Journal of Generalized Lie Theory and Applications Vol. 3 (2009), No. 3, 197–259

On the structure of left and right F-, SM-, and E-quasigroups

Victor SHCHERBACOV

Institute of Mathematics and Computer Science, Academy of Sciences of Moldova, str. Academiei 5, MD-2028 Chisinau, Moldova

E-mail: scerb@math.md

Abstract

It is proved that any left F-quasigroup is isomorphic to the direct product of a left F-quasigroup with a unique idempotent element and isotope of a special form of a left distributive quasigroup. The similar theorems are proved for right F-quasigroups, left and right SM- and E-quasigroups. Information on simple quasigroups from these quasigroup classes is given; for example, finite simple F-quasigroup is a simple group or a simple medial quasigroup. It is proved that any left F-quasigroup is isotopic to the direct product of a group and a left S-loop. Some properties of loop isotopes of F-quasigroups (including M-loops) are pointed out. A left special loop is an isotope of a left F-quasigroup if and only if this loop is isotopic to the direct product of a group and a left S-loop (this is an answer to Belousov "1a" problem). Any left E-quasigroup is isotopic to the direct product of an abelian group and a left S-loop (this is an answer to Kinyon-Phillips 2.8(1) problem). As corollary it is obtained that any left FESM-quasigroup is isotopic to the direct product of an abelian group and a left S-loop (this is an answer to Kinyon-Phillips 2.8(2) problem). New proofs of some known results on the structure of commutative Moufang loops are presented.

2000 MSC: 20N05

Contents

1 Introduction

Murdoch introduced F-quasigroups in [77]. At this time, Sushkevich studied quasigroups with the weak associative properties [116, 117]. Their name F-quasigroups obtained in an article of Belousov [8]. Later Belousov and his pupils Golovko and Florja, Ursul, Kepka, Kinyon, Phillips, Sabinin, Sbitneva, Sabinina, and many other mathematicians studied F-quasigroups and left F-quasigroups [12, 15, 16, 28, 34, 35, 36, 41, 42, 53, 55, 86, 87]. In [55, 57, 58] it is proved that any F-quasigroup is linear over a Moufang loop. The structure of F-quasigroups also is described in [55, 57, 58].

Left and right SM-quasigroups (semimedial quasigroups) are defined by Kepka. In [49] Kepka has called these quasigroups LWA-quasigroups and RWA-quasigroups, respectively. SM-quasigroups are connected with trimedial quasigroups. These quasigroup classes are studied in [6, 49, 50, 52, 63, 64, 106, 107]. Kinyon and Phillips have defined and studied left and right E-quasigroups [64].

Main idea of this paper is to use quasigroup endomorphisms by the study of structure of quasigroups with some generalized distributive identities. This idea has been used by the study of many loop and quasigroup classes, for example, by the study of commutative Moufang loops, commutative diassociative loops, CC-loops (LK-loops), F-quasigroups, SM-quasigroups, trimedial quasigroups, and so on [4, 7, 12, 23, 24, 25, 61, 62, 65, 83, 84]. This idea is clearly expressed in Shchukin's book [106].

Using language of identities of quasigroups with three operations in signature, i.e., of quasigroups of the form $(Q, \cdot, /, \backslash)$, we can say that we study some quasigroups from the following quasigroup classes: (i) $(xy)\backslash(xy) = (x\backslash x) \cdot (y\backslash y)$; (ii) $(xy)/(xy) = (x/x) \cdot (y/y)$; (iii) $(xy) \cdot (xy) = (xx) \cdot (yy)$.

This paper is connected with the following problems.

Problem 1 (Belousov Problem 1a [12, 55, 98])**.** Find necessary and sufficient conditions that a left special loop is isotopic to a left F-quasigroup.

Problem 1a has been solved partially by Florea and Ursul [34, 36]. They proved that a left F-quasigroup with IP-property is isotopic to an A-loop.

Problem 2 (Problem 2.8 from [64])**.** (1) Characterize the loop isotopes of quasigroups satisfying (E_l).

(2) Characterize the loop isotopes of quasigroups satisfying (E_l), (S_l), and (F_l).

Problem 3. It is easy to see that in loops $1 \cdot ab = 1a \cdot 1b$. Describe quasigroups with the property $f(ab) = f(a)f(b)$ for all $a, b \in Q$, where $f(a)$ is left local identity element of a (see [99, p. 12]).

The results of this paper were presented at the conference LOOPS'07 (August 19–24, 2007, Prague). In order to make the reading of this paper more or less easy we give some necessary preliminary results and quit detailed proofs.

1.1 Quasigroups

Let $(Q,\cdot)$ be a groupoid (be a magma in alternative terminology). As usual, the map $L_a : Q \to Q, L_a x = a\cdot x$ for all $x \in Q$, is a left translation of the groupoid $(Q,\cdot)$ relative to a fixed element $a \in Q$; the map $R_a : Q \to Q$, $R_a x = x\cdot a$, is a right translation.

Definition 1.1. A groupoid $(G,\cdot)$ is said to be a *division groupoid* if the mappings L_x and R_x are surjective for every $x \in G$.

In a division groupoid $(G,\cdot)$, any from equations $a\cdot x = b$ and $y\cdot a = b$ has at least one solution for any fixed $a, b \in Q$, but we cannot guarantee that these solutions are unique solutions.

Definition 1.2. A groupoid $(G,\cdot)$ is said to be a *cancellation groupoid* if $a\cdot b = a\cdot c \Rightarrow b = c$, $b\cdot a = c\cdot a \Rightarrow b = c$ for all $a, b, c \in G$.

If any from equations $a\cdot x = b$ and $y\cdot a = b$ has a solution in a cancellation groupoid $(G,\cdot)$ for some fixed $a, b \in Q$, then this solution is unique. In other words, in a cancellation groupoid, the mappings L_x and R_x are injective for every $x \in G$.

Definition 1.3. A groupoid $(Q,\cdot)$ is called a *quasigroup* if, for all $a, b \in Q$, there exist unique solutions $x, y \in Q$ to the equations $x\cdot a = b$ and $a\cdot y = b$, i.e., in this case any right and any left translation of the groupoid $(Q,\cdot)$ is a bijection of the set Q.

Remark 1.4. Any division cancellation groupoid is a quasigroup and vice versa.

A sub-object $(H,\cdot)$ of a quasigroup $(Q,\cdot)$ is closed relative to the operation $\cdot$, i.e., if $a, b \in H$, then $a\cdot b \in H$.

We denote by S_Q the group of all bijections (permutations in finite case) of a set Q.

Definition 1.5. A groupoid (Q,A) is an *isotope of a groupoid* (Q,B) if there exist permutations μ_1, μ_2, μ_3 of the set Q such that $A(x_1,x_2) = \mu_3^{-1}B(\mu_1 x_1, \mu_2 x_2)$ for all $x_1, x_2 \in Q$. We also can say that a groupoid (Q,A) is an *isotopic image of a groupoid* (Q,B). The triple (μ_1,μ_2,μ_3) is called an *isotopy (isotopism)*.

We will write this fact also in the form $(Q,A) = (Q,B)T$, where $T = (\mu_1,\mu_2,\mu_3)$ [12, 15, 83].

If only the fact will be important that binary groupoids $(Q,\circ)$ and $(Q,\cdot)$ are isotopic, then we will use the record $(Q,\cdot) \sim (Q,\circ)$.

Definition 1.6. Isotopy of the form $(\mu_1,\mu_2,\varepsilon)$ is called a *principal isotopy*.

Remark 1.7. Up to isomorphism any isotopy is a *principal isotopy*. Indeed, $T = (\mu_1,\mu_2,\mu_3) = (\mu_1\mu_3^{-1}, \mu_2\mu_3^{-1}, \varepsilon)(\mu_3,\mu_3,\mu_3)$.

We have the following definition of a quasigroup.

Definition 1.8 (see [14, 30, 76])**.** A binary groupoid (Q,A) such that in the equality $A(x_1,x_2) = x_3$ knowledge of any 2 elements of x_1, x_2, x_3 uniquely specifies the remaining one is called a *binary quasigroup*.

From Definition 1.8, it follows that with any quasigroup (Q,A) it is possible to associate more $(3!-1) = 5$ quasigroups, the so-called parastrophes of quasigroup (Q,A):

$$\begin{aligned} A\big(x_1,x_2\big) = x_3 &\Longleftrightarrow A^{(12)}\big(x_2,x_1\big) = x_3 \Longleftrightarrow A^{(13)}\big(x_3,x_2\big) = x_1 \\ &\Longleftrightarrow A^{(23)}\big(x_1,x_3\big) = x_2 \Longleftrightarrow A^{(123)}\big(x_2,x_3\big) = x_1 \\ &\Longleftrightarrow A^{(132)}\big(x_3,x_1\big) = x_2 \end{aligned}$$

We will denote

- the operation of (12)-parastrophe of a quasigroup $(Q, \cdot)$ by $\ast$;
- the operation of (13)-parastrophe of a quasigroup $(Q, \cdot)$ by $/$;
- the operation of (23)-parastrophe of a quasigroup $(Q, \cdot)$ by $\backslash$;
- the operation of (123)-parastrophe of a quasigroup $(Q, \cdot)$ by $//$;
- the operation of (132)-parastrophe of the quasigroup $(Q, \cdot)$ by $\backslash\backslash$.

We have defined left and right translations of a groupoid and, therefore, of a quasigroup. But for quasigroups it is possible to define the third kind of translations. If $(Q, \cdot)$ is a quasigroup, then the map $P_a : Q \to Q$, $x \cdot P_a x = a$ for all $x \in Q$, is called a *middle translation* [13, 104].

In Table 1 connections between different kinds of translations in different parastrophes of a quasigroup $(Q, \cdot)$ are given. This table in fact is there in [13]; see also [31, 94].

Kinds	$\varepsilon = \cdot$	$(12) = \ast$	$(13) = /$	$(23) = \backslash$	$(123) = //$	$(132) = \backslash\backslash$
R	R	L	R^{-1}	P	P^{-1}	L^{-1}
L	L	R	P^{-1}	L^{-1}	R^{-1}	P
P	P	P^{-1}	L^{-1}	R	L	R^{-1}
R^{-1}	R^{-1}	L^{-1}	R	P^{-1}	P	L
L^{-1}	L^{-1}	R^{-1}	P	L	R	P^{-1}
P^{-1}	P^{-1}	P	L	R^{-1}	L^{-1}	R

Table 1

In Table 1, for example, $R^{(23)} = R^{\backslash} = P^{(\cdot)}$.

If $T = (\alpha_1, \alpha_2, \alpha_3)$ is an isotopy, σ is a parastrophy, then we define $T^\sigma = (\alpha_{\sigma^{-1}1}, \alpha_{\sigma^{-1}2}, \alpha_{\sigma^{-1}3})$.

Lemma 1.9. *In a quasigroup (Q, A): $(AT)^\sigma = A^\sigma T^\sigma$, $(T_1T_2)^\sigma = T_1^\sigma T_2^\sigma$ [12, 14].*

Definition 1.10. An element $f(b)$ of a quasigroup $(Q, \cdot)$ is called left local identity element of an element $b \in Q$, if $f(b) \cdot b = b$, in other words, $f(b) = b/b$.

An element $e(b)$ of a quasigroup $(Q, \cdot)$ is called right local identity element of an element $b \in Q$, if $b \cdot e(b) = b$, in other words, $e(b) = b\backslash b$.

An element $s(b)$ of a quasigroup $(Q, \cdot)$ is called middle local identity element of an element $b \in Q$, if $b \cdot b = s(b)$ [93, 94].

An element e is a left (right) identity element for quasigroup $(Q, \cdot)$ which means that $e = f(x)$ for all $x \in Q$ (resp., $e = e(x)$ for all $x \in Q$). A quasigroup with the left (right) identity element will be called a *left (right) loop*.

The fact that an element e is an *identity element* of a quasigroup $(Q, \cdot)$ means that $e(x) = f(x) = e$ for all $x \in Q$, i.e., all left and right local identity elements in the quasigroup $(Q, \cdot)$ coincide [12].

Connections between different kinds of local identity elements in different parastrophes of a quasigroup $(Q, \cdot)$ are given in Table 2 [93, 94].

In Table 2, for example, $s^{(123)} = e^{(\cdot)}$.

Remark 1.11. We notice that in [6, 106] the mapping s is denoted by β.

	ε	(12)	(13)	(23)	(123)	(132)
f	f	e	s	f	e	s
e	e	f	e	s	s	f
s	s	s	f	e	f	e

Table 2

Definition 1.12. A quasigroup $(Q, \cdot)$ with an identity element $e \in Q$ is called a *loop*.

Quasigroup isotopy of the form $(R_a^{-1}, L_b^{-1}, \varepsilon)$ is called an LP-isotopy. Any LP-isotopic image of a quasigroup is a loop [12, 15].

Lemma 1.13 (see [15, Lemma 1.1])**.** *Let $(Q, +)$ be a loop and $(Q, \cdot)$ a quasigroup. If $(Q, +) = (Q, \cdot)(\alpha, \beta, \varepsilon)$, then $(\alpha, \beta, \varepsilon) = (R_a^{-1}, L_b^{-1}, \varepsilon)$ for some translations of $(Q, \cdot)$.*

Lemma 1.14. *If $(Q, \cdot)$ is a quasigroup, $(H, \cdot)$ is its subquasigroup, $a, b \in H$, then $(H, \cdot)T$ is a subloop of the loop $(Q, \cdot)T$, where T is an isotopy of the form $(R_a^{-1}, L_b^{-1}, \varepsilon)$.*

Proof. We have that $R_a|_H, L_b|_H$ are translations of $(H, \cdot)$, since $a, b \in H$. □

We define the following mappings of a quasigroup $(Q, \cdot)$: $f : x \mapsto f(x)$, $f(x) \cdot x = x$ for all $x \in Q$; $e : x \mapsto e(x)$, $x \cdot e(x) = x$ for all $x \in Q$; $s : x \mapsto s(x)$, $s(x) = x \cdot x$ for all $x \in Q$.

Definition 1.15 (see [12, 15, 21, 27, 32, 33, 83, 104])**.** An algebra $(Q, \cdot, \backslash, /)$ is called a quasigroup, if on the set Q there exist operations "$\backslash$" and "$/$" such that in $(Q, \cdot, \backslash, /)$ identities

$$x \cdot (x \backslash y) = y \tag{1.1}$$

$$(y/x) \cdot x = y \tag{1.2}$$

$$x \backslash (x \cdot y) = y \tag{1.3}$$

$$(y \cdot x)/x = y \tag{1.4}$$

are fulfilled.

Lemma 1.16. *(1) Any sub-object of a quasigroup $(Q, \cdot)$ is a cancellation groupoid.*

(2) Any sub-object of a quasigroup $(Q, \cdot, \backslash, /)$ is a subquasigroup.

(3) Any subquasigroup of a quasigroup $(Q, \cdot)$ is a subquasigroup in $(Q, \cdot, \backslash, /)$ and, vice versa, any subquasigroup of a quasigroup $(Q, \cdot, \backslash, /)$ is a subquasigroup in $(Q, \cdot)$.

Proof. (1) If $a, b, c \in H$, then from $a \cdot b = a \cdot c$ follows $b = c$, since $(H, \cdot) \subseteq (Q, \cdot)$. Similarly from $b \cdot a = c \cdot a$ follows $b = c$.

(2) and (3), see [12, 27, 71, 83]. □

Left, middle, and right nuclei of a loop $(Q, \cdot)$ are defined in the following way:

$$N_l = \{a \in Q \mid a \cdot xy = ax \cdot y, x,\ y \in Q\}$$

$$N_m = \{a \in Q \mid xa \cdot y = x \cdot ay, x,\ y \in Q\}$$

$$N_r = \{a \in Q \mid xy \cdot a = x \cdot ya, x,\ y \in Q\}$$

Nucleus of a loop is defined in the following way: $N = N_l \cap N_m \cap N_r$ [12, 24]. Bruck defined a center of a loop $(Q, \cdot)$ as $C(Q, \cdot) = N \cap Z$, where

$$Z = \big\{a \in Q \mid a \cdot x = x \cdot a \ \forall x \in Q\big\}$$

Information on quasigroup nuclei can be found in [99].

