better understanding, we have to recognize differences between heat and temperature. Heat is the flow of thermal energy from a warmer place to a cooler place. Thus, the term heat is used to describe the energy transferred through the heating process. On the other hand, temperature is a physical property of matter that describes the hotness or coldness of an object or environment. Therefore, no heat would be exchanged between bodies of the same temperature [1].

The one-dimensional heat equation that we are going to see in this study is given by the formula

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is a function of temperature, c^2 is the constant thermal conductivity of the materials, t is time and x is a spatial variable. Here the "one-dimensional" refers to the fact that we are considering only one spatial dimension [2].

In this study, we focus on the derivation of one-dimensional heat equation and its solution using methods of separation of variables, Fourier series and Fourier transforms along with its numerical analysis using MATLAB.

Derivation of Heat Equation in One Dimension

In this section, we will derive a one-dimensional heat equation which governs the temperature in a body in space. We obtain this model of temperature distribution under the following physical assumptions.

Physical Assumptions

1. We consider a thin rod of length L, made of homogenous material (material properties are translational invariant) and the rod is perfectly insulated along its length so that heat can flow only through its ends (see Figure 1).

2. The *specific heat* σ and the *density* ρ of the material of the rod are constant. No heat is produced or disappears in the body.

3. Experiments show that, in a body, heat flows in the direction of decreasing temperature, and the heat flow Q is proportional to the temperature gradient, that is, $Q = -k\frac{\partial u}{\partial x} = -ku_x(x,t)$ (in one dimension), where k is the thermal conductivity of the material (solid) and the negative sign denotes that the heat flux vector is in the direction of decreasing temperature, $u(x,t)$ is the temperature at a point x and time *t*.

4. The *thermal conductivity* k is constant, as is the case for homogeneous material and nonextreme temperatures.

Depending on the given physical assumptions and Figure 1 above, we can derive a formula of the heat equation as follows.

Let $u(x,t)$ be the temperature of the homogenous thin rod at a distance x at time t. We consider an infinitesimal piece from the rod with length $[x, x+\Delta x]$. If A is the cross-section of the rod and ρ is the density of the material of the rod, then the infinitesimal volume is given by $\Delta V = A\Delta x$ and the corresponding infinitesimal mass is $\Delta m = \rho A \Delta x$. Then, the amount of heat for the volume element is $Q = \sigma \Delta m\, u(x,t)$, where σ is the specific heat of the material of the rod (bar).

At time $t + \Delta t$, the amount of heat is

$$Q_1 = \sigma \Delta m u(x, t+\Delta t).$$

$$Change\ in\ heat = Q_1 - Q = \sigma\Delta m\, u(x,t+\Delta t) - \sigma \Delta m\, u(x,t)$$

$$= \sigma\rho A[u(x,t+\Delta t) - u(x,t)]\Delta x$$

Depending on an assumption (3), the change in heat must be equal to the heat flowing in at x, minus the heat flowing out at $x + \Delta x$, during the time interval Δt, that is,

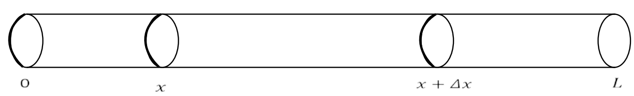

Figure 1. A thin homogenous rod of length L, perfectly insulated along its length.

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$\sigma\rho A[u(x,t+\Delta t)-u(x,t)]\Delta x=[-ku_x(x,t)-(-ku_x(x+\Delta x,t))]\Delta t.$

Then after dividing both sides by $\sigma\rho A\Delta x\Delta t$ we obtain

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}=\left(\frac{k}{\sigma\rho A}\right)\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}$$

Taking the limit on both sides as $\Delta t\to 0$and $\Delta x\to 0$, and by applying the definition of derivative we obtain:

$$\lim_{\Delta t\to 0}\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}=\left(\frac{k}{\sigma\rho A}\right)\lim_{\Delta x\to 0}\frac{u_x(x+\Delta x,t)-u_x(x,t)}{\Delta x}$$

$$\frac{\partial u(x,t)}{\partial t}=\left(\frac{k}{\sigma\rho A}\right)\frac{\partial^2 u(x,t)}{\partial x^2}$$

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which gives the required one-dimensional heat (diffusion) equation determining the heat flow through a small thin rod. Here, $c^2=\frac{k}{\sigma\rho A}$ is called the constant thermal conductivity [2, 3].