1.2 Autotopisms

Definition 1.17. An autotopism (sometimes we will call autotopism an autotopy) is an isotopism of a quasigroup $(Q,\cdot)$ into itself, i.e., a triple (α,β,γ) of permutations of the set Q is an autotopy if the equality $x\cdot y=\gamma^{-1}(\alpha x\cdot\beta y)$ is fulfilled for all $x,y\in Q$.

Definition 1.18. The third component of any autotopism is called a *quasiautomorphism.*

By $\mathrm{Top}(Q,\cdot)$ we will denote the group of all autotopies of a quasigroup $(Q,\cdot)$.

Theorem 1.19 (see [12, 15, 14])**.** *If quasigroups $(Q,\cdot)$ and $(Q,\circ)$ are isotopic with isotopy T, i.e., $(Q,\cdot)=(Q,\circ)T$, then $\mathrm{Top}(Q,\cdot)=T^{-1}\,\mathrm{Top}(Q,\circ)T$.*

Lemma 1.20 (see [12, 15])**.** *If $(Q,\cdot)$ is a loop, then any its autotopy has the form $(R_a^{-1},L_b^{-1},\varepsilon)(\gamma,\gamma,\gamma)$.*

Proof. Let $T=(\alpha,\beta,\gamma)$ be an autotopy of a loop $(Q,\cdot)$, i.e., $\alpha x\cdot\beta y=\gamma(x\cdot y)$. If we put $x=1$, then we obtain $\alpha 1\cdot\beta y=\gamma y$, $\gamma=L_{\alpha 1}\beta$, $\beta=L_{\alpha 1}^{-1}\gamma$. If we put $y=1$, then, by analogy, we obtain, $\alpha=R_{\beta 1}^{-1}\gamma$. Then $T=(R_{\beta 1}^{-1}\gamma,L_{\alpha 1}^{-1}\gamma,\gamma)=(R_k^{-1},L_d^{-1},\varepsilon)(\gamma,\gamma,\gamma)$, where $\beta 1=k$, $\alpha 1=d$. $\square$

We can obtain more detailed information on autotopies of a group and, since autotopy groups of isotopic quasigroups are isomorphic, on autotopies of quasigroups that are some group isotopes.

Theorem 1.21 (see [15])**.** *Any autotopy of a group $(Q,+)$ has the form*

$$\big(L_a\delta,R_b\delta,L_aR_b\delta\big)$$

where L_a is a left translation of the group $(Q,+)$, R_b is a right translation of this group, δ is an automorphism of $(Q,+)$.

Corollary 1.22. *(1) If $L_a\delta=L_aR_b\delta$, then $R_b=\varepsilon$. (2) If $R_b\delta=L_aR_b\delta$, then $L_a=\varepsilon$. (3) If $L_a\delta=R_b\delta$, then $a\in C(Q,+)$.*

Proof. (3) We have $a+\delta x+a+\delta y=a+a+\delta x+\delta y$, $\delta x+a=a+\delta x$ for all $x\in Q$. $\square$

Corollary 1.23. *Any group quasiautomorphism has the form $L_d\varphi$, where $\varphi\in\mathrm{Aut}(Q,+)$ [96].*

Proof. We have $L_aR_b\,\delta x=a+\delta x+b=a+b-b+\delta x+b=L_{a+b}I_b\,\delta x=L_d\,\varphi$, where $d=a+b$, $\varphi=I_b\delta$, $I_b\,x=-b+x+b$. $\square$

Lemma 1.24. *(1) If $x\cdot y=\alpha x*y$, where $(Q,*)$ is an idempotent quasigroup, α is a permutation of the set Q, then $\mathrm{Aut}(Q,\cdot)=C_{\mathrm{Aut}(Q,*)}(\alpha)=\{\tau\in\mathrm{Aut}(Q,*)\mid\tau\alpha=\alpha\tau\}$, in particular, $\mathrm{Aut}(Q,\cdot)\subseteq\mathrm{Aut}(Q,*)$.*

(2) If $x\cdot y=x*\beta y$, where $(Q,*)$ is an idempotent quasigroup, β is a permutation of the set Q, then $\mathrm{Aut}(Q,\cdot)=C_{\mathrm{Aut}(Q,*)}(\beta)=\{\tau\in\mathrm{Aut}(Q,*)\mid\tau\beta=\beta\tau\}$, in particular, $\mathrm{Aut}(Q,\cdot)\subseteq\mathrm{Aut}(Q,*)$ (see [72, Corollary 12]).

Proof. (1) We give a sketch of the proof. If $\varphi\in\mathrm{Aut}(Q,\cdot)$, then $\varphi(x\cdot y)=\varphi(\alpha x*y)=\varphi x\cdot\varphi y=\alpha\varphi x*\varphi y$. If $y=\alpha x$, then $\varphi\alpha x=\alpha\varphi x*\varphi\alpha x$, $\varphi\alpha=\alpha\varphi$, $\varphi(\alpha x*y)=\varphi\alpha x*\varphi y$.

(2) The proof of Case (2) is similar to the proof of Case (1). $\square$

1.3 Quasigroup classes

Definition 1.25. A quasigroup $(Q, \cdot)$ is

- *medial*, if $xy \cdot uv = xu \cdot yv$ for all $x, y, u, v \in Q$;
- *left distributive*, if $x \cdot uv = xu \cdot xv$ for all $x, u, v \in Q$;
- *right distributive*, if $xu \cdot v = xv \cdot uv$ for all $x, u, v \in Q$;
- *distributive*, if it is left and right distributive;
- *idempotent*, if $x \cdot x = x$ for all $x \in Q$;
- *unipotent*, if there exists an element $a \in Q$ such that $x \cdot x = a$ for all $x \in Q$;
- *left semi-symmetric*, if $x \cdot xy = y$ for all $x, y \in Q$;
- *TS-quasigroup*, if $x \cdot xy = y, xy = yx$ for all $x, y \in Q$;
- *left F-quasigroup*, if $x \cdot yz = xy \cdot e(x)z$ for all $x, y, z \in Q$;
- *right F-quasigroup*, if $xy \cdot z = xf(z) \cdot yz$ for all $x, y, z \in Q$;
- *left semimedial or middle F-quasigroup*, if $s(x) \cdot yz = xx \cdot yz = xy \cdot xz$ for all $x, y, z \in Q$;
- *right semimedial*, if $zy \cdot s(x) = zx \cdot yx$ for all $x, y, z \in Q$;
- *F-quasigroup*, if it is left and right F-quasigroup;
- *left E-quasigroup*, if $x \cdot yz = f(x)y \cdot xz$ for all $x, y, z \in Q$;
- *right E-quasigroup*, if $zy \cdot x = zx \cdot ye(x)$ for all $x, y, z \in Q$;
- *E-quasigroup*, if it is left and right E-quasigroup;
- *LIP-quasigroup*, if there exists a permutation λ of the set Q such that $\lambda x \cdot (x \cdot y) = y$ for all $x, y \in Q$;
- *RIP-quasigroup*, if there exists a permutation ρ of the set Q such that $(x \cdot y) \cdot \rho y = x$ for all $x, y \in Q$;
- *IP-quasigroup*, if it is LIP- and RIP-quasigroup.

A quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \beta y + c$, where $(Q, +)$ is a group, $\varphi \in \mathrm{Aut}(Q, +)$, β is a permutation of the set Q, is called a *left linear quasigroup*; a quasigroup $(Q, \cdot)$ of the form $x \cdot y = \alpha x + \psi y + c$, where $(Q, +)$ is a group, $\psi \in \mathrm{Aut}(Q, +)$, α is a permutation of the set Q, is called a *right linear quasigroup* [114, 118].

Definition 1.26. A loop $(Q, \cdot)$ is

- *Bol loop (left Bol loop)*, if $x(y \cdot xz) = (x \cdot yx)z$ for all $x, y, z \in Q$;
- *Moufang loop*, if $x(yz \cdot x) = xy \cdot zx$ for all $x, y, z \in Q$;
- *commutative Moufang loop (CML)*, if $xx \cdot yz = xy \cdot xz$ for all $x, y, z \in Q$;
- *left M-loop*, if $x \cdot (y \cdot z) = (x \cdot (y \cdot I\varphi x)) \cdot (\varphi x \cdot z)$ for all $x, y, z \in Q$, where φ is a mapping of the set Q, $x \cdot Ix = 1$ for all $x \in Q$;
- *right M-loop*, if $(y \cdot z) \cdot x = (y \cdot \psi x) \cdot ((I^{-1}\psi x \cdot z) \cdot x)$ for all $x, y, z \in Q$, where ψ is a mapping of the set Q;
- *M-loop*, if it is left M- and right M-loop;
- *left special*, if $S_{a,b} = L_b^{-1} L_a^{-1} L_{ab}$ is an automorphism of $(Q, \cdot)$ for any pair $a, b \in Q$ [110];
- *right special*, if $T_{a,b} = R_b^{-1} R_a^{-1} R_{ba}$ is an automorphism of $(Q, \cdot)$ for any pair $a, b \in Q$ [110].

In [12] the left special loop is called *special*. In [50, 106, 64] left semimedial quasigroups are studied. A quasigroup is trimedial if and only if it is satisfies left and right E-quasigroup equality [64]. Information on properties of trimedial quasigroups is there in [63].

Every semimedial quasigroup is isotopic to a commutative Moufang loop [50]. In the trimedial case the isotopy has a more restrictive form [50].

In a quasigroup $(Q, \cdot, \backslash, /)$ the equalities $x \cdot yz = xy \cdot e(x)z$, $xy \cdot z = xf(z) \cdot yz$, $x \cdot yz = f(x)y \cdot xz$ and $zy \cdot x = zx \cdot ye(x)$ take the form $x \cdot yz = xy \cdot (x\backslash x)z$, $xy \cdot z = x(z/z) \cdot yz$, $x \cdot yz = (x/x)y \cdot xz$ and $zy \cdot x = zx \cdot y(x\backslash x)$, respectively, and they are identities in $(Q, \cdot, \backslash, /)$.

Therefore any subquasigroup of a left F-quasigroup $(Q, \cdot, \backslash, /)$ is a left F-quasigroup; any homomorphic image of a left F-quasigroup $(Q, \cdot, \backslash, /)$ is a left F-quasigroup [27, 71]. It is clear that the situation is the same for right F-quasigroups, left and right E- and SM-quasigroups.

Lemma 1.27. *Any medial quasigroup $(Q, \cdot)$ is both a left and right F-, SM-, and E-quasigroup.*

Proof. Equality $x \cdot uv = xu \cdot e(x)v$ follows from medial identity $xy \cdot uv = xu \cdot yv$ by $y = e(x)$. Respectively, by $u = e(x)$ we have $xy \cdot e(x)v = x \cdot yv$, i.e., $(Q, \cdot)$ is a left F-quasigroup in these cases, and so on. $\square$

Lemma 1.28. *(1) Any left distributive quasigroup $(Q, \cdot)$ is a left F-, SM-, and E-quasigroup.*

(2) Any right distributive quasigroup $(Q, \cdot)$ is a right F-, SM-, and E-quasigroup.

Proof. (1) It is easy to see that $(Q, \cdot)$ is idempotent quasigroup. Therefore $x \cdot x = x/x = x\backslash x = x$. Then $x \cdot yz = xy \cdot xz = ((x/x) \cdot y) \cdot xz = ((x\backslash x) \cdot y) \cdot xz = xy \cdot (x/x)z = xy \cdot (x\backslash x)z = xx \cdot yz$.

(2) The proof of this case is similar to the proof of Case (1). $\square$

Lemma 1.29. *A quasigroup $(Q, \cdot)$ in which*

(1) the equality $x \cdot yz = xy \cdot \delta(x)z$ is true for all $x, y, z \in Q$, where δ is a map of the set Q, is a left F-quasigroup [15];

(2) the equality $xy \cdot z = x\delta(z) \cdot yz$ is true for all $x, y, z \in Q$, where δ is a map of the set Q, is a right F-quasigroup;

(3) the equality $\delta(x) \cdot yz = xy \cdot xz$ is true for all $x, y, z \in Q$, where δ is a map of the set Q, is a left semimedial quasigroup;

(4) the equality $zy \cdot \delta(x) = zx \cdot yx$ is true for all $x, y, z \in Q$, where δ is a map of the set Q, is a right semimedial quasigroup;

(5) the equality $x \cdot yz = \delta(x)y \cdot xz$ is true for all $x, y, z \in Q$, where δ is a map of the set Q, is a left E-quasigroup;

(6) the equality $zy \cdot x = zx \cdot y\delta(x)$ is true for all $x, y, z \in Q$, where δ is a map of the set Q, is a right E-quasigroup.

Proof. (1) If we take $y = e(x)$, then we have $x \cdot e(x)z = x \cdot \delta(x)z$, $e(x) = \delta(x)$. Cases (2)–(6) are proved similarly. $\square$

Theorem 1.30 (Toyoda Theorem [12, 15, 22, 78, 103, 119])**.** *Any medial quasigroup $(Q, \cdot)$ can be presented in the form $x \cdot y = \varphi x + \psi y + a$, where $(Q, +)$ is an Abelian group, φ, ψ are automorphisms of $(Q, +)$ such that $\varphi\psi = \psi\varphi$, a is some fixed element of the set Q and vice versa.*

Theorem 1.31 (Belousov Theorem [9, 12, 15])**.** *Any distributive quasigroup $(Q, \circ)$ can be presented in the form $x \circ y = \varphi x + \psi y$, where $(Q, +)$ is a commutative Moufang loop, $\varphi, \psi \in \mathrm{Aut}(Q, +)$, $\varphi, \psi \in \mathrm{Aut}(Q, \cdot)$, $\varphi\psi = \psi\varphi$.*

A left (right) F-quasigroup is isotopic to a left (right) M-loop [15, 42]. A left (right) F-quasigroup is isotopic to a left (right) special loop [8, 16, 12, 41]. An F-quasigroup is isotopic to a Moufang loop [55].

If a loop $(Q, \circ)$ is isotopic to a left distributive quasigroup $(Q, \cdot)$ with isotopy the form $x \circ y = R_a^{-1}x \cdot L_a^{-1}y$, then $(Q, \circ)$ will be called a *left S-loop*. Loop $(Q, \circ)$ and quasigroup $(Q, \cdot)$ are said to be *related*.

If a loop $(Q, \circ)$ is isotopic to a right distributive quasigroup $(Q, \cdot)$ with isotopy the form $x \circ y = R_a^{-1}x \cdot L_a^{-1}y$, then $(Q, \circ)$ will be called a *right S-loop*.

Definition 1.32. An automorphism ψ of a loop $(Q, \circ)$ is called *complete*, if there exists a permutation φ of the set Q such that $\varphi x \circ \psi x = x$ for all $x \in Q$. Permutation φ is called a complement of automorphism ψ.

The following theorem is proved in [17].

Theorem 1.33. *A loop $(Q, \circ)$ is a left S-loop, if and only if there exists a complete automorphism ψ of the loop $(Q, \circ)$ such that at least one of the following conditions is fulfilled:*

(a) $\varphi(x \circ \varphi^{-1}y) \circ (\psi x \circ z) = x \circ (y \circ z)$;

(b) $L^{\circ}_{x,y}\psi = \psi L^{\circ}_{x,y}$ and $\varphi x \circ (\psi x \circ y) = x \circ y$ for all $x, y \in Q$, $x, y \in Q$, $L^{\circ}_{x,y} \in LI(Q, \circ)$.

Thus $(Q, \cdot)$, where $x \cdot y = \varphi x \circ \psi y$, is a left distributive quasigroup which corresponds to the loop $(Q, \circ)$.

Remark 1.34. In [17, 81] a left S-loop is called an *S-loop*.

A left distributive quasigroup $(Q, \cdot)$ with identity $x \cdot xy = y$ is isotopic to a left Bol loop [12, 15, 16]. Last results of Nagy [79] let us hope on progress in researches of left distributive quasigroups. Some properties of distributive and left distributive quasigroups are described in [38, 39, 40, 115].

Theorem 1.35. *Any loop which is isotopic to a left F-quasigroup is a left M-loop (see [15, Theorem 3.17, p. 109]).*

Theorem 1.36 (Generalized Albert Theorem)**.** *Any loop isotopic to a group is a group [2, 3, 12, 15, 68, 83, 99].*

1.4 Congruences and homomorphisms

Results of this subsection are standard, well-known [12, 24, 83, 71, 27], and slightly adapted for our aims.

A binary relation φ on a set Q is a subset of the cartesian product $Q \times Q$ [21, 71, 82].

If φ and ψ are binary relations on Q, then their product is defined in the following way: $(a, b) \in \varphi \circ \psi$, if there is an element $c \in Q$ such that $(a, c) \in \varphi$ and $(c, b) \in \psi$. The last condition is written also in such form $a\,\varphi\, c\,\psi\, b$.

Theorem 1.37. *Let S be a nonempty set and let $\sim$ be a relation between elements of S that satisfies the following properties:*

(1) (Reflexive) $a \sim a$ for all $a \in S$.

(2) (Symmetric) If $a \sim b$, then $b \sim a$.

(3) (Transitive) If $a \sim b$ and $b \sim c$, then $a \sim c$.

Then $\sim$ yields a natural partition of S, where $\bar{a} = \{x \in S \mid x \sim a\}$ is the cell containing a for all $a \in S$. Conversely, each partition of S gives rise to a natural relation $\sim$ satisfying the reflexive, symmetric, and transitive properties if $a \sim b$ is defined to mean that $a \in \bar{b}$ [44].

Definition 1.38. A relation $\sim$ on a set S satisfying the reflexive, symmetric, and transitive properties is called an equivalence relation on S. Each cell $\bar{a}$ in the natural partition given by an equivalence relation is an equivalence class.

Definition 1.39. An equivalence θ is a congruence of a groupoid $(Q, \cdot)$, if the following implications are true for all $x, y, z \in Q$: $x\theta y \Rightarrow (z \cdot x)\theta(z \cdot y), x\theta y \Rightarrow (x \cdot z)\theta(y \cdot z)$ [29].

In other words, equivalence θ is a *congruence* of $(Q, \cdot)$ if and only if θ is a subalgebra of $(Q \times Q, (\cdot, \cdot))$. Therefore we can formulate Definition 1.39 in the following form.

Definition 1.40 (see [29]). An equivalence θ is a congruence of a groupoid $(Q, \cdot)$, if the following implication is true for all $x, y, w, z \in Q$: $x\theta y \wedge w\theta z \Rightarrow (x \cdot w)\,\theta\,(y \cdot z)$.

Definition 1.41 (see [12, 15]). A congruence θ of a quasigroup $(Q, \cdot)$ is *normal*, if the following implications are true for all $x, y, z \in Q$: $(z \cdot x)\theta(z \cdot y) \Rightarrow x\theta y, (x \cdot z)\theta(y \cdot z) \Rightarrow x\theta y$.

Definition 1.42. An equivalence θ is a congruence of a quasigroup $(Q, \cdot, /, \backslash)$, if the following implications are true for all $x, y, z \in Q$:

$$\begin{aligned}
&x\theta y \Longrightarrow (z \cdot x)\theta(z \cdot y), x\theta y \Longrightarrow (x \cdot z)\theta(y \cdot z)\\
&x\theta y \Longrightarrow (z/x)\theta(z/y), x\theta y \Longrightarrow (x/z)\theta(y/z)\\
&x\theta y \Longrightarrow (z\backslash x)\theta(z\backslash y), x\theta y \Longrightarrow (x\backslash z)\theta(y\backslash z)
\end{aligned}$$

One from the most important properties of e-quasigroup $(Q, \cdot, \backslash, /)$ is the following property.

Lemma 1.43 (see [12, 15, 21, 71]). *Any congruence of a quasigroup $(Q, \cdot, \backslash, /)$ is a normal congruence of quasigroup $(Q, \cdot)$; any normal congruence of a quasigroup $(Q, \cdot)$ is a congruence of quasigroup $(Q, \cdot, \backslash, /)$.*

Definition 1.44. If θ is a binary relation on a set Q, α is a permutation of the set Q and from $x\theta y$ it follows $\alpha x\theta\alpha y$ and $\alpha^{-1}x\theta\alpha^{-1}y$ for all $(x, y) \in \theta$, then we will say that the permutation α is an *admissible* permutation relative to the binary relation θ [12, 100].

Moreover, we will say that a binary relation θ admits a permutation α.

Lemma 1.45 (see [13]). *Any normal quasigroup congruence is admissible relative to any left, right, and middle quasigroup translation.*

Proof. The fact that any normal quasigroup congruence is admissible relative to any left and right quasigroup translation follows from Definitions 1.39 and 1.41.

Let θ be a normal congruence of a quasigroup $(Q, \cdot)$. Prove the following implication:

$$a\theta b \longrightarrow P_c a\,\theta\,P_c b \tag{1.5}$$

If $P_c a = k$, then $a \cdot k = c$, $k = a\backslash c$, $k = R_c^{\backslash} a$. Similarly if $P_c b = m$, then $b \cdot m = c$, $m = b\backslash c$, $m = R_c^{\backslash} b$. Since θ is a congruence of quasigroup $(Q, \cdot, \backslash, /)$ (Lemma 1.43), then implication (1.5) is true.

Implication

$$a\theta b \longrightarrow P_c^{-1}a\,\theta\,P_c^{-1}b \tag{1.6}$$

is proved in a similar way. If $P_c^{-1}a = k$, then $k \cdot a = c$, $k = c/a$, $k = L_c^{/}a$. Similarly if $P_c^{-1}b = m$, then $m \cdot b = c$, $m = c/b$, $m = L_c^{/}b$. Since θ is a congruence of quasigroup $(Q, \cdot, \backslash, /)$ (Lemma 1.43), then implication (1.6) is true. □

Corollary 1.46. *If θ is a normal quasigroup congruence of a quasigroup Q, then θ is a normal congruence of any parastrophe of Q [13].*

Proof. The proof follows from Lemma 1.45 and Table 1. □

In Lemma 1.48 we will use the following fact about quasigroup translations and normal quasigroup congruences.

Lemma 1.47. *If $a\theta b$, $c\theta d$, then $R_a^{-1}c\,\theta\,R_b^{-1}d$.*

Proof. If $ac\,\theta\,bd$ and $c\theta d$, then $a\theta b$. Indeed, if $c\theta d$, then $ac\theta ad$. If $ac\,\theta\,bd$ and $ac\theta ad$, then $bd\theta ad$, and, finally, $a\theta b$. In other words, if $R_c a\,\theta\,R_d b$ and $c\theta d$, then $a\theta b$.

Since $a\theta b$ and θ is a normal quasigroup congruence, we have $c\,\theta\,d \Leftrightarrow R_a R_a^{-1}c\,\theta\,R_b R_b^{-1}d \Leftrightarrow R_a^{-1}c\,\theta\,R_b^{-1}d$. □

We give a sketched proof of the following well-known fact [27, 70, 108]. We follow [108].

Lemma 1.48. *Normal quasigroup congruences commute in pairs.*

Proof. Let θ_1 and θ_2 be normal congruences of a quasigroup $(Q, \cdot)$. Then $a(\theta_1 \circ \theta_2)b$ means that there exists an element $c \in Q$ such that $a\theta_1 c$ and $c\theta_2 b$.

Further, we have

$$\begin{array}{ll} a\theta_2 a, & a\theta_2 a \\ c\theta_2 b, & L_c^{-1}c\theta_2 L_c^{-1}b \\ b\theta_2 b, & L_c^{-1}b\theta_2 L_c^{-1}b \end{array}$$

Then

$$R^{-1}_{L_c^{-1}c}a \cdot L_c^{-1}b\,\theta_2\,R^{-1}_{L_c^{-1}b}a \cdot L_c^{-1}b = a$$

From relations

$$\begin{array}{ll} a\theta_1 a, & a\theta_1 a \\ a\theta_1 c, & L_c^{-1}a\theta_1 L_c^{-1}c \\ b\theta_1 b, & L_c^{-1}b\theta_1 L_c^{-1}b \end{array}$$

we obtain

$$b = R^{-1}_{L_c^{-1}a}a \cdot L_c^{-1}b\,\theta_1\,R^{-1}_{L_c^{-1}c}a \cdot L_c^{-1}b$$

Therefore, $a(\theta_2 \circ \theta_1)b$. □

See [37] for additional information on permutability of quasigroup congruences.

Definition 1.49 (see [83])**.** If $(Q, \cdot)$ and $(H, \circ)$ are binary quasigroups, h is a single-valued mapping of Q into H such that $h(x_1 \cdot x_2) = hx_1 \circ hx_2$, then h is called a *homomorphism (a multiplicative homomorphism)* of $(Q, \cdot)$ into $(H, \circ)$ and the set $\{hx \mid x \in Q\}$ is called *homomorphic image* of $(Q, \cdot)$ under h.

In case $(Q, \cdot) = (H, \circ)$ a homomorphism is also called an *endomorphism*, and an isomorphism is referred to as an automorphism.

Lemma 1.50. *(1) Any homomorphic image of a quasigroup $(Q, \cdot)$ is a division groupoid [5, 24].*

(2) Any homomorphic image of a quasigroup $(Q, \cdot, \backslash, /)$ is a quasigroup [27, 71].

Proof. (1) Let $h(a), h(b) \in h(Q)$. We demonstrate that solution of equation $h(a) \circ x = h(b)$ lies in $h(Q)$. Consider the equation $a \cdot y = b$. Denote solution of this equation by c. Then $h(c)$ is solution to the equation $h(a) \circ x = h(b)$. Indeed, $h(a) \circ h(c) = h(a \cdot c) = h(b)$. For equation $x \cdot h(a) = h(b)$, the proof is similar.

(2) see [12, 15, 27, 71, 83]. □

Let h be a homomorphism of a quasigroup $(Q, \cdot)$ onto a groupoid $(H, \circ)$. Then h induces a congruence $\operatorname{Ker} h = \theta$ (the kernel of h) in the following way, $x\,\theta\,y$ if and only if $h(x) = h(y)$ [15, 83].

If θ is a normal congruence of a quasigroup $(Q, \cdot)$, then θ determines natural homomorphism h ($h(a) = \theta(a)$) of $(Q, \cdot)$ onto some quasigroup $(Q', \circ)$ by the rule $\theta(x) \circ \theta(y) = \theta(x \cdot y)$, where $\theta(x), \theta(y), \theta(x \cdot y) \in Q/\theta$ [15, 83].

Theorem 1.51 (see [15],[83, Theorem I.7.2])**.** *If h is a homomorphism of a quasigroup $(Q, \cdot)$ onto a quasigroup $(H, \circ)$, then h determines a normal congruence θ on $(Q, \cdot)$ such that $Q/\theta \cong (H, \circ)$, and vice versa, a normal congruence θ induces a homomorphism from $(Q, \cdot)$ onto $(H, \circ) \cong Q/\theta$.*

A subquasigroup $(H, \cdot)$ of a quasigroup $(Q, \cdot)$ is *normal* ($(H, \cdot) \trianglelefteq (Q, \cdot)$), if $(H, \cdot)$ is an equivalence class (in other words, a coset class) of a normal congruence.

Lemma 1.52. *An equivalence class $\theta(h) = H$ of a congruence θ of a quasigroup $(Q, \cdot)$ is a sub-object of $(Q, \cdot)$ if and only if $(h \cdot h)\,\theta\,h$.*

Proof. We recall by Lemma 1.16 any quasigroup sub-object is a cancellation groupoid. The proof is similar to the proof of Lemma 1.9 from [15]. If $a\,\theta\,h$ and $b\,\theta\,h$, then $ab\,\theta\,h^2$; moreover $h^2\,\theta\,h$, since $ab \in H$. Conversely, let $h^2\,\theta\,h$. If $a, b \in H$, then $a\,\theta\,h$ and $b\,\theta\,h$, $ab\,\theta\,h^2\theta\,h$. Then $ab \in H$. □

Lemma 1.53 (see [12], [15, Lemma 1.9])**.** *An equivalence class $\theta(h)$ of a normal congruence θ of a quasigroup $(Q, \cdot)$ is a subquasigroup of $(Q, \cdot)$ if and only if $(h \cdot h)\,\theta\,h$.*

Lemma 1.54. *If h is an endomorphism of a quasigroup $(Q, \cdot)$, then $(hQ, \cdot)$ is a subquasigroup of $(Q, \cdot)$.*

Proof. We rewrite the proof from [15, p. 33] for slightly more general case. Prove that $(hQ, \cdot)$ is a subquasigroup of quasigroup $(Q, \cdot)$. Let $h(a)$, $h(b) \in h(Q)$. We demonstrate that solution of equation $h(a) \cdot x = h(b)$ lies in $h(Q)$. Consider the equation $a \cdot y = b$. Denote solution of this equation by c. Then $h(c)$ is solution of equation $h(a) \cdot x = h(b)$. Indeed, $h(a) \cdot h(c) = h(a \cdot c) = h(b)$.

It is easy to see that this is a unique solution. Indeed, if $h(a) \cdot c_1 = h(b)$, then $h(a) \cdot h(c) = h(a) \cdot c_1$. Since $h(a), h(c), c_1$ are elements of quasigroup $(Q, \cdot)$, then $h(c) = c_1$.

For equation $x \cdot h(a) = h(b)$, the proof is similar. $\square$

Remark 1.55. It is possible to give the following proof of Lemma 1.54. The $(hQ, \cdot)$ is a cancellation groupoid, since it is a sub-object of the quasigroup $(Q, \cdot)$ (Lemma 1.16). From the other side $(hQ, \cdot)$ is a division groupoid, since it is a homomorphic image of $(Q, \cdot)$ (Lemma 1.50). Therefore by Remark 1.4 $(hQ, \cdot)$ is a subquasigroup of the quasigroup $(Q, \cdot)$.

Corollary 1.56. *(1) Any subquasigroup $(H, \cdot)$ of a left F-quasigroup $(Q, \cdot)$ is a left F-quasigroup.*

(2) Any endomorphic image of a left F-quasigroup $(Q, \cdot)$ is a left F-quasigroup.

Proof. (1) If $a \in H$, then the solution of equation $a \cdot x = a$, $x = e(a)$ also is in H.

(2) From Case (1) and Lemma 1.54 it follows that any endomorphic image of a left F-quasigroup $(Q, \cdot)$ is a left F-quasigroup. $\square$

Remark 1.57. The same situation is for right F-quasigroups, left and right E-, and SM-quasigroups and all combinations of these properties.

Corollary 1.58. *If h is an endomorphism of a quasigroup $(Q, \cdot)$, then h is an endomorphism of the quasigroups $(Q, *)$, $(Q, /)$, $(Q, \backslash)$, $(Q, //)$, $(Q, \backslash\backslash)$, i.e., from $h(x \cdot y) = h(x) \cdot h(y)$ we obtain that*

(1) $h(x * y) = h(x) * h(y)$;
(2) $h(x/y) = h(x)/h(y)$;
(3) $h(x\backslash y) = h(x)\backslash h(y)$;
(4) $h(x//y) = h(x)//h(y)$;
(5) $h(x\backslash\backslash y) = h(x)\backslash\backslash h(y)$.

Proof. From Lemma 1.54 we have that $(hQ, \cdot)$ is a subquasigroup of $(Q, \cdot)$.

(1) If we pass from the quasigroup $(Q, \cdot)$ to quasigroup $(Q, *)$, then subquasigroup $(hQ, *)$ of the quasigroup $(Q, *)$ will correspond to the subquasigroup $(hQ, \cdot)$. Indeed, any subquasigroup of the quasigroup $(Q, \cdot)$ is closed relative to parastrophe operations $*, /, \backslash, //, \backslash\backslash$ of the quasigroup $(Q, \cdot)$. Further we have $h(x * y) = h(y \cdot x) = h(y) \cdot h(x) = h(x) * h(y)$.

(2) If we pass from the quasigroup $(Q, \cdot)$ to quasigroup $(Q, /)$, then subquasigroup $(hQ, /)$ of the quasigroup $(Q, /)$ will correspond to the subquasigroup $(hQ, \cdot)$.

Let $z = x/y$, where $x, y \in Q$. Then from definition of the operation $/$ it follows that $x = zy$. Then $h(x) = h(z)h(y)$, $h(x/y) = h(z) = h(x)/h(y)$ (see [71, p. 96, Theorem 1]).