Solution of Heat Equation using the Methods of Separation of Variables and Fourier Series

We shall solve Eq. (1) by using methods of separation of variables and Fourier series, for some important types of boundary conditions (BC) and initial conditions (IC). We begin with the case in which the ends $x=0$ and $x=L$ of the rod (bar) are kept at temperature zero, so that we have the boundary conditions (BC)

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Furthermore, the initial temperature in the rod at time $t=0$ is given, say, $f(x)$, so that we have the initial condition (IC)

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Due to Eq. (2), $f(0)=0$ and $f(L)=0$.

We shall determine a solution $u(x,t)$ of Eq. (1) satisfying Eq. (2) and Eq. (3).

Let $u(x,t)=X(x)T(t)$ be a solution of Eq. (1). Substituting $\frac{\partial u}{\partial t}=X(x)T'(t)$ and $\frac{\partial^2 u}{\partial x^2}=X''(x)\,T(t)$ we get:

$$X(x)T'(t)=c^2X''(x)T(t),$$

$$\frac{X''}{X}=\frac{1}{c^2}\frac{T'}{T}$$

From this equality, we deduce that both functions must be equal to some constant k as one of them is a function of x only and the other is a function of t, unless the equality may not hold. Hence

$$\frac{X''}{X}=\frac{1}{c^2}\frac{T'}{T}=k$$

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Now, in order to solve Eq. (4) we will have the following three cases.

Case I: k is positive (= λ^2, say).

Eq. (4) takes the form

$$X''-\lambda^2X=0, T'-(\lambda c)^2T=0.$$

For $X''-\lambda^2X=0$ we try to find a solution of the form $X(x)=e^{mx}$ and obtain an auxiliary equation $m^2-\lambda^2=0$, which implies $m_1=\lambda, m_2=-\lambda$ Therefore, its solution is

$X(x)=Ae^{m_1x}+A'e^{m_2x}=Ae^{\lambda x}+A'e^{-\lambda x}$ where A, A' are arbitrary constants.

Note that from [4],we have that for $X(x)=Ae^{\lambda x}+A'e^{-\lambda x}$, if we choose $A=A'=\frac{1}{2}$ and $A=\frac{1}{2}, A'=-\frac{1}{2}$, we get a particular solution $X=\frac{1}{2}\left(e^{\lambda x}+e^{-\lambda x}\right)=\cosh\lambda x$ and $X=\frac{1}{2}\left(e^{\lambda x}-e^{-\lambda x}\right)=\sinh\lambda x$.

Since $\cosh\lambda x$ and $\sinh\lambda x$ are linearly independent on any interval of x-axis, an alternative form for the general solution of $X''-\lambda^2X=0$ is $X=A_1\cosh\lambda x+A_2\sinh\lambda x$.

Similarly,

$$T'-(\lambda c)^2T=0 \Leftrightarrow \int\frac{dT}{T}=\int(\lambda c)^2\,dt \Rightarrow T(t)=A_3e^{(\lambda c)^2t}.$$

Hence, the general solution of the heat equation, Eq. (1), is

$$u(x,t)=(C_1\cosh(\lambda x)+C_2\sinh(\lambda x))e^{(\lambda c)^2t},$$

where $C_1=A_1A_3, C_2=A_2A_3$.

From the boundary condition on Eq. (2) it follows that

$$u(0,t)=X(0)T(t)=0 \Rightarrow (C_1\cosh(0)+C_2\sinh(0))e^{(\lambda c)^2t}=0 \Rightarrow C_1=0.$$

To obtain C_2 we use

$$u(L,t)=X(L)T(t)=0$$

$$(C_1\cosh(\lambda L)+C_2\sinh(\lambda L))e^{\lambda^2t}=0$$

$$C_2\sinh(\lambda L)=0, (\because C_1=0)$$

$$C_2=0, (\because \sinh\lambda L\neq 0).$$

Therefore,

$$u(x,t)=0.$$

Case II: $k=0$.

Eq. (4) takes the form

$$\frac{d^2X}{dx^2}=0, \frac{dT}{dt}=0$$

$$X(x)=A_4x+A_5, T(t)=A_6$$

where A_4, A_5 and A_6 are arbitrary constants.

A general solution of the heat equation for this case is

$$u(x,t)=C_3x+C_4, C_3=A_4A_6, C_4=A_5A_6.$$

To determine C_3 and C_4 we use Eq. (2) as follows:

$$u(0,t)=X(0)T(t)=0 \Rightarrow C_4=0,$$

$$u(L,t)=X(L)T(t)=0 \Rightarrow C_3L=0 \Rightarrow C_3=0, (\because L\neq 0).$$

Hence

$$u(x,t)=0.$$

Case III: k is negative ($k=-\lambda^2$, say).

Eq. (4) takes the form

$$X''+\lambda^2X=0, T'+(\lambda c)^2T=0.$$

For $X''+\lambda^2X=0$ the auxiliary equation is $m^2+\lambda^2=0\Rightarrow m_1=\lambda i, m_2=-\lambda i$, which is a conjugate complex root with $\alpha=0, \beta=\lambda$, ($m=\alpha\pm i\beta$).

Hence, its general solution is of the form

$$X=e^{\alpha x}(A_7\cos\beta x+A_8\sin\beta x),$$

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where A_7, A_8 are arbitrary constants.

Similarly,

$$T'+(\lambda c)^2T=0 \Leftrightarrow \int\frac{dT}{T}=-\int(\lambda c)^2\,dt$$

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where A_9 is constant. From the boundary conditions it follows that

$$u(0,t)=X(0)T(t)=0, u(L,t)=X(L)T(t)=0$$

$$\Rightarrow X(0), X(L)=0 \text{ or } T(t)=0.$$

Since $T(t) = A_9 e^{-\lambda^2 c^2 t} = 0$ would give $u(x,t) = 0$, which is a trivial solution, we restrict $T \neq 0$ and require $X(0) = 0, X(L) = 0$. Hence,

$X(0) = 0 \Rightarrow A_7 = 0, X(L) = 0 \Rightarrow A_8 \sin \lambda L = 0, (\because A_7 = 0)$.

Here we will still make a restriction $A_8 \neq 0$ to get $u(x,t) \neq 0$.Thus,

$sin\lambda L = 0 \Rightarrow \lambda L = n\pi,$

$\lambda = \frac{n\pi}{L}, n = 1,2,3,\cdots$

Setting $A_8 = 1$,we will obtain infinitely many solutions of Eq.(5), that is $X(x) = X_n(x)$, where

$X_n = \sin\frac{n\pi x}{L}, n = 1,2,3,\cdots$

These solutions satisfy Eq.(2). Note that for negative integer $n$we obtain essentially the same solutions, except for a minus sign, because $\sin(-\alpha) = -\sin\alpha$.

We now substitute $k = -\lambda^2 = -\left(\frac{n\pi}{L}\right)^2$in Eq.(6) and again obtain an infinitely many solutions,

$T_n(t) = B_n e^{-\left(\frac{nc\pi}{L}\right)^2 t}, (n = 1,2,3,\cdots)$

where B_n is a constant. Hence the functions

$Eq.($ Error! Bookmark not defined.$) u_n(x,t) = X_n(x)T_n(t) = B_n sin($

are solutions of the heat equation, (Eq. (1)), satisfying Eq.(2).