The remaining cases are proved in the similar way. $\square$

Lemma 1.59 (see [12, 102])**.** *If $(Q, \cdot)$ is a finite quasigroup, then any of its congruences is normal, any of its homomorphic images is a quasigroup.*

Lemma 1.60. *Let $(Q, \cdot)$ be a quasigroup.*

- *If f is an endomorphism of $(Q, \cdot)$, then $f(e(x)) = e(f(x))$, $f(s(x)) = s(f(x))$ for all $x \in Q$;*
- *If e is an endomorphism of $(Q, \cdot)$, then $e(f(x)) = f(e(x))$, $e(s(x)) = s(e(x))$ for all $x \in Q$;*

- *If s is an endomorphism of $(Q,\cdot)$, then $s(f(x)) = f(s(x))$, $s(e(x)) = e(s(x))$ for all $x \in Q$ ([64, Lemma 2.4.]).*

Proof. We will use Corollary 1.58.

- If f is an endomorphism, then $f(e(x)) = f(x\backslash x) = f(x)\backslash f(x) = e(f(x))$, $f(s(x)) = f(x)\cdot f(x) = s(f(x))$.
- If e is an endomorphism, then $e(f(x)) = e(x)/e(x) = f(e(x))$, $e(s(x)) = e(x)\cdot e(x) = s(e(x))$.
- If s is an endomorphism, then $s(f(x)) = s(x)/s(x) = f(s(x))$, $s(e(x)) = s(x)\backslash s(x) = s(e(x))$.

This proves the lemma. $\square$

The group $M(Q,\cdot) = \langle L_a, R_b \mid a, b \in Q\rangle$, where $(Q,\cdot)$ is a quasigroup, is called multiplication group of quasigroup.

The group $\mathbb{I}_h = \{\alpha \in M(Q,\cdot) \mid \alpha h = h\}$ is called inner mapping group of a quasigroup $(Q,\cdot)$ relative to an element $h \in Q$. Group $\mathbb{I}_h$ is stabilizer of a fixed element h by action $(\alpha : x \longmapsto \alpha(x)$ for all $\alpha \in M(Q,\cdot)$, $x \in Q)$ of group $M(Q,\cdot)$ on the set Q. In loop case usually it is studied the group $\mathbb{I}_1(Q,\cdot) = \mathbb{I}(Q,\cdot)$, where 1 is the identity element of a loop $(Q,\cdot)$.

Theorem 1.61. *A subquasigroup H of a quasigroup Q is normal if and only if $\mathbb{I}_k H \subseteq H$ for a fixed element $k \in H$ [12].*

In [12, p. 59] the following key lemma is proved.

Lemma 1.62. *Let θ be a normal congruence of a quasigroup $(Q,\cdot)$. If a quasigroup $(Q,\circ)$ is isotopic to $(Q,\cdot)$ and the isotopy (α,β,γ) is admissible relative to θ, then θ is a normal congruence also in $(Q,\circ)$.*

For our aims we will use the following theorem.

Theorem 1.63 (see [80, 60, 94, 96])**.** *Let $(Q,+)$ be an IP-loop, $x\cdot y = (\varphi x + \psi y) + c$, where $\varphi,\psi \in \mathrm{Aut}(Q,+)$, $a \in C(Q,+)$, θ be a normal congruence of $(Q,+)$. Then θ is normal congruence of $(Q,\cdot)$ if and only if $\varphi\,|_{\mathrm{Ker}\,\theta}$, $\psi\,|_{\mathrm{Ker}\,\theta}$ are automorphisms of* $\mathrm{Ker}\,\theta$.

We denote by $\mathrm{nCon}(Q,\cdot)$ the set of all normal congruences of a quasigroup $(Q,\cdot)$.

Corollary 1.64. *If $(Q,\cdot)$ is a quasigroup, $(Q,+)$ is a loop of the form $x + y = R_a^{-1}x\cdot L_b^{-1}y$ for all $x, y \in Q$, then* $\mathrm{nCon}(Q,\cdot) \subseteq \mathrm{nCon}(Q,+)$.

Proof. If θ is a normal congruence of a quasigroup $(Q,\cdot)$, then, since θ is admissible relative to the isotopy $T = (R_a^{-1}, L_b^{-1}, \varepsilon)$, θ is also a normal congruence of a loop $(Q,+)$. $\square$

In loop case situation with normality of subloops is well known and more near to the group case [24, 83, 12, 15]. As usual a subloop $(H,+)$ of a loop $(Q,+)$ is normal, if $H = \theta(0) = \mathrm{Ker}\,\theta$, where $\theta(0)$ is an equivalence class of a normal congruence θ that contains identity element of $(Q,+)$ [12, 83]. We will name congruence θ and subloop $(H,+)$ by *corresponding*.

Example 1.65. In the group S_3 ($S_3 = \langle a, b \mid a^3 = b^2 = 1,\ bab = a^{-1}\rangle$, $S_3 \cong Z_3 \rtimes Z_2$) there exists endomorphism h ($h(a) = 1$, $h(b) = b$) such that $h(S_3) = \langle b\rangle \cong Z_2$, $\mathrm{Ker}\,h = \langle a\rangle \cong Z_3$ and $Z_2 \ntrianglelefteq S_3$.

Example 1.66. In the cyclic group $(Z_4, +)$, $Z_4 = \{0, 1, 2, 3\}$, there exists endomorphism h ($h(x) = x + x$) such that $h(Z_4) = \operatorname{Ker} h = \{0, 2\}$. The endomorphism h defines normal congruence θ with the following coset classes: $\theta(0) = \{0, 2\}$ and $\theta(1) = \{1, 3\}$. It is clear that $Z_4/\theta \cong Z_2$.

Definition 1.67. A normal subloop $(H, +)$ of a loop $(Q, +)$ is admissible relative to a permutation α of the set Q if and only if the corresponding to $(H, +)$ normal congruence θ is admissible relative to α.

Definition 1.68. A quasigroup $(Q, \cdot)$ is *simple* if its only normal congruences are the diagonal $\hat{Q} = \{(q, q) \mid q \in Q\}$ and universal $Q \times Q$.

Definition 1.69. We will name a subloop $(H, +)$ of a loop $(Q, +)\alpha$-invariant relative to a permutation α of the set Q, if $\alpha H = H$.

We will name a loop $(Q, +)\alpha$-simple if only identity subloop and the loop $(Q, +)$ are invariant relative to the permutation α of the set Q.

We will name a quasigroup $(Q, \cdot)\alpha$-simple relative to the permutation α of the set Q, if only the diagonal and universal congruences are admissible relative to α.

Corollary 1.70. *Let $(Q, \cdot) = (Q, +)(\alpha, \beta, \varepsilon)$, where $(Q, +)$ is a loop, $\alpha, \beta \in S_Q$. If $(Q, +)$ does not contain normal subloops admissible relative to permutations α, β, then quasigroup $(Q, \cdot)$ is simple.*

Proof. The proof follows from Lemmas 1.62 and 1.13 and Corollary 1.64. □

1.5 Direct products

Definition 1.71. If $(Q_1, \cdot)$, $(Q_2, \circ)$ are binary quasigroups, then their *(external) direct product* $(Q, *) = (Q_1, \cdot) \times (Q_2, \circ)$ is the set of all ordered pairs (a', a'') where $a' \in Q_1$, $a'' \in Q_2$, and where the operation in $(Q, *)$ is defined componentwise, that is, $(a_1 * a_2) = (a'_1 \cdot a'_2, a''_1 \circ a''_2)$.

Direct product of quasigroups is studied in many articles and books; see, for example, [18, 19, 29, 45, 108, 80]. The concept of direct product of quasigroups was used already in [78]. In group case it is possible to find these definitions, for example, in [44].

In [27, 108, 109] there is a definition of the (internal) direct product of Ω-algebras. We recall that any quasigroup is an Ω-algebra.

Let U and W be equivalence relations on a set A, let $U \vee W = \{(x, y) \in A^2 \mid \exists n \in N, \exists t_0, t_1, \dots, t_{2n} \in A,\ x = t_0 U t_1 W t_2 U \cdots U t_{2n-1} W t_{2n} = y\}$. $U \vee W$ is an equivalence relation on A called the join of U and W. If U and W are equivalence relations on A for which $U \circ W = W \circ U$, then $U \circ W = U \vee W$, U and W are said to commute [108].

If A is an Ω-algebra and U, W are congruences on A, then $U \vee W$, and $U \cap W$ are also congruences on A.

Definition 1.72 (see [108, 109]). If U and W are congruences on the algebra A which commute and for which $U \cap W = \hat{A} = \{(a, a) \mid \forall a \in A\}$, then the join $U \circ W = U \vee W$ of U and W is called *direct product $U \sqcap W$ of U and W.*

The following theorem establishes the connection between concepts of internal and external direct products of Ω-algebras.

Theorem 1.73 (see [108, p. 16], [109]). *An Ω-algebra A is isomorphic to a direct product of Ω-algebras B and C with isomorphism φ, i.e., $\varphi : A \to B \times C$, if and only if there exist such congruences U and W of A that $A^2 = U \sqcap W$.*

We will use the following easy proved fact.

Lemma 1.74. *If a loop Q is isomorphic to the direct product of the loops A and B, then $C(Q) \cong C(A) \times C(B)$.*

Lemma 1.75. *If a left F-quasigroup Q is isomorphic to the direct product of a left F-quasigroup A and a quasigroup B, then B also is a left F-quasigroup.*

Proof. Indeed, if $q = (a, b)$, where $q \in Q$, $a \in A$, $b \in B$, then $e(q) = (e(a), e(b))$. $\square$

Remark 1.76. An analog of Lemma 1.75 is true for right F-quasigroups, left and right SM- and E-quasigroups.

There exist various approaches to the concept of semidirect product of quasigroups [89, 88, 26, 120]. By an analogy with group case [47] we give the following definition of the semidirect product of quasigroups. Main principe is that a semidirect product is a cartesian product as a set [120].

Definition 1.77 (see [121])**.** Let Q be a quasigroup, A a normal subquasigroup of Q (i.e., $A \trianglelefteq Q$) and B a subquasigroup of Q. A quasigroup Q is the semidirect product of quasigroups A and B, if there exists a homomorphism $h : Q \to B$ which is the identity on B and whose kernel is A, i.e., A is a coset class of the normal congruence $\operatorname{Ker} h$. We will denote this fact as follows: $Q \cong A \leftthreetimes B$.

Remark 1.78. From results of Mal'tsev [70], see, also, [100], it follows that normal subquasigroup A is a coset class of only one normal congruence of the quasigroup Q.

Lemma 1.79. *If a quasigroup Q is the semidirect product of quasigroups A and B, $A \trianglelefteq Q$, then there exists an isotopy T of Q such that QT is a loop and $QT \cong AT \leftthreetimes BT$.*

Proof. If we take isotopy of the form $(R_a^{-1}, L_a^{-1}, \varepsilon)$, where $a \in A$, then we have that QT is a loop, AT is its normal subloop (Lemma 1.62, Remark 1.45). Further we have that BT is a loop since $BT \cong QT/AT$. Therefore BT is a subloop of the loop QT, since the set B is a subset of the set Q. $\square$

Corollary 1.80. *If a quasigroup Q is the direct product of quasigroups A and B, then there exists an isotopy $T = (T_1, T_2)$ of Q such that $QT \cong AT_1 \times BT_2$ is a loop.*

Proof. The proof follows from Lemma 1.79. $\square$

Lemma 1.81. *(1) If a linear left loop $(Q, \cdot)$ with the form $x \cdot y = x + \psi y$, where $(Q, +)$ is a group, $\psi \in \operatorname{Aut}(Q, +)$, is the semidirect product of a normal subgroup $(H, \cdot) \trianglelefteq (Q, \cdot)$ and a subgroup $(K, \cdot) \subseteq (Q, \cdot)$, $H \cap K = 0$, then $(Q, \cdot) = (Q, +)$.*

(2) If a linear right loop $(Q, \cdot)$ with the form $x \cdot y = \varphi x + y$, where $(Q, +)$ is a group, $\varphi \in \operatorname{Aut}(Q, +)$, is the semidirect product of a normal subgroup $(H, \cdot) \trianglelefteq (Q, \cdot)$ and a subgroup $(K, \cdot) \subseteq (Q, \cdot)$, $H \cap K = 0$, then $(Q, \cdot) = (Q, +)$.

Proof. (1) Since $(Q, \cdot)$ is the semidirect product of a normal subgroup $(H, \cdot)$ and a subgroup $(K, \cdot)$, then we can write any element a of the loop $(Q, \cdot)$ in a unique way as a pair $a = (k, 0) \cdot (0, h)$, where $(k, 0) \in (K, \cdot)$, $(0, h) \in (H, \cdot)$. We notice $\psi(k, 0) = (k, 0)$, $\psi(0, h) = (0, h)$, since $(K, \cdot)$, $(H, \cdot)$ are subgroups of the left loop $(Q, \cdot)$. Indeed, from $(k_1 \cdot k_2) \cdot k_3 = k_1 \cdot (k_2 \cdot k_3)$ for all $k_1, k_2, k_3 \in K$ we have $k_1 + \psi k_2 + \psi k_3 = k_1 + \psi k_2 + \psi^2 k_3$, $k_3 = \psi k_3$ for all $k_3 \in K$.

Further we have $\psi a = \psi((k, 0) \cdot (0, h)) = \psi((k, 0) + \psi(0, h)) = \psi(k, 0) + \psi^2(0, h) = (k, 0) + \psi(0, h) = ((k, 0) \cdot (0, h)) = a$, $\psi = \varepsilon$, $(Q, \cdot) = (Q, +)$.

(2) This case is proved similarly to Case (1). $\square$

Example 1.82. Medial quasigroup $(Z_9, \circ)$, $x \circ y = x + 4 \cdot y$, where $(Z_9, +)$ is the cyclic group, $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, demonstrates that some restrictions in Lemma 1.81 are essential.

1.6 Parastrophe invariants and isostrophisms

Parastrophe invariants and isostrophisms are studied in [13].

Lemma 1.83. *If a quasigroup Q is the direct product of a quasigroup A and a quasigroup B, then $Q^\sigma = A^\sigma \times B^\sigma$, where σ is a parastrophy.*

Proof. From Theorem 1.73 it follows that the direct product $A \times B$ defines two quasigroup congruences. From Theorem 1.51 it follows that these congruences are normal. By Corollary 1.46 these congruences are invariant relative to any parastrophy of the quasigroup Q. □

Lemma 1.84. *If Q is a quasigroup and $\alpha \in \operatorname{Aut}(Q)$, then $\alpha \in \operatorname{Aut}(Q^\sigma)$, where σ is a parastrophy.*

Proof. It is easy to check [93, 94]. □

Lemma 1.85. *(1) A quasigroup $(Q, \cdot)$ is a left F-quasigroup if and only if its (12)-parastrophe is a right F-quasigroup.*

(2) A quasigroup $(Q, \cdot)$ is a left E-quasigroup if and only if its (12)-parastrophe is a right E-quasigroup.

(3) A quasigroup $(Q, \cdot)$ is a left SM-quasigroup if and only if its (12)-parastrophe is a right SM-quasigroup.

(4) A quasigroup $(Q, \cdot)$ is a left distributive quasigroup if and only if its (12)-parastrophe is a right distributive quasigroup.

(5) A quasigroup $(Q, \cdot)$ is a left distributive quasigroup if and only if its (23)-parastrophe is a left distributive quasigroup.

(6) A quasigroup $(Q, \cdot)$ is a left SM-quasigroup if and only if $(Q, \backslash)$ is a left F-quasigroup.

(7) A quasigroup $(Q, \cdot)$ is a right SM-quasigroup if and only if $(Q, /)$ is a right F-quasigroup.

(8) A quasigroup $(Q, \cdot)$ is a left E-quasigroup if and only if $(Q, \backslash)$ is a left E-quasigroup (see [64, Lemma 2.2]).

(9) A quasigroup $(Q, \cdot)$ is a right E-quasigroup if and only if $(Q, /)$ is a right E-quasigroup (see [64, Lemma 2.2]).