Here, Eq.(7) is a solution of Eq. (1) that satisfies the boundary conditions given on Eq.(2) but we didn't use an initial condition given on Eq.(3). To obtain a solution that also satisfies the initial condition on Eq.(3), we consider a series of Eq.(7). That is,

$Eq.($ Error! Bookmark not defined.$) u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) =$

From the initial condition, Eq.(3), we have

$u(x,0) = \sum B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

Hence for Eq.(8) to satisfy Eq.(3),B_n's must be the coefficients of the Fourier sine series. Thus,

$Eq.($ Error! Bookmark not defined.$) B_n = \frac{2}{L}\int_0^L f(x)$ ↕

The solution of our problem can be established, assuming that $f(x)$ is piecewise continuous on the interval $0 \leq x \leq L$.We can generalize this method by the following theorem [3, 5, 6].

Theorem 1: The nontrivial Fourier series solution of the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ with the boundary condition (BC)

$u(0,t) = 0, u(L,t) = 0 \ \forall t \geq 0$

and initial condition (IC)

$u(x,0) = f(x)$

is given by the formula

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{nc\pi}{L}\right)^2 t}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, (0 \leq x \leq L, n = 1,2,3,\cdots)$$

Example 1:A thin bar of length π units is placed in boiling water (temperature $100°C$). After reaching 100 °C throughout, the bar is removed from the boiling water. With the lateral sides kept insulated, suddenly, at time $t = 0$ the ends are immersed in a medium with constant freezing temperature $0°C$. Taking $c = 1$, find the temperature $u(x,t)$ for $t \geq 0$.

Solution: The problem that we need to solve is

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 \leq x \leq \pi, t \geq 0,$

BC: $u(0,t) = 0$ and $u(\pi,t) = 0, t > 0,$

IC: $u(x,0) = 100, 0 \leq x \leq \pi$.

From Theorem 1, we have

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\, e^{-n^2 t}$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} 100\sin(nx)\,dx = \frac{200}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{200}{n\pi}(1 - \cos n\pi).$$

But $\cos n\pi = (-1)^n$ implies

$$1 - \cos n\pi = \begin{cases} 2, if\ n\ is\ odd \\ 0, if\ n\ is\ even\end{cases} \Rightarrow B_n = \begin{cases}\frac{400}{n\pi}, n = 2k+1 \\ 0, n = 2k\end{cases}, k = 1,2,\cdots$$

Hence, the required temperature distribution of the bar for all $t \geq 0$ is

$$u(x,t) = \frac{400}{\pi}\sum_{k=1}^{\infty}\frac{e^{-(2k+1)^2 t}}{2k+1}\sin(2k+1)x.$$

If we substitute a given value of t into this series solution, we will obtain a function of x alone. This function gives the temperature distribution of the bar at the given time t.

Numerical Solution of Example 1 by its graph

Apart from above method we used to obtain analytical series solution of Example1, we can use numerical simulations to visualize a physical meaning and behavior of the solution obtained by sketching its graph. In the following, we will introduce the solution procedures by using a syntax of partial differential equation solver pdepe() provided in MATLAB PDE Toolbox. We might prepare the following MATLAB function and place in the @sym directory.

```
function [c,f,s] = pdex1pde(x,t,u,DuDx) %pdex1pde is the function name, where as the function c,
%f and s can be calculated from the heat equation
c = 1;
f = DuDx;
s = 0;
```

```
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t) % syntax of the boundary condition, for left and % right bounds
pl = ul;
ql = 0;
pr = ur;
qr = 0;
```

```
function u0 = pdex1ic(x) % syntax of the initial condition
u0 = 100;
```

```
x=0:.5:pi; % 0 < x < pi
t=0:0.5:4; % time interval with 0.5 step size.
m=0;
sol = pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t);
u = sol(:,:,1); % Extract the first solution component as u.
surf(x,t,u) % A surface plot is often a good way to study a solution.
title('Fig. 2: Numerical solution surface')
xlabel('Distance x')
ylabel('Time t')
zlabel('u(x,t)')
% A solution profile can also be illuminating.
figure
plot(x,u(end,:))
title('Fig. 3: Temperature distribution in a bar for t>0')
xlabel('Distance x')
ylabel('u(x,t)')
grid
```

Figures 2 & 3 shows the temperature distribution of the rod at the taken values of t in an interval $[0,4]$. Figure 2 illustrates that for $t=0$, the temperature distribution in the rod (with ends held at $0°C$) is the same with the given initial temperature $u(x,0)=100°C$ as it was expected; and for small values of $t\in(0,4)$, the temperature in the bar close to the initial temperature and for large values of $t\in(0,4]$, the temperature decays to $0°C$. Figure 3 simply illuminates a solution profile of Figure 2.

Solution of Heat Equation using the Method of Fourier Transform

Our discussion of heat equation, (Eq. (1)), here is extended to rods (bars) of infinite length, which are good models of very long bars or wires. Let us illustrate the method by solving Eq. (1) for a bar that extends to infinity on both sides and is laterally insulated. Then we do not have boundary conditions, but only the initial condition (IC)

$$u(x,0)=f(x), -\infty<x<\infty$$

where $f(x)$ is the given initial temperature of the bar.

Let $\hat{u}(\omega,t)=\mathcal{F}(u(x,t);\ x\to\omega)$ denote the Fourier transform of $u(x,t)$, regarded as a function of x. Then, the heat equation becomes,

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Recall that $\mathcal{F}\left(f^{(n)}(x);x\to\omega\right)=(i\omega)^n\mathcal{F}(f(x))$. Therefore,

$$\mathcal{F}\left(\frac{\partial^n u(x,t)}{\partial x^n};x\to\omega\right)=(i\omega)^n\mathcal{F}(u(x,t);\ x\to\omega)$$

and

$$\mathcal{F}(u(x,t);\ x\to\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx.$$

Hence,

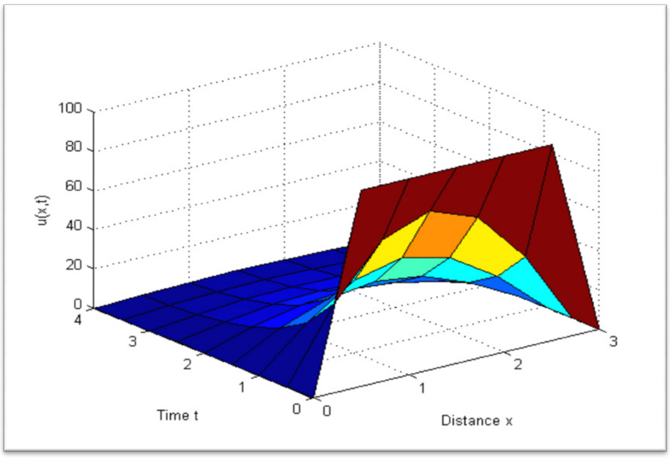

Figure 2. Numerical solution surface.

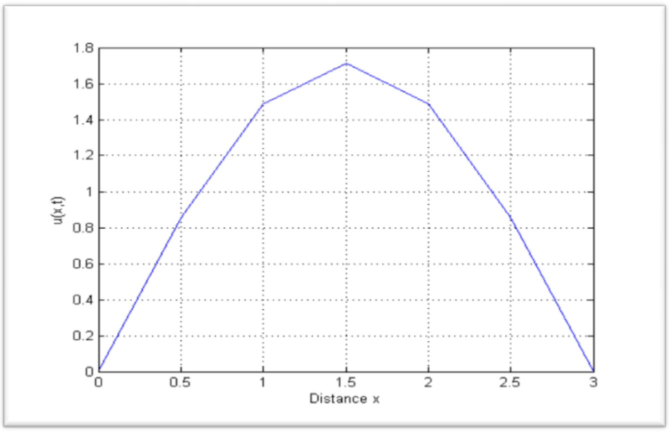

Figure 3: Temperature distribution in a bar for t>0.

$$\mathcal{F}\left(\frac{\partial^2 u(x,t)}{\partial x^2};x\to\omega\right)=(i\omega)^2\mathcal{F}(u(x,t);\ x\to\omega)=(i\omega)^2\hat{u}(\omega,t)=-\omega^2\hat{u}(\omega,t).$$

and

$$\mathcal{F}(u_t(x,t);\ x\to\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\big(u(x,t)\big)e^{-i\omega x}dx.$$

Now, assuming that we may interchange the order of differentiation and integration, we have

$$\mathcal{F}(u_t(x,t))=\widehat{(u_t)}(\omega,t)=\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx\right)=\frac{\partial}{\partial t}(\hat{u}(\omega,t))=\hat{u}_t(\omega,t).$$

Here the symbol $\widehat{(u_t)}$ denotes the transform of the derivatives u_t and $\hat{u}_t$ denotes the derivative of the transform $\hat{u}$.