Proof. It is easy to check Cases (1)–(4).

(5) The fulfilment in a quasigroup $(Q, \cdot)$ of the left distributive identity is equivalent to the fact that in this quasigroup any left translation L_x is an automorphism of this quasigroup. Indeed, we can rewrite left distributive identity in such manner $L_x yz = L_x y \cdot L_x z$. Using Table 1 we have that $L_x^{\backslash} = L_x^{-1}$. Thus by Lemma 1.84 $L_x^{\backslash} \in \operatorname{Aut}(Q, \backslash)$. Therefore, if $(Q, \cdot)$ is a left distributive quasigroup, then $(Q, \backslash)$ also is a left distributive quasigroup and vice versa.

(6) Let $(Q, \backslash)$ be a left F-quasigroup. Then

$$x\backslash(y\backslash z) = (x\backslash y)\backslash(e^{(\backslash)}(x)\backslash z) = v$$

If $x\backslash(y\backslash z) = v$, then $x \cdot v = (y\backslash z)$, $y \cdot (x \cdot v) = z$. We notice, if $x\backslash e^{(\backslash)}(x) = x$, then $e^{(\backslash)}(x) = x \cdot x \stackrel{\text{def}}{=} s(x)$. See Table 2.

We can rewrite equality $(x\backslash y)\backslash(e^{(\backslash)}(x)\backslash z) = v$ in the form $(x\backslash y)\cdot v = s(x)\backslash z$, $s(x)\cdot((x\backslash y)\cdot v) = z$. Now we have the equality $s(x)\cdot((x\backslash y)\cdot v) = y\cdot(x\cdot v)$. If we denote $(x\backslash y)$ by u, then $x \cdot u = y$.

Therefore we can rewrite equality $s(x)\cdot((x\backslash y)\cdot v) = y\cdot(x\cdot v)$ in the form $s(x)\cdot(u\cdot v) = (x\cdot u)\cdot(x\cdot v)$, i.e., in the form $(x\cdot x)\cdot(u\cdot v) = (x\cdot u)\cdot(x\cdot v)$.

In a similar way it is possible to check the converse: if $(Q, \backslash)$ is a left SM-quasigroup, then $(Q, \cdot)$ is a left F-quasigroup.

Cases (7)–(9) are proved in a similar way. □

Corollary 1.86. *If $(Q, \cdot)$ is a group, then*

(1) $(Q, \backslash)$ is a left SM-quasigroup;

(2) $(Q, /)$ is a right SM-quasigroup.

Proof. (1) Any group is a left F-quasigroup since in this case $e(x) = 1$ for all $x \in Q$. Therefore we can use Lemma 1.85(6).

(2) We can use Lemma 1.85(7). □

Definition 1.87 (see [14])**.** A quasigroup (Q, B) is an isostrophic image of a quasigroup (Q, A) if there exists a collection of permutations $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) = (\sigma, T)$, where $\sigma \in S_3$, $T = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_1, \alpha_2, \alpha_3$ are permutations of the set Q such that

$$B\big(x_1, x_2\big) = A\big(x_1, x_2\big)(\sigma, T) = A^{\sigma}\big(x_1, x_2\big)T = \alpha_3^{-1}A\big(\alpha_1 x_{\sigma^{-1}1}, \alpha_2 x_{\sigma^{-1}2}\big)$$

for all $x_1, x_2 \in Q$.

A collection of permutations $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) = (\sigma, T)$ will be called an *isostrophism* or an *isostrophy* of a quasigroup (Q, A). We can rewrite equality from Definition 1.87 in the form $(A^{\sigma})T = B$.

Lemma 1.88 (see [14])**.** *An isostrophic image of a quasigroup is a quasigroup.*

Proof. The proof follows from the fact that any parastrophic image of a quasigroup is a quasigroup and any isotopic image of a quasigroup is a quasigroup. □

From Lemma 1.88 it follows that it is possible to define the multiplication of isostrophies of a quasigroup operation defined on a set Q.

Definition 1.89. If (σ, S) and (τ, T) are isostrophisms of a quasigroup (Q, A), then

$$(\sigma, S)(\tau, T) = \big(\sigma\tau, S^{\tau}T\big)$$

where $A^{\sigma\tau} = (A^{\sigma})^{\tau}$ and $(x_1, x_2, x_3)(S^{\tau}T) = ((x_1, x_2, x_3)S^{\tau})T$ for any quasigroup triplet (x_1, x_2, x_3) [105].

Slightly other operation on the set of all isostrophies (multiplication of quasigroup isostrophies) is defined in [14]. Definition from [69] is very close to Definition 1.89. See, also, [13, 48].

Corollary 1.90. *One has $(\varepsilon, S)(\tau, \varepsilon) = (\tau, S^{\tau}) = (\tau, \varepsilon)(\varepsilon, S^{\tau})$.*

Lemma 1.91. *One has $(\sigma, S)^{-1} = (\sigma^{-1}, (S^{-1})^{\sigma^{-1}})$.*

Proof. Let $S = (\alpha_1, \alpha_2, \alpha_3)$ be an isotopy of a quasigroup A, $S^{-1} = (\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})$, $S^\sigma = (\alpha_{\sigma^{-1}1}, \alpha_{\sigma^{-1}2}, \alpha_{\sigma^{-1}3})$. Then

$$(\sigma, S)\Big(\sigma^{-1}, (S^{-1})^{\sigma^{-1}}\Big) = \Big(\varepsilon', S^{\sigma^{-1}}(S^{-1})^{\sigma^{-1}}\Big) \overset{\text{(Lemma 1.9)}}{=} \Big(\varepsilon', (SS^{-1})^{\sigma^{-1}}\Big) = \big(\varepsilon', (\varepsilon, \varepsilon, \varepsilon)\big)$$

□

1.7 Group isotopes and identities

Information for this subsection has been taken from [1, 10, 11, 14, 67, 114, 118]. We formulate famous Four quasigroups theorem [1, 10, 14, 114] as follows.

Theorem 1.92. *A quadruple (f_1, f_2, f_3, f_4) of binary quasigroup operation defined on a non-empty set Q is the general solution of the generalized associativity equation*

$$A_1\big(A_2(x, y), z\big) = A_3\big(x, A_4(y, z)\big)$$

if and only if there exists a group $(Q, +)$ and permutations $\alpha, \beta, \gamma, \mu, \nu$ of the set Q such that $f_1(t, z) = \mu t + \gamma z$, $f_2(x, y) = \mu^{-1}(\alpha x + \beta y)$, $f_3(x, u) = \alpha x + \nu u$, $f_4(y, z) = \nu^{-1}(\beta y + \gamma z)$.

Lemma 1.93 (see Belousov criteria [11])**.** *If in a group $(Q, +)$ the equality $\alpha x + \beta y = \gamma y + \delta x$ holds for all $x, y \in Q$, where $\alpha, \beta, \gamma, \delta$ are some fixed permutations of Q, then $(Q, +)$ is an Abelian group.*

There exists also the following corollary adapted for our aims from results of Sokhatskii (see [114, Theorem 6.7.2]).

Corollary 1.94. *If in a principal group isotope $(Q, \cdot)$ of a group $(Q, +)$ the equality $\alpha x \cdot \beta y = \gamma y \cdot \delta x$ holds for all $x, y \in Q$, where $\alpha, \beta, \gamma, \delta$ are some fixed permutations of Q, then $(Q, +)$ is an Abelian group.*

Proof. If $x \cdot y = \xi x + \chi y$, then we can rewrite the equality $\alpha x \cdot \beta y = \gamma y \cdot \delta x$ in the form $\xi\alpha x + \chi\beta y = \xi\gamma y + \chi\delta x$. Now we can apply the Belousov criteria (Lemma 1.93). □

Lemma 1.95. *(1) For any principal group isotope $(Q, \cdot)$ there exists its form $x \cdot y = \alpha x + \beta y$ such that $\alpha\, 0 = 0$ [111].*

(2) For any principal group isotope $(Q, \cdot)$ there exists its form $x \cdot y = \alpha x + \beta y$ such that $\beta\, 0 = 0$.

(3) For any right linear quasigroup $(Q, \cdot)$ there exists its form $x \cdot y = \alpha x + \psi y + c$ such that $\alpha 0 = 0$.

(4) For any left linear quasigroup $(Q, \cdot)$ there exists its form $x \cdot y = \varphi x + \beta y + c$ such that $\beta 0 = 0$.

(5) For any left linear quasigroup $(Q, \cdot)$ with idempotent element 0 there exists its form $x \cdot y = \varphi x + \beta y$ such that $\beta 0 = 0$.

(6) For any right linear quasigroup $(Q, \cdot)$ with idempotent element 0 there exists its form $x \cdot y = \alpha x + \psi y$ such that $\alpha 0 = 0$.

Proof. (1) We have $x \cdot y = \alpha x + \beta y = R_{-\alpha 0}\alpha x + L_{\alpha 0}\beta y = \alpha' x + \beta' y$, $\alpha' 0 = 0$.

(2) We have $x \cdot y = \alpha x + \beta y = R_{\beta 0}\alpha x + L_{-\beta 0}\beta y = \alpha' x + \beta' y$, $\beta' 0 = 0$.

(3) We have $x \cdot y = \alpha x + \psi y + c = R_{-\alpha 0}\alpha x + I_{\alpha 0}\psi y + \alpha 0 + c = \alpha' x + \psi' y + c'$, where $I_{\alpha 0}\psi y = \alpha 0 + \psi y - \alpha 0$, $\alpha' 0 = 0$. Since $I_{\alpha 0}$ is an inner automorphism of the group $(Q, +)$, we obtain $I_{\alpha 0}\psi \in \operatorname{Aut}(Q, +)$.

(4) We have $x \cdot y = \varphi x + \beta y + c = \varphi x + \beta y - \beta 0 + \beta 0 + c = \varphi x + R_{-\beta 0}\beta y + \beta 0 + c = \varphi x + \beta' y + c'$, where $\beta' = R_{-\beta 0}\beta$, $c' = \beta 0 + c$.

(5) If $x \cdot y = \varphi x + \beta y + c$, then $0 = 0 \cdot 0 = \varphi\, 0 + \beta\, 0 + c = \beta\, 0 + c$, $\beta\, 0 = -c$. Therefore $x \cdot y = \varphi x + R_c\beta y = \varphi x + \beta' y$ and $\beta' 0 = R_c \beta 0 = -c + c = 0$.

(6) If $x \cdot y = \alpha x + \psi y + c$, then $0 = 0 \cdot 0 = \alpha\, 0 + \psi\, 0 + c = \alpha\, 0 + c$, $\alpha\, 0 = -c$. Therefore $x \cdot y = \alpha x + c - c + \psi y + c = R_c\alpha x + I_{-c}\psi y = R_c\alpha' x + \psi' y$ and $\alpha' 0 = R_c\alpha 0 = -c + c = 0$. Moreover, ψ' is an automorphism of $(Q, +)$ as the product of two automorphisms of the group $(Q, +)$. □

Lemma 1.96. *For any left linear quasigroup $(Q, \cdot)$ there exists its form such that $x \cdot y = \varphi x + \beta y$.*

For any right linear quasigroup $(Q, \cdot)$ there exists its form such that $x \cdot y = \alpha x + \psi y$.

Proof. We can rewrite the form $x \cdot y = \varphi x + \beta y + c$ of a left linear quasigroup $(Q, \cdot)$ as follows: $x \cdot y = \varphi x + R_c\beta y = \varphi x + \beta' y$, where $\beta' = R_c\beta$.

We can rewrite the form $x \cdot y = \alpha x + \psi y + c$ of a right linear quasigroup $(Q, \cdot)$ as follows: $x \cdot y = \alpha x + c - c + \psi y + c = R_c\alpha x + I_c\psi y = \alpha' x + \psi' y$, where $I_{-c}\psi y = -c + \psi y + c$. □

Classical criteria of a linearity of a quasigroup are given by Belousov in [11]. We give a partial case of Sokhatskii result (see [112], [113, Theorem 3], [114, Theorem 6.8.6]).

We recall that up to isomorphism every isotope is principal (Remark 1.7).

Theorem 1.97. *Let $(Q, \cdot)$ be a principal isotope of a group $(Q, +)$, $x \cdot y = \alpha x + \beta y$.*

If $(\alpha_1 x \cdot \alpha_2 y) \cdot a = \alpha_3 x \cdot \alpha_4 y$ is true for all $x, y \in Q$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are permutations of the set Q, a is a fixed element of the set Q, then $(Q, \cdot)$ is a left linear quasigroup.

If $a \cdot (\alpha_1 x \cdot \alpha_2 y) = \alpha_3 x \cdot \alpha_4 y$ is true for all $x, y \in Q$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are permutations of the set Q, a is a fixed element of the set Q, then $(Q, \cdot)$ is a right linear quasigroup.

Proof. We follow [114]. By Lemma 1.95 quasigroup $(Q, \cdot)$ can have the form $x \cdot y = \alpha x + \beta y$ over a group $(Q, +)$ such that $\alpha\, 0 = 0$. If we pass in the equality $(\alpha_1 x \cdot \alpha_2 y) \cdot a = \alpha_3 x \cdot \alpha_4 y$ to the operation "+", then we obtain $\alpha(\alpha\alpha_1 x + \beta\alpha_2 y) + \beta a = \alpha\alpha_3 x + \beta\alpha_4 y$, $\alpha(x + y) = \alpha\alpha_3\alpha_1^{-1}\alpha^{-1}x + \beta\alpha_4\alpha_2^{-1}\beta^{-1}y - \beta a$.

Then the permutation α is a group quasiautomorphism. It is known that any group quasiautomorphism has the form $L_a\varphi$, where $\varphi \in \mathrm{Aut}(Q, +)$. See [15, 12] or Corollary 1.23. Therefore $\alpha \in \mathrm{Aut}(Q, +)$, since $\alpha 0 = 0$.

By Lemma 1.95 there exists the form $x \cdot y = \alpha x + \beta y$ of quasigroup $(Q, \cdot)$ such that $\beta\, 0 = 0$. If we pass in the equality $a \cdot (\alpha_1 x \cdot \alpha_2 y) = \alpha_3 x \cdot \alpha_4 y$ to the operation "+", then we obtain $\alpha a + \beta(\alpha\alpha_1 x + \beta\alpha_2 y) = \alpha\alpha_3 x + \beta\alpha_4 y$, $\beta(x + y) = -\alpha a + \alpha\alpha_3\alpha_1^{-1}\alpha^{-1}x + \beta\alpha_4\alpha_2^{-1}\beta^{-1}y$.

Then the permutation β is a group quasiautomorphism. Therefore $\beta \in \mathrm{Aut}(Q, +)$, since $\beta 0 = 0$. □

Corollary 1.98. *(1) If a left F-quasigroup (E-quasigroup, SM-quasigroup) is a group isotope, then this quasigroup is right linear.*

(2) If a right F-quasigroup (E-quasigroup, SM-quasigroup) is a group isotope, then this quasigroup is left linear [113].

Proof. The proof follows from Theorem 1.97. □

Lemma 1.99. *(1) If in a right linear quasigroup $(Q, \cdot)$ over a group $(Q, +)$ the equality $k \cdot yx = xy \cdot b$ holds for all $x, y \in Q$ and fixed $k, b \in Q$, then $(Q, +)$ is an Abelian group.*

(2) If in a left linear quasigroup $(Q, \cdot)$ over a group $(Q, +)$ the equality $k \cdot yx = xy \cdot b$ holds for all $x, y \in Q$ and fixed $k, b \in Q$, then $(Q, +)$ is an Abelian group.

Proof. (1) By Lemma 1.96 we can take the following form of $(Q,\cdot)$: $x\cdot y = \alpha x + \psi y$. Thus we have $\alpha k + \psi(\alpha y + \psi x) = \alpha(\alpha x + \psi y) + \psi b$, $\alpha k + \psi\alpha y + \psi^2 x - \psi b = \alpha(\alpha x + \psi y)$, $\alpha(x+y) = \alpha k + \psi\alpha\psi^{-1} y + \psi^2\alpha^{-1} x - \psi b$. Therefore α is a quasiautomorphism of the group $(Q,+)$. Let $\alpha = L_d\varphi$, where $\varphi \in \mathrm{Aut}(Q,+)$.