Thus, Eq. (10) becomes

$$\hat{u}_t(\omega,t)=-c^2\omega^2\hat{u}(\omega,t).$$

Since this equation involves only a derivative with respect to t but none with respect to ω, this is a first order ordinary DE, with t as the independent variable and ω as a parameter. Hence

$$\frac{d}{dt}\hat{u}(\omega,t)=-c^2\omega^2\hat{u}(\omega,t)\ \Rightarrow\frac{d\hat{u}}{\hat{u}}=-c^2\omega^2dt\Rightarrow\int\frac{d\hat{u}}{\hat{u}}=-c^2\omega^2\int dt$$

$$\Rightarrow \ln\hat{u}=-c^2\omega^2t+g_1(\omega)$$

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where $g(\omega)=e^{g_1(\omega)}$ is an arbitrary function of ω.

The Fourier transform of the initial condition $u(x,0)=f(x)$ yields:

$$\mathcal{F}(u(x,0);\ x\to\omega)=\mathcal{F}\big(f(x)\big)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,0)e^{-i\omega x}dx=\mathcal{F}\big(f(x)\big)$$

$$\hat{u}(\omega,0)=\mathcal{F}\big(f(x)\big).$$

Substituting this in Eq. (11) we obtain that

$$\hat{u}(\omega,0)=g(\omega)=\mathcal{F}\big(f(x)\big).\ (\because t=0)$$

Hence Eq. (11) becomes

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But we know that the Fourier inverse of $\hat{u}(\omega,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx$ is

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(\omega,t)e^{i\omega x}d\omega.$$

Therefore, by substituting Eq. (12) in this equation, we will obtain

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-c^2\omega^2t}\mathcal{F}\big(f(x)\big)e^{i\omega x}d\omega\ or$$

Eq.(Error! Bookmark not defined.) $u(x,t)=\frac{\ }{\sqrt{\ }}$

where $\hat{f}(\omega)$ is another notation for $\mathcal{F}\big(f(x)\big)$.

In Eq. (13) we may insert the Fourier transform $\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-i\omega v}dv$ and exchange the order of integration as follows:

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-i\omega v}dv\right)e^{-c^2\omega^2t}e^{i\omega x}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(v)\left[\int_{-\infty}^{\infty}e^{-c^2\omega^2t}e^{i(\omega x-\omega v)}d\omega\right]dv.$$

By Euler formula, the integral of the inner function becomes

$$\int_{-\infty}^{\infty}e^{-c^2\omega^2t}[\cos(\omega x-\omega v)+i\sin(\omega x-\omega v)]d\omega$$

$$=\int_{-\infty}^{\infty}e^{-c^2\omega^2t}\cos(\omega x-\omega v)\,d\omega+i\underbrace{\int_{-\infty}^{\infty}e^{-c^2\omega^2t}\sin(\omega x-\omega v)\,d\omega}_{=0}.$$

Since $e^{-c^2\omega^2 t}\sin(\omega x - \omega v)$ is an odd function of ω, its integral is 0 and as $e^{-c^2\omega^2 t}\cos(\omega x - \omega v)$ is even function of ω, its integral from $-\infty$ to ∞ equals twice the integral from 0 to ∞. Hence

$Eq.($ Error! Bookmark not defined.$)$ $u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} e^{-}\right.$

Then we can evaluate the inner integral of Eq.(14) by using the formula

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This takes the form of our inner integral if we choose $\omega = \frac{s}{c\sqrt{t}}$ as a new variable of integration and set $b = \frac{x-v}{2c\sqrt{t}}$. Then $2bs = \omega(x - v)$ and $ds = c\sqrt{t}d\omega$ so that Eq.(15) becomes

$$\int_0^{\infty} e^{-c^2\omega^2 t}\cos(\omega x - \omega v)c\sqrt{t}d\omega = \frac{\sqrt{\pi}}{2}e^{-\left(\frac{x-v}{2c\sqrt{t}}\right)^2} \text{ or}$$

$$c\sqrt{t}\int_0^{\infty} e^{-c^2\omega^2 t}\cos(\omega x - \omega v)\,d\omega = \frac{\sqrt{\pi}}{2}\exp\left(-\frac{(x-v)^2}{4c^2t}\right) \text{ or}$$

$$\int_0^{\infty} e^{-c^2\omega^2 t}\cos(\omega x - \omega v)\,d\omega = \frac{\sqrt{\pi}}{2c\sqrt{t}}\exp\left(-\frac{(x-v)^2}{4c^2t}\right)$$

By inserting this result into Eq.(14) we obtain the representation

$Eq.($ Error! Bookmark not defined.$)$ $u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty}$

Taking $z = \frac{v-x}{2c\sqrt{t}}$ as a variable of integration, we get the alternative form

$Eq.($ Error! Bookmark not defined.$)$ $u(x,t) = -$

If $f(x)$ is bounded for all values of x and integrable in every finite interval, it can be shown that the function on Eq.(16) or Eq.(17) satisfies the heat equation and its initial condition. Hence Eq.(16) or Eq.(17) is the required solution [2, 6, 7, 8]. We can generalize this method by the following theorem.

Theorem 2: The nontrivial Fourier transform solution of the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ with the initial condition (IC)

$$u(x,0) = f(x)$$

is given by the formula

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left(-\frac{(x-v)^2}{4c^2t}\right)dv,$$

where $f(v)$ is the given initial temperature.

Moreover, to get a most simplified answer we might substitute $z = \frac{v-x}{2c\sqrt{t}}$ and use

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2cz\sqrt{t})e^{-z^2}dz, .$$

Example 2: Solve the heat problem on the infinite line with $c = 1$ and initial temperature distribution $f(x) = 100$ if $|x| < 1$ and 0 otherwise.

Solution: In order to solve this problem, we will use Theorem 2.

Since the given initial temperature is $f(v) = \begin{cases} 100, -1 < v < 1 \\ 0, v < -1, v > 1 \end{cases}$, we might break up the limit of integral into three and obtain

$$u(x,t) = \frac{100}{2\sqrt{\pi t}}\int_{-1}^{1} exp\left(-\frac{(x-v)^2}{4t}\right)dv.$$

To simplify this, we will substitute $z = \frac{v-x}{2c\sqrt{t}}$ and use the following conversions with $c = 1$.

$$z = \frac{v-x}{2c\sqrt{t}} \Rightarrow dz = \frac{1}{2c\sqrt{t}}dv, v = -1 \Rightarrow z = \frac{-1-x}{2c\sqrt{t}}, v = 1 \Rightarrow z = \frac{1-x}{2c\sqrt{t}}.$$

Hence

$$u(x,t) = \frac{100}{\sqrt{\pi}}\int_{(-1-x)/2\sqrt{t}}^{(1-x)/2\sqrt{t}} e^{-z^2}dz$$

This integral is not an elementary function, but can be expressed in terms of the error function (or an integral of a Gaussian function) which is given by the formula [2]

$$\text{erf}(w) = \frac{2}{\sqrt{\pi}}\int_0^{w} e^{-z^2}dz, \forall w.$$

That is,

$$u(x,t) = \frac{100}{\sqrt{\pi}}\int_{(-1-x)/2\sqrt{t}}^{0} e^{-z^2}dz + \frac{100}{\sqrt{\pi}}\int_0^{(1-x)/2\sqrt{t}} e^{-z^2}dz$$

$$= 50\left[\frac{2}{\sqrt{\pi}}\int_0^{(1-x)/2\sqrt{t}} e^{-z^2}dz - \frac{2}{\sqrt{\pi}}\int_0^{(-1-x)/2\sqrt{t}} e^{-z^2}dz\right]$$

$$= 50\left[\text{erf}\left(\frac{1-x}{2\sqrt{t}}\right) - \text{erf}\left(\frac{-1-x}{2\sqrt{t}}\right)\right], t > 0$$