Further we have $d + \varphi x + \varphi y = \alpha k + \psi\alpha\psi^{-1} y + \psi^2\alpha^{-1} x - \psi b$, $L_d\varphi x + \varphi y = L_{\alpha k}\psi\alpha\psi^{-1} y + R_{-\psi b}\psi^2\alpha^{-1} x$. Finally, we can apply Lemma 1.93.

Case (2) is proved in a similar way. □

Quasigroup $(Q,\cdot)$ with equality $xy\cdot z = x\cdot(y\circ z)$ for all $x,y,z\in Q$ is called "quasigroup which fulfills Sushkevich postulate A".

Quasigroup $(Q,\cdot)$ with equality $x\cdot yz = (x\circ y)\cdot z$ for all $x,y,z\in Q$ will be called "quasigroup which fulfills Sushkevich postulate A*".

Theorem 1.100. *(1) If quasigroup $(Q,\cdot)$ fulfills Sushkevich postulate A, then $(Q,\cdot)$ is isotopic to the group $(Q,\circ)$, $(Q,\cdot) = (Q,\circ)(\varphi,\varepsilon,\varphi)$ (see [15, Theorem 1.7]).*

(2) If quasigroup $(Q,\cdot)$ fulfills Sushkevich postulate A, then $(Q,\cdot)$ is isotopic to the group $(Q,\circ)$, $(Q,\cdot) = (Q,\circ)(\varepsilon,\psi,\psi)$.*

Proof. Case (1) is proved in [15].

The proof of Case (2) is similar to the proof of Case (1). It is easy to see that $(Q,\circ)$ is quasigroup. Indeed, if $z=c$, then we have $x\cdot R^{\cdot}_c y = R^{\cdot}_c(x\circ y)$, $(Q,\circ)$ is isotope of quasigroup $(Q,\cdot)$. Therefore $(Q,\circ)$ is a quasigroup. Moreover, $x\cdot y = R_c(x\circ R_c^{-1}y)$, $(Q,\cdot) = (Q,\circ)(\varepsilon,\psi,\psi)$, where $\psi = R_c^{-1}$.

Quasigroup $(Q,\circ)$ is a group. It is possible to use Theorem 1.92 but we give direct proof similar to the proof from [15]. We have $(x\circ(y\circ z))\cdot w = x\cdot((y\circ z)\cdot w) = x\cdot(y\cdot(z\cdot w)) = (x\circ y)\cdot(z\cdot w) = ((x\circ y)\circ z)\cdot w$, $x\circ(y\circ z) = (x\circ y)\circ z$. □

Quasigroup $(Q,\cdot)$ with generalized identity $xy\cdot z = x\cdot y\delta(z)$, where δ is a fixed permutation of the set Q, is called "quasigroup which fulfills Sushkevich postulate B".

Quasigroup $(Q,\cdot)$ with generalized identity $x\cdot yz = (\delta(x)\cdot y)\cdot z$, where δ is a fixed permutation of the set Q, will be called "quasigroup which fulfills Sushkevich postulate B*".

It is easy to see that any quasigroup with postulate B (B*) is a quasigroup with postulate A (A*).

Theorem 1.101. *(1) If quasigroup $(Q,\cdot)$ fulfills Sushkevich postulate B, then $(Q,\cdot)$ is isotopic to the group $(Q,\circ)$, $(Q,\cdot) = (Q,\circ)(\varepsilon,\psi,\varepsilon)$, where $\psi\in\mathrm{Aut}(Q,\circ)$, $\psi\in\mathrm{Aut}(Q,\cdot)$ (see [15, Theorem 1.8]).*

(2) If quasigroup $(Q,\cdot)$ fulfills Sushkevich postulate B, then $(Q,\cdot)$ is isotopic to the group $(Q,\circ)$, $(Q,\cdot) = (Q,\circ)(\varphi,\varepsilon,\varepsilon)$, where $\varphi\in\mathrm{Aut}(Q,\circ)$, $\varphi\in\mathrm{Aut}(Q,\cdot)$.*

Proof. Case (1) is proved in [15]. It is easy to see that quasigroup $(Q,\cdot)$ has the right identity element, i.e., $(Q,\cdot)$ is right loop. Indeed, $x\cdot 0 = x\circ\psi 0 = x$ for all $x\in Q$, where 0 is zero of group $(Q,\circ)$.

(2) The proof of Case (2) is similar to the proof of Case (1). Here we give the direct proof because the book [15] is rare. Since the quasigroup $(Q,\cdot)$ fulfills postulates A* and B*, then by Theorem 1.100(2), groupoid (magma) $(Q,\circ)$, $x\circ y = \delta(x)\cdot y$, is a group and $(Q,\cdot) = (Q,\circ)(\delta^{-1},\varepsilon,\varepsilon)$. By the same theorem $(Q,\cdot) = (Q,\circ)(\varepsilon,\psi,\psi)$. Therefore (δ,ψ,ψ) is an autotopy of the group $(Q,\circ)$. By Corollary 1.22 $\delta\in\mathrm{Aut}(Q,\circ)$. Therefore $\varphi = \delta^{-1}\in\mathrm{Aut}(Q,\circ)$. It is easy to see that $(Q,\cdot)$ is left loop. □

2 Direct decompositions

2.1 Left and right F-quasigroups

In order to study the structure of left F-quasigroups we will use approach from [78, 102]. As usual $e(e(x)) = e^2(x)$, and so on.

Lemma 2.1 (see [77, 15])**.** *(1) In a left F-quasigroup $(Q, \cdot)$ the map e^i is an endomorphism of $(Q, \cdot)$, $e^i(Q, \cdot)$ is a subquasigroup of quasigroup $(Q, \cdot)$ for all suitable values of the index i.*

(2) In a right F-quasigroup $(Q, \cdot)$ the map f^i is an endomorphism of $(Q, \cdot)$, $f^i(Q, \cdot)$ is a subquasigroup of quasigroup $(Q, \cdot)$ for all suitable values of the index i

Proof. (1) From identity $x \cdot yz = xy \cdot e(x)z$ by $z = e(y)$ we have $xy = xy \cdot e(x)e(y)$, i.e., $e(x \cdot y) = e(x) \cdot e(y)$. Further we have $e^2(x \cdot y) = e(e(x \cdot y)) = e(e(x) \cdot e(y)) = e^2(x) \cdot e^2(y)$ and so on. Therefore e^m is an endomorphism of the quasigroup $(Q, \cdot)$. The fact that $e^m(Q, \cdot)$ is a subquasigroup of quasigroup $(Q, \cdot)$ follows from Lemma 1.54.

(2) The proof is similar. □

The proof of the following lemma has taken from [15, p. 33].

Lemma 2.2. *(1) Endomorphism e of a left F-quasigroup $(Q, \cdot)$ is zero endomorphism, i.e., $e(x) = k$ for all $x \in Q$, if and only if left F-quasigroup $(Q, \cdot)$ is a right loop, isotope of a group $(Q, +)$ of the form $(Q, \cdot) = (Q, +)(\varepsilon, \psi, \varepsilon)$, where $\psi \in \mathrm{Aut}(Q, +)$, $k = 0$.*

(2) Endomorphism f of a right F-quasigroup $(Q, \cdot)$ is zero endomorphism, i.e., $f(x) = k$ for all $x \in Q$, if and only if right F-quasigroup $(Q, \cdot)$ is a left loop, isotope of a group $(Q, +)$ of the form $(Q, \cdot) = (Q, +)(\varphi, \varepsilon, \varepsilon)$, where $\varphi \in \mathrm{Aut}(Q, +)$, $k = 0$.

Proof. (1) We can rewrite equality $x \cdot yz = xy \cdot L_k z$ in the form $xy \cdot z = x(yL_k^{-1}z) = x(y \cdot \delta z)$, where $\delta = L_k^{-1}$. Therefore Sushkevich postulate B is fulfilled in $(Q, \cdot)$ and we can apply Theorem 1.101. Further we have $x \cdot 0 = x + \psi 0 = x$. From the other side $x \cdot k = x$. Therefore, $k = 0$. It is easy to see that the converse also is true.

(2) We can use the "mirror" principles. □

Lemma 2.3 (see [15])**.** *(1) The endomorphism e of a left F-quasigroup $(Q, \cdot)$ is a permutation of the set Q if and only if quasigroup $(Q, \circ)$ of the form $x \circ y = x \cdot e(y)$ is a left distributive quasigroup and $e \in \mathrm{Aut}(Q, \circ)$.*

(2) The endomorphism f of a right F-quasigroup $(Q, \cdot)$ is a permutation of the set Q if and only if quasigroup $(Q, \circ)$ of the form $x \circ y = f(x) \cdot y$ is a right distributive quasigroup and $f \in \mathrm{Aut}(Q, \circ)$.

Proof. (1) Prove that $(Q, \circ)$ is left distributive. We have

$$\begin{aligned} x \circ (y \circ z) &= x \cdot e\big(y \cdot e(z)\big) = x \cdot \big(e(y) \cdot e^2(z)\big) = \big(x \cdot e(y)\big) \cdot \big(e(x) \cdot e^2(z)\big) \\ &= \big(x \cdot e(y)\big) \cdot e\big(x \cdot e(z)\big) = (x \circ y) \circ (x \circ z) \end{aligned} \tag{2.1}$$

Prove that $e \in \mathrm{Aut}(Q, \circ)$. We have $e(x \circ y) = e(x \cdot e(y)) = e(x) \cdot e^2(y) = e(x) \circ e(y)$ [72].

Conversely, let $(Q, \cdot)$ be an isotope of the form $x \cdot y = x \circ \psi(y)$, where $\psi \in \mathrm{Aut}(Q, \circ)$, of a left distributive quasigroup $(Q, \circ)$. The fact that $\psi \in \mathrm{Aut}(Q, \cdot)$ follows from Lemma 1.24.

We can use equalities (2.1) by the proving that $(Q, \cdot)$ is a left F-quasigroup. The fact that $\psi = e^{-1}$ follows from Lemma 1.29.

(2) The proof is similar. □

In a left F-quasigroup $(Q, \cdot)$ define the following (maybe infinite) chain:

$$Q \supset e(Q) \supset e^2(Q) \supset \dots \supset e^m(Q) \supset \cdots \tag{2.2}$$

Definition 2.4. Chain (2.2) becomes stable means that there exists a number m (finite or infinite) such that $e^m(Q) = e^{m+1}(Q) = e^{m+2}(Q) \dots$. We notice, in other words

$$e^m(Q) = \bigcap_{i=1}^{\infty} e^i(Q) = \lim_{i\to\infty} e^i(Q)$$

In this case we will say that endomorphism e has the order m.

Lemma 2.5. *In any left F-quasigroup Q chain (2.2) becomes stable, i.e., the map $e|_{e^m(Q)}$ is an automorphism of quasigroup $e^m(Q)$.*

Proof. We have two cases. (1) Chain (2.2) becomes stable on a finite step m. It is clear that in this case $e|_{e^m(Q)}$ is an automorphism of $e^m(Q, \cdot)$.

(2) Prove that chain (2.2) will be stabilized on the step $m = \infty$, if it is not stabilized on a finite step m. Denote $\bigcap_{i=1}^{\infty} e^i(Q)$ by C.

Notice, if $A \subseteq Q$, $B \subseteq Q$, then $e(A \cap B) \subseteq e(A) \cap e(B)$. Indeed, if $x \in A \cap B$, then $e(x) \in e(A \cap B)$. If $x \in A \cap B$, then $x \in A$ and $x \in B$. Therefore $e(x) \in e(A)$ and $e(x) \in e(B)$, $e(x) \in e(A) \cap e(B)$, $e(A \cap B) \subseteq e(A) \cap e(B)$. Then

$$e(C) = e\left(\bigcap_{i=1}^{\infty} e^i(Q)\right) \subseteq \bigcap_{i=1}^{\infty} e^{i+1}(Q) = C$$

Prove that $e(C) = C$. Any element $c \in C$ has the form $c = \lim_{i\to\infty} e^i(a)$, where $a \in Q$. Then $e(c) = e(\lim_{i\to\infty} e^i(a)) = \lim_{i\to\infty} e^{i+1}(a) \in C$ for any $c \in C$. Therefore there does not exist element x of the set C such that $e(x) \notin e(C)$.

Therefore for any m (finite or infinite) $e|_{e^m(Q)}$ is an automorphism of $e^m(Q, \cdot)$. □

Example 2.6. Quasigroup $(Z, \cdot)$, where $x \cdot y = -x + y$, $(Z, +)$ is infinite cyclic group, is medial, unipotent, left F-quasigroup such that $e(x) = x + x = 2x$. Notice in this case Ker $e = \{0\}$. In [71, p. 59] a mapping similar to the mapping e is called isomorphism and the embedding of an algebra in its subalgebra.

Theorem 2.7. *(1) Any left F-quasigroup $(Q, \cdot)$ has the following structure:*

$$(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$$

where $(A, \circ)$ is a quasigroup with a unique idempotent element; $(B, \cdot)$ is isotope of a left distributive quasigroup $(B, \star)$, $x \cdot y = x \star \psi y$ for all $x, y \in B$, $\psi \in \mathrm{Aut}(B, \cdot)$, $\psi \in \mathrm{Aut}(B, \star)$.

(2) Any right F-quasigroup $(Q, \cdot)$ has the following structure:

$$(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$$

where $(A, \circ)$ is a quasigroup with a unique idempotent element; $(B, \cdot)$ is isotope of a right distributive quasigroup $(B, \star)$, $x \cdot y = \varphi x \star y$ for all $x, y \in B$, $\varphi \in \mathrm{Aut}(B, \cdot)$, $\varphi \in \mathrm{Aut}(B, \star)$.

Proof. The proof of this theorem mainly repeats the proof of Theorem 6 from [102].

If the map e is a permutation of the set Q, then by Lemma 2.3 $(Q, \cdot)$ is isotope of left distributive quasigroup.

If $e(Q) = k$, where k is a fixed element of the set Q, then the quasigroup $(Q, \cdot)$ is a quasigroup with right identity element k, i.e., it is a right loop, which is isotopic to a group $(Q, +)$ (Lemma 2.2).

Let us suppose that $e^m = e^{m+1}$, where $m > 1$.

From Lemma 2.5 it follows that $e^m(Q, \cdot) = (B, \cdot)$ is a subquasigroup of quasigroup $(Q, \cdot)$. It is clear that $(B, \cdot)$ is a left F-quasigroup in which the map $\overline{e} = e|_{e^m(Q)}$ is a permutation of the set $B \subset Q$. In other words, $e(B, \cdot) = (B, \cdot)$.

Define binary relation δ on quasigroup $(Q, \cdot)$ by the following rule: $x\delta y$ if and only if $e^m(x) = e^m(y)$. Define binary relation ρ on quasigroup $(Q, \cdot)$ by the rule $x\rho y$ if and only if $B \cdot x = B \cdot y$, i.e., for any $b_1 \in B$ there exists exactly one element $b_2 \in B$ such that $b_1 \cdot x = b_2 \cdot y$ and, vice versa, for any $b_2 \in B$ there exists exactly one element $b_1 \in B$ such that $b_1 \cdot x = b_2 \cdot y$.

From Theorem 1.51 and Lemma 1.54 it follows that δ is a normal congruence.

It is easy to check that binary relation ρ is equivalence relation (see Theorem 1.37).

We prove that binary relation ρ is a congruence, i.e., that the following implication is true: $x_1\rho y_1,\ x_2\rho y_2, \Rightarrow (x_1 \cdot x_2)\rho(y_1 \cdot y_2)$.

Using the definition of relation ρ we can rewrite the last implication in the following equivalent form: if

$$B \cdot x_1 = B \cdot y_1, \quad B \cdot x_2 = B \cdot y_2 \tag{2.3}$$

then $B \cdot (x_1 \cdot x_2) = B \cdot (y_1 \cdot y_2)$.