Numerical Solution of Example 2 by its graph

Here, we use the preceding solution of *Example2* to analyze the behavior of the solution from its graph which might be sketched using MATLAB program and use the following MATLAB code and obtain Fig. 4 below.

```
syms x
E1=50*(erf((1-x)/(2*sqrt(0.00000000001)))-erf((-1-x)/
(2*sqrt(0.00000000001)))); % evaluate the %function value
E2=50*(erf((1-x)/(2*sqrt(1/10)))-erf((-1-x)/(2*sqrt(1/10))));
E3=50*(erf((1-x)/(2*sqrt(1/2)))-erf((-1-x)/(2*sqrt(1/2))));
E4=50*(erf((1-x)/(2*sqrt(1)))-erf((-1-x)/(2*sqrt(1))));
E5=50*(erf((1-x)/(2*sqrt(10)))-erf((-1-x)/(2*sqrt(10))));
ezplot(E1); % draw the curve
hold on; % To reserve current axis
ezplot(E2);
hold on;
ezplot(E3);
hold on;
ezplot(E4);
hold on;
ezplot(E5);
hold off;
X=-6:0.05:6; % specify the vector with a step-size of 0.05
E1X=double(subs(E1,x,X)); % To evaluate symbolic expression numerically
E2X=double(subs(E2,x,X));
E3X=double(subs(E3,x,X));
E4X=double(subs(E4,x,X));
E5X=double(subs(E5,x,X));
plot(X,E1X,'r-.pentagram',X, E2X,'k',X,E3X,'c',X, E4X,'b',X, E5X,'--'); %
To produce a %multicolored graph that indicates the difference between
E1,E2, E3, E4 and E5
title('Fig.4: Solution u(x,t) of Example 2');
legend('t=0','t=1/10','t=1/2','t=1','t=10');
xlabel('x');
ylabel('u(x,t)');
grid % To add grids to the curve
```

Figure 4 shows the temperature distribution in Example 2 at various values of t. The graphs show that for small values of t, the temperature in the bar close to the initial temperature distribution, and as t increases, the temperature spreads through the bar and eventually approximate to 0 (or reaches the equilibrium temperature of 0°C).

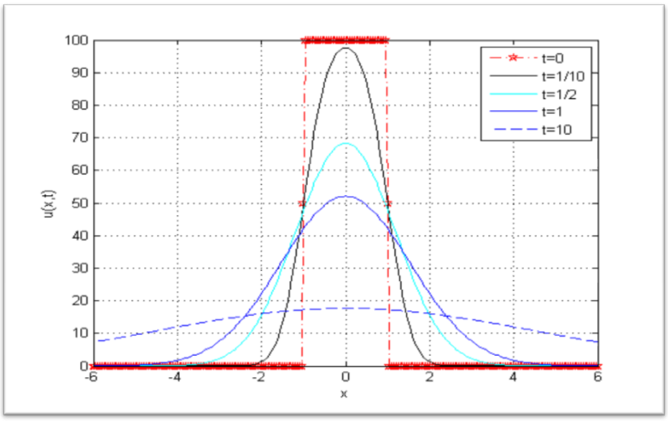

Figure 4: Solution u(x,t) of Example 2.

Conclusion

Many physical phenomena that interrelated to temperature distribution can be modeled using one dimensional heat equation as we have seen in this paper. In many cases analytical solutions are not enough to visualize the behavior of the solutions. Thus, we might depend on numerical solutions to obtain more information on the inherent problems. In this paper we have observed how to derive and solve one dimensional heat equation. Furthermore, by using MATLAB program we have provided the tangible understanding on the examples raised in the paper.

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