If we multiply both sides of equalities (2.3), respectively, then we obtain the following equality:

$$\left(\overset{x}{B} \cdot \overset{y}{x_1}\right) \cdot \left(\overset{e(x)}{B} \cdot \overset{z}{x_2}\right) = \left(B \cdot y_1\right) \cdot \left(B \cdot y_2\right)$$

Using left F-quasigroup equality $(x \cdot yz = xy \cdot e(x)z)$ from the right to the left and, taking into consideration that if $x \in B$, then $e(x) \in B$, i.e., $eB = B$, we can rewrite the last equality in the following form:

$$B \cdot \left(x_1 \cdot x_2\right) = B \cdot \left(y_1 \cdot y_2\right)$$

since $(B, \cdot)$ is a subquasigroup and, therefore, $B \cdot B = B$. Thus the binary relation ρ is a congruence.

Prove that $\delta \cap \rho = \hat{Q} = \{(x, x) \mid \forall x \in Q\}$. From reflexivity of relations δ, ρ it follows that $\delta \cap \rho \supseteq \hat{Q}$.

Let $(x, y) \in \delta \cap \rho$, i.e., let $x\ \delta\ y$ and $x\ \rho\ y$ where $x, y \in Q$. Using the definitions of relations δ, ρ we have $e^m(x) = e^m(y)$ and $(B, \cdot) \cdot x = (B, \cdot) \cdot y$. Then there exist $a, b \in B$ such that $a \cdot x = b \cdot y$. Applying to both sides of last equality the map e^m we obtain $e^m(a) \cdot e^m(x) = e^m(b) \cdot e^m(y)$, $e^m(a) = e^m(b)$, $a = b$, since the map $e^m|_B$ is a permutation of the set B. If $a = b$, then from equality $a \cdot x = b \cdot y$ we obtain $x = y$.

Prove that $\delta \circ \rho = Q \times Q$. Let a, c be any fixed elements of the set Q. We prove the equality if it will be shown that there exists element $y \in Q$ such that $a\delta y$ and $y\rho c$.

From definition of congruence δ we have that condition $a\delta y$ is equivalent to equality $e^m(a) = e^m(y)$. From definition of congruence ρ it follows that condition $y\rho c$ is equivalent to the following condition: $y \in \rho(c) = B \cdot c$.

We prove the equality if it will be shown that there exists element $y \in B \cdot c$ such that $e^m(a) = e^m(y)$. Such element y there exists since $e^m(B \cdot c) = e^m(B) \cdot e^m(c) = B = e^m(Q)$.

Prove that $\rho \circ \delta = Q \times Q$. Let a, c be any fixed elements of the set Q. We prove the equality if it will be shown that there exists element $y \in Q$ such that $a \rho y$ and $y \delta c$.

From definition of congruence δ we have that condition $y \delta c$ is equivalent to equality $e^m(c) = e^m(y)$. From definition of congruence ρ it follows that condition $a \rho y$ is equivalent to the following condition: $y \in \rho(a) = B \cdot a$.

We prove the equality if it will be shown that there exists element $y \in B \cdot a$ such that $e^m(c) = e^m(y)$. Such element y there exists since $e^m(B \cdot a) = e^m(B) \cdot e^m(a) = B = e^m(Q)$.

Therefore $\rho \circ \delta = Q \times Q = \delta \circ \rho$, $\delta \cap \rho = \hat{Q}$ and we can use Theorem 1.73. Now we can say that quasigroup $(Q, \cdot)$ is isomorphic to the direct product of a quasigroup $(Q, \cdot)/\delta \cong (B, \cdot)$ (Theorem 1.51) and a division groupoid $(Q, \cdot)/\rho \cong (A, \circ)$ [5, 24].

From Definition 1.71 it follows, if $(Q, \cdot) \cong (B, \cdot) \times (A, \circ)$, where $(Q, \cdot)$, $(B, \cdot)$ are quasigroups, then $(A, \circ)$ also is a quasigroup. Then by Theorem 1.51 the congruence ρ is normal, $(B, \cdot) \trianglelefteq (Q, \cdot)$.

Left F-quasigroup equality holds in quasigroup $(B, \cdot)$ since $(B, \cdot) \subseteq (Q, \cdot)$.

If the quasigroups $(Q, \cdot)$ and $(B, \cdot)$ are left F-quasigroups, $(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$, then $(A, \circ)$ also is a left F-quasigroup (Lemma 1.75).

Prove that the quasigroup $(A, \circ) \cong (Q, \cdot)/(B, \cdot)$, where $e^m(Q, \cdot) = (B, \cdot)$, has a unique idempotent element.

We can identify elements of quasigroup $(Q, \cdot)/(B, \cdot)$ with cosets of the form $B \cdot c$, where $c \in Q$.

From properties of quasigroup $(A, \circ)$ we have that $e^m(A) = a$, where the element a is a fixed element of the set A that corresponds to the coset class B. Further, taking into consideration the properties of endomorphism e of the quasigroup $(A, \circ)$, we obtain $e^{m+1}A = e(e^m A) = e(a) = a$. Therefore $e(a) = a$, i.e., the element a is an idempotent element of quasigroup $(A, \circ)$.

Prove that there exists exactly one idempotent element in quasigroup $(A, \circ)$. Suppose that there exists an element c of the set A such that $c \circ c = c$, i.e., such that $e(c) = c$. Then we have $e^m(c) = c = a$, since $e^m(A) = a$.

The fact that $(B, \cdot)$ is isotope of a left distributive quasigroup $(B, \star)$ follows from Lemma 2.3.

Properties of right F-quasigroups coincide with the "mirror" properties of left F-quasigroups. □

We notice, in finite case all congruences are normal and permutable (Lemmas 1.59 and 1.48). Therefore for finite case Theorem 2.7 can be proved in more short way.

We add some details on the structure of left F-quasigroup $(Q, \cdot)$. By $e^j(Q, \cdot)$ we denote endomorphic image of the quasigroup $(Q, \cdot)$ relative to the endomorphism e^j.

Corollary 2.8. *If $(Q, \cdot)$ is a left F-quasigroup, then $e^m(Q, \cdot) \trianglelefteq (Q, \cdot)$.*

Proof. This follows from the fact that the binary relation ρ from Theorem 2.7 is a normal congruence in $(Q, \cdot)$ and subquasigroup $e^m(Q, \cdot) = (B, \cdot)$ is an equivalence class of ρ. □

Remark 2.9. For brevity we will denote the endomorphism $e|_{e^j(Q,\cdot)}$ such that

$$e|_{e^j(Q,\cdot)} : e^j(Q, \cdot) \longrightarrow e^{j+1}(Q, \cdot)$$

by e_j, the endomorphism $f|_{f^j(Q,\cdot)}$ by f_j, the endomorphism $s|_{s^j(Q,\cdot)}$ by s_j.

Corollary 2.10. *If $(Q,\cdot)$ is a left F-quasigroup with an idempotent element, then equivalence class (cell) $\bar{a}$ of the normal congruence* $\mathrm{Ker}\, e_j$ *containing an idempotent element $a \in Q$ forms linear right loop $(\bar{a},\cdot)$ for all suitable values of j.*

Proof. By Lemma 1.53 $(\bar{a},\cdot)$ is a quasigroup. From properties of the endomorphism e we have that in $(\bar{a},\cdot)$ endomorphism e is zero endomorphism. Therefore in this case we can apply Lemma 2.2. Then $(\bar{a},\cdot)$ is isotopic to a group with isotopy of the form $(\varepsilon,\psi,\varepsilon)$, where $\psi \in \mathrm{Aut}(\bar{a},\cdot)$. □

Corollary 2.11. *If $(Q,\cdot)$ is a right F-quasigroup, then $f^m(Q,\cdot) \trianglelefteq (Q,\cdot)$.*

Proof. The proof is similar to the proof of Corollary 2.8. □

Corollary 2.12. *If $(Q,\cdot)$ is a right F-quasigroup with an idempotent element, then equivalence class $\bar{a}$ of the normal congruence* $\mathrm{Ker}\, f_j$ *containing an idempotent element $a \in Q$ forms linear left loop $(\bar{a},\cdot)$ for all suitable values of j.*

Proof. The proof is similar to the proof of Corollary 2.10. □

2.2 Left and right SM- and E-quasigroups

We can formulate theorem on the structure of left semimedial quasigroup using connections between a quasigroup and its (23)-parastrophe (Lemma 1.85), but in order to have more information about left semimedial quasigroup we prefer to give direct formulations some results from Section 2.1.

Lemma 2.13. *(1) In a left semimedial quasigroup $(Q,\cdot)$ the map s^i is an endomorphism of $(Q,\cdot)$, $s^i(Q,\cdot)$ is a subquasigroup of quasigroup $(Q,\cdot)$ for all suitable values of the index i [77, 15].*

(2) In a right semimedial quasigroup $(Q,\cdot)$ the map s^i is an endomorphism of $(Q,\cdot)$, $s^i(Q,\cdot)$ is a subquasigroup of quasigroup $(Q,\cdot)$ for all suitable values of the index i.

(3) In a left E-quasigroup $(Q,\cdot)$ the map f^i is an endomorphism of $(Q,\cdot)$, $f^i(Q,\cdot)$ is a subquasigroup of quasigroup $(Q,\cdot)$ for all suitable values of the index m [64].

(4) In a right E-quasigroup $(Q,\cdot)$ the map e^i is an endomorphism of $(Q,\cdot)$, $e^i(Q,\cdot)$ is a subquasigroup of quasigroup $(Q,\cdot)$ for all suitable values of the index m [64].

Proof. (1) From identity $xx \cdot yz = xy \cdot xz$ by $z = y$ we have $xx \cdot yy = xy \cdot xy$, i.e., $s(x) \cdot (y) = s(x \cdot y)$. Therefore s^i is an endomorphism of the quasigroup $(Q,\cdot)$.

The fact that $s^i(Q,\cdot)$ is a subquasigroup of quasigroup $(Q,\cdot)$ follows from Lemma 1.54.

(2) The proof of Case (2) is similar to the proof of Case (1), thus we omit it.

(3) From identity $x \cdot yz = f(x)y \cdot xz$ by $y = f(y)$, $z = y$ we have $xy = f(x)f(y) \cdot xy$. But $f(xy) \cdot xy = xy$. Therefore $f(x) \cdot f(y) = f(x \cdot y)$ [64].

(4) From identity $zy \cdot x = zx \cdot ye(x)$ by $y = e(y)$, $z = y$ we have $yx = yx \cdot e(y)e(x)$. □

Theorem 2.14. *(1) If the endomorphism s of a left semimedial quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $s(x) = 0$ for all $x \in Q$, then $(Q,\cdot)$ is an unipotent quasigroup, $(Q,\cdot) \cong (Q,\circ)$, where $x \circ y = -\varphi x + \varphi y$, $(Q,+)$ is a group, $\varphi \in \mathrm{Aut}(Q,+)$.*

(2) If the endomorphism s of a right semimedial quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $s(x) = 0$ for all $x \in Q$, then $(Q,\cdot)$ is an unipotent quasigroup, $(Q,\cdot) \cong (Q,\circ)$, where $x \circ y = \varphi x - \varphi y$, $(Q,+)$ is a group, $\varphi \in \mathrm{Aut}(Q,+)$.

(3) If the endomorphism f of a left E-quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $f(x) = 0$ for all $x \in Q$, then up to isomorphism $(Q,\cdot)$ is a left loop, $x\cdot y = \alpha x+y$, $(Q,+)$ is an Abelian group, $\alpha 0 = 0$.

(4) If the endomorphism e of a right E-quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $e(x) = 0$ for all $x \in Q$, then up to isomorphism $(Q,\cdot)$ is a right loop, $x \cdot y = x + \beta y$, $(Q,+)$ is an Abelian group, $\beta 0 = 0$.

Proof. (1) We can rewrite equality $xx \cdot yz = xy \cdot xz$ in the form $k \cdot yz = xy \cdot xz$, where $s(x) = k$ for all $x \in Q$. If we denote xz by v, then $z = x\backslash v$ and equality $k \cdot yz = xy \cdot xz$ takes the form $k \cdot (y \cdot (x\backslash v)) = xy \cdot v$, $k \cdot (y \cdot (x\backslash v)) = (y * x) \cdot v$.

Then the last equality has the form $A_1(y, A_2(x, v)) = A_3(A_4(y, x), v)$, where A_1, A_2, A_3, A_4 are quasigroup operations, namely, $A_1(y, t) = L_k(y\cdot t)$, $t = A_2(x, v) = x\backslash v$, $A_3(u, v) = u\cdot v$, $u = A_4(y, x) = x \cdot y$.

From Four quasigroups theorem (Theorem 1.92) it follows that quasigroup $(Q,\cdot)$ is an isotope of a group $(Q,+)$.

If in the equality $k \cdot yz = xy \cdot xz$ we fix the variable $x = b$, then we obtain the equality $k \cdot yz = by \cdot bz$, $k \cdot yz = L_b y \cdot L_b z$. From Theorem 1.97 it follows that $(Q,\cdot)$ is a right linear quasigroup.

If in $k \cdot yz = xy \cdot xz$ we put $x = z$, then we obtain $k \cdot yx = xy \cdot k$. From Lemma 1.99 it follows that $(Q,+)$ is a commutative group.

From Lemma 1.95 we have that there exists a group $(Q,+)$ such that $x \cdot y = \alpha x + \psi y + c$, where α is a permutation of the set Q, $\alpha 0 = 0$, $\psi \in \mathrm{Aut}(Q,+)$.

Further we have $s(0) = k = 0 \cdot 0 = c$, $k = c$. Then $s(x) = k = x \cdot x = \alpha x + \psi x + k$. Therefore $\alpha x + \psi x = 0$ for all $x \in Q$. Then $\alpha = I\psi$, where $x + I(x) = 0$ for all $x \in Q$. Therefore α is an antiautomorphism of the group $(Q,+)$, $x \cdot y = I\psi x + \psi y + k$.

Finally $L_k^{-1}(L_k x \cdot L_k y) = L_k^{-1}(I\psi x + I\psi k + \psi k + \psi y + k) = L_k^{-1}(I\psi x + \psi y + k) = -k + I\psi x + k - k + \psi y + k = I_k I\psi x + I_k\psi y = II_k\psi x + I_k\psi y = x \circ y$, where $I_k x = -k + x + k$ is an inner automorphism of $(Q,+)$. It is easy to see that $s^{\circ}(x) = 0$ for all $x \in Q$.

Below we will suppose that any left semimedial quasigroup $(Q,\cdot)$ with zero endomorphism s is an unipotent quasigroup with the form $x \cdot y = -\varphi x + \varphi y$, where $(Q,+)$ is a group, $\varphi \in \mathrm{Aut}(Q,+)$.

(2) We can rewrite equality $zy\cdot s(x) = zx\cdot yx$ in the form $zy\cdot k = zx\cdot yx$, where $s(x) = k$. If we denote zx by v, then $z = v/x$ and the equality $zy \cdot k = zx \cdot yx$ takes the form $((v/x)y)k = v \cdot yx = v \cdot (x * y)$.

We rewrite the last equality in the form $A_1(A_2(v, x), y) = A_3(v, A_4(x, y))$, where A_1, A_2, A_3, A_4 are quasigroup operations, namely, $A_1(t, y) = R_k(t \cdot y)$, $t = A_2(v, x) = v/x$, $A_3(v, u) = v \cdot u$, $u = A_4(x, y) = x * y$.

From Four quasigroups theorem it follows that quasigroup $(Q,\cdot)$ is an isotope of a group $(Q,+)$.

If in the equality $zy \cdot k = zx \cdot yx$ we fix the variable $x = b$, then we obtain the equality $zy \cdot k = zb \cdot yb$, $zy \cdot k = R_b z \cdot R_b y$. From Theorem 1.97 it follows that $(Q,\cdot)$ is a left linear quasigroup.

If in the equality $zy \cdot k = zx \cdot yx$ we put $x = z$, then we obtain $xy \cdot k = k \cdot yx$. Thus from Lemma 1.99 it follows that $(Q,+)$ is a commutative group.

From Lemma 1.95 we have that there exists a group $(Q,+)$ such that $x \cdot y = \varphi x + \beta y + c$, where β is a permutation of the set Q, $\beta 0 = 0$, $\varphi \in \mathrm{Aut}(Q,+)$.

Further we have $s(0) = k = 0 \cdot 0 = c$, $k = c$. Then $s(x) = k = x \cdot x = \varphi x + \beta x + k$. Therefore $\varphi x + \beta x = 0$ for all $x \in Q$. Then $\beta = I\varphi$, where $Ix + x = 0$ for all $x \in Q$.

Therefore $\beta = I\varphi \in \mathrm{Aut}(Q,+)$, $x \cdot y = \varphi x - \varphi y + k$.

We have $R_k^{-1}(R_k x \cdot R_k y) = R_k^{-1}(\varphi x + \varphi k - \varphi k - \varphi y + k) = \varphi x - \varphi y + k - k = \varphi x - \varphi y = x \circ y$. It is easy to see that $s^{\circ}(x) = 0$ for all $x \in Q$.

Below we will suppose that any right semimedial quasigroup $(Q,\cdot)$ with zero endomorphism s is an unipotent quasigroup with the form $x \cdot y = \varphi x - \varphi y$, where $(Q,+)$ is a group, $\varphi \in \mathrm{Aut}(Q,+)$.

(3) We can rewrite the equality $x \cdot yz = f(x)y \cdot xz$ in the form $x \cdot yz = ky \cdot xz = y \cdot xz$, $x \cdot (z * y) = xz * y$, where $f(x) = k$ for all $x \in Q$.

Then $A_1(x, A_2(z,y)) = A_3(A_4(x,z), y)$, where A_1, A_2, A_3, A_4 are quasigroup operations, namely, $A_1(x,t) = x \cdot t$, $t = A_2(z,y) = z * y$, $A_3(u,y) = u * y$, $u = A_4(x,z) = x \cdot z$. From Four quasigroups theorem it follows that quasigroup $(Q,\cdot)$ is a group isotope.

If in the equality $x \cdot yz = y \cdot xz$ we fix variable z, i.e., if we take $z = a$, then we have $x \cdot R_a y = y \cdot R_a x$. From Corollary 1.94 it follows that the group $(Q,+)$ is commutative.

If in the equality $x \cdot yz = y \cdot xz$ we fix variable x, i.e., if we take $x = a$, then we have $a \cdot yz = y \cdot az$, $a \cdot (yz) = y \cdot L_a z$. The application of Theorem 1.97 to the last equality gives us that $(Q,\cdot)$ is a right linear quasigroup, i.e., $x \cdot y = \alpha x + \psi y + c$.

Then $f(x) \cdot x = k \cdot x = \alpha k + \psi x + c = x$. By $x = 0$ we have $\alpha k + \psi 0 + c = 0$, $\alpha k = -c$. Therefore, $k \cdot x = x = \psi x$ for all $x \in Q$. Then $\psi = \varepsilon$, $x \cdot y = \alpha x + y + c = L_c \alpha x + y$ for all $x, y \in Q$. In other words, $x \cdot y = \alpha x + y$ for all $x, y \in Q$.

Further let $a + \alpha 0 = 0$. Then $L_a^{-1}(L_a \alpha x + L_a y) = -a + a + \alpha x + a + y = a + \alpha x + y = \alpha' x + y = x \circ y$, where $\alpha' = L_a \alpha$ and $\alpha' 0 = 0$.

(4) Case (4) is a “mirror” case of Case (3), but we give the direct proof. We can rewrite equality $zy \cdot x = zx \cdot ye(x)$ in the form $zy \cdot x = zx \cdot yk = zx \cdot y$, $(y * z) \cdot x = y * zx$, where $e(x) = k$.

Then $A_1(A_2(y,z), x) = A_3(y, A_4(z,x))$, where A_1, A_2, A_3, A_4 are quasigroup operations, namely, $A_1(t,x) = t \cdot x$, $t = A_2(y,z) = y * z$, $A_3(y,v) = y * v$, $v = A_4(z,x) = z \cdot x$.

From Four quasigroups theorem it follows that quasigroup $(Q,\cdot)$ is an isotope of a group $(Q,+)$.

If in the equality $zy \cdot x = zx \cdot y$ we fix variable z, i.e., if we take $z = a$, then we have $L_a y \cdot x = L_a x \cdot y$. From Corollary 1.94 it follows that the group $(Q,+)$ is commutative.

If in the equality $zy \cdot x = zx \cdot y$ we fix variable x, i.e., if we take $x = a$, then we have $zy \cdot a = za \cdot y$, $zy \cdot a = R_a z \cdot y$. The application of Theorem 1.97 to the last equality gives us that $(Q,\cdot)$ is a left linear quasigroup, i.e., $x \cdot y = \varphi x + \beta y + c$.

Then $x \cdot e(x) = x \cdot k = \varphi x + \beta k + c = x$. By $x = 0$ we have $\varphi 0 + \beta k + c = 0$, $\beta k = -c$. Therefore $x \cdot k = x = \varphi x$ for all $x \in Q$. Then $\varphi = \varepsilon$, $x \cdot y = x + \beta y + c = x + R_c \beta y$ for all $x, y \in Q$. In other words, $x \cdot y = x + \beta y$ for all $x, y \in Q$.

Further let $a + \beta 0 = 0$. Then $L_a^{-1}(L_a x + L_a \beta y) = -a + a + x + a + \beta y = x + a + \beta y = x + \beta' y = x \circ y$, where $\beta' = L_a \beta$ and $\beta' 0 = 0$. □

In proof of the following lemma we use ideas from [15].

Lemma 2.15. *(1) If the endomorphism s of a left semimedial quasigroup $(Q,\cdot)$ is a permutation of the set Q, then quasigroup $(Q,\circ)$ of the form $x \circ y = s^{-1}(x \cdot y)$ is a left distributive quasigroup and $s \in \mathrm{Aut}(Q,\circ)$.*

(2) If the endomorphism s of a right semimedial quasigroup $(Q,\cdot)$ is a permutation of the set Q, then quasigroup $(Q,\circ)$ of the form $x \circ y = s^{-1}(x \cdot y)$ is a right distributive quasigroup and $s \in \mathrm{Aut}(Q,\circ)$.

(3) If the endomorphism f of a left E-quasigroup $(Q, \cdot)$ is a permutation of the set Q, then quasigroup $(Q, \circ)$ of the form $x \circ y = f(x) \cdot y$ is a left distributive quasigroup and $f \in \mathrm{Aut}(Q, \circ)$.

(4) If the endomorphism e of a right E-quasigroup $(Q, \cdot)$ is a permutation of the set Q, then quasigroup $(Q, \circ)$ of the form $x \circ y = x \cdot e(y)$ is a right distributive quasigroup and $e \in \mathrm{Aut}(Q, \circ)$.

Proof. (1) We prove that $(Q, \circ)$ is left distributive. It is clear that $s^{-1} \in \mathrm{Aut}(Q, \cdot)$. We have

$$x \circ (y \circ z) = s^{-1}\big(x \cdot s^{-1}(y \cdot z)\big)$$
$$(x \circ y) \circ (x \circ z) = s^{-2}\big((x \cdot y) \cdot (x \cdot z)\big) = s^{-2}\big(s(x) \cdot (y \cdot z)\big) = s^{-1}\big(x \cdot s^{-1}(y \cdot z)\big)$$

Prove that $s \in \mathrm{Aut}(Q, \circ)$. We have $s(x \circ y) = x \cdot y$, $s(x) \circ s(y) = s^{-1}(s(x) \cdot s(y)) = x \cdot y$. See also [72].

(2) We prove that $(Q, \circ)$ is right distributive. It is clear that $s^{-1} \in \mathrm{Aut}(Q, \cdot)$. We have

$$(x \circ y) \circ z = s^{-1}\big(s^{-1}(x \cdot y) \cdot z\big)$$
$$(x \circ z) \circ (y \circ z) = s^{-2}\big((x \cdot z) \cdot (y \cdot z)\big) = s^{-2}\big((x \cdot y) \cdot s(z)\big) = s^{-1}\big(s^{-1}(x \cdot y) \cdot z\big)$$

Prove that $s \in \mathrm{Aut}(Q, \circ)$. We have $s(x \circ y) = x \cdot y$, $s(x) \circ s(y) = s^{-1}(s(x) \cdot s(y)) = x \cdot y$.

(3) If the endomorphism f is a permutation of the set Q, then $f, f^{-1} \in \mathrm{Aut}(Q, \cdot)$. We have

$$x \circ (y \circ z) = f(x) \cdot \big(f(y) \cdot z\big)$$
$$(x \circ y) \circ (x \circ z) = f\big(f(x) \cdot y\big) \cdot \big(f(x) \cdot z\big) = \big(f^2(x) \cdot f(y)\big) \cdot \big(f(x) \cdot z\big) = f(x) \cdot \big(f(y) \cdot z\big)$$

Prove that $f \in \mathrm{Aut}(Q, \circ)$. We have $f(x \circ y) = f(f(x) \cdot y) = f^2(x) \cdot f(y) = f(x) \circ f(y)$.

(4) If the endomorphism e is a permutation of the set Q, then $e, e^{-1} \in \mathrm{Aut}(Q, \cdot)$. We have

$$(x \circ y) \circ z = \big(x \cdot e(y)\big) \cdot e(z)$$
$$(x \circ z) \circ (y \circ z) = \big(x \cdot e(z)\big) \cdot e\big(y \cdot e(z)\big) = \big(x \cdot e(z)\big) \cdot \big(e(y) \cdot e^2(z)\big) = \big(x \cdot e(y)\big) \cdot e(z)$$

Prove that $e \in \mathrm{Aut}(Q, \circ)$. We have $e(x \circ y) = e(x \cdot e(y)) = e(x) \cdot e^2(y) = e(x) \circ e(y)$. □

Remark 2.16. By the proof of Lemma 2.15 it is possible to use Lemma 2.3 and parastrophe invariant arguments.

Theorem 2.17. *(1) Every left SM-quasigroup $(Q, \cdot)$ has the following structure:*

$$(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$$

where $(A, \circ)$ is a quasigroup with a unique idempotent element and there exists a number m such that $|s^m(A, \circ)| = 1$; $(B, \cdot)$ is an isotope of a left distributive quasigroup $(B, \star)$, $x \cdot y = s(x \star y)$ for all $x, y \in B$, $s \in \mathrm{Aut}(B, \cdot)$, $s \in \mathrm{Aut}(B, \star)$.

(2) Every right SM-quasigroup $(Q, \cdot)$ has the following structure:

$$(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$$

where $(A, \circ)$ is a quasigroup with a unique idempotent element and there exists an ordinal number m such that $|s^m(A, \circ)| = 1$; $(B, \cdot)$ is an isotope of a right distributive quasigroup $(B, \star)$, $x \cdot y = s(x \star y)$ for all $x, y \in B$, $s \in \mathrm{Aut}(B, \cdot)$, $s \in \mathrm{Aut}(B, \star)$.

(3) Every left E-quasigroup $(Q, \cdot)$ has the following structure:

$$(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$$

where $(A, \circ)$ is a quasigroup with a unique idempotent element and there exists a number m such that $|f^m(A, \circ)| = 1$; $(B, \cdot)$ is an isotope of a left distributive quasigroup $(B, \star)$, $x \cdot y = f^{-1}(x) \star y$ for all $x, y \in B$, $f \in \mathrm{Aut}(B, \cdot)$, $f \in \mathrm{Aut}(B, \star)$.

(4) Every right E-quasigroup $(Q, \cdot)$ has the following structure:

$$(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$$

where $(A, \circ)$ is a quasigroup with a unique idempotent element and there exists a number m such that $|e^m(A, \circ)| = 1$; $(B, \cdot)$ is an isotope of a right distributive quasigroup $(B, \star)$, $x \cdot y = x \star e^{-1}(y)$ for all $x, y \in B$, $e \in \mathrm{Aut}(B, \cdot)$, $e \in \mathrm{Aut}(B, \star)$.

Proof. The proof is similar to the proof of Theorem 2.7. It is possible also to use parastrophe invariance ideas. $\square$

Corollary 2.18. *If $(Q, \cdot)$ is a left SM-quasigroup, then $s^m(Q, \cdot) \trianglelefteq (Q, \cdot)$; if $(Q, \cdot)$ is a right SM-quasigroup, then $s^m(Q, \cdot) \trianglelefteq (Q, \cdot)$; if $(Q, \cdot)$ is a left E-quasigroup, then $f^m(Q, \cdot) \trianglelefteq (Q, \cdot)$; if $(Q, \cdot)$ is a right E-quasigroup, then $e^m(Q, \cdot) \trianglelefteq (Q, \cdot)$.*

Corollary 2.19. *If $(Q, \cdot)$ is a left SM-quasigroup with an idempotent element, then equivalence class $\bar{a}$ of the normal congruence $\mathrm{Ker}\, s_j$ containing an idempotent element $a \in Q$ forms an unipotent quasigroup $(\bar{a}, \cdot)$ isotopic to a group with isotopy of the form $(-\psi, \psi, \varepsilon)$, where $\psi \in \mathrm{Aut}(\bar{a}, \cdot)$ for all suitable values of j.*

If $(Q, \cdot)$ is a right SM-quasigroup with an idempotent element, then equivalence class $\bar{a}$ of the normal congruence $\mathrm{Ker}\, s_j$ containing an idempotent element $a \in Q$ forms an unipotent quasigroup $(\bar{a}, \cdot)$ isotopic to a group with isotopy of the form $(\varphi, -\varphi, \varepsilon)$, where $\varphi \in \mathrm{Aut}(\bar{a}, \cdot)$ for all suitable values of j.

If $(Q, \cdot)$ is a left E-quasigroup with an idempotent element, then equivalence class $\bar{a}$ of the normal congruence $\mathrm{Ker}\, f_j$ containing an idempotent element $a \in Q$ forms a left loop isotopic to an Abelian group with isotopy of the form $(\alpha, \varepsilon, \varepsilon)$ for all suitable values of j.

If $(Q, \cdot)$ is a right E-quasigroup with an idempotent element, then equivalence class $\bar{a}$ of the normal congruence $\mathrm{Ker}\, e_j$ containing an idempotent element $a \in Q$ forms a right loop isotopic to an Abelian group with isotopy of the form $(\varepsilon, \beta, \varepsilon)$ for all suitable values of j.

Proof. Mainly the proof repeats the proof of Corollary 2.10. It is possible to use Theorem 2.14. $\square$

2.3 CML as an SM-quasigroup

In this subsection we give information (mainly well known) about commutative Moufang loops (CML) which is possible to obtain from the fact that a loop $(Q, \cdot)$ is left semimedial if and only if it is a commutative Moufang loop. Novelty of information from this subsection is in the fact that some well-known theorems about CML are obtained quit easy using quasigroup approach.

Lemma 2.20. *(1) A left F-loop is a group. (2) A right F-loop is a group. (3) A loop $(Q, \cdot)$ is left semimedial if and only if it is a commutative Moufang loop. (4) A loop $(Q, \cdot)$ is right semimedial if and only if it is a commutative Moufang loop. (5) A left E-loop $(Q, \cdot)$ is a commutative group. (6) A right E-loop $(Q, \cdot)$ is a commutative group.*

Proof. (1) From $x \cdot yz = xy \cdot e(x)z$ we have $x \cdot yz = x \cdot yz$.

(2) From $xy \cdot z = xf(z) \cdot yz$ we have $xy \cdot z = x \cdot yz$.

(3) We use the proof from [12, p. 99]. Let $(Q, \cdot)$ be a left semimedial loop. If $y = 1$, then we have $x^2 \cdot z = x \cdot xz$. If $z = 1$, then $x^2 y = xy \cdot x$. Then $x \cdot xy = xy \cdot x$. If we denote xy by y, then we obtain that $xy = yx$, i.e., the loop $(Q, \cdot)$ is commutative.

It is clear that a commutative Moufang loop is left semimedial.

(4) For the proof of Case (3) it is possible to use “mirror” principles. We give the direct proof. Let $(Q, \cdot)$ be a right semimedial loop, i.e., $zy \cdot x^2 = zx \cdot yx$ for all $x, y, z \in Q$. If $y = 1$, the we have $z \cdot x^2 = zx \cdot x$.

If $z = 1$, then $yx^2 = x \cdot yx$. Then $zx \cdot x = x \cdot zx$. If we denote zx by z, then we obtain that $zx = xz$, i.e., the loop $(Q, \cdot)$ is commutative. Moreover, we have $x^2 \cdot yz = zy \cdot x^2$, $xy \cdot zx = xz \cdot yx$.

It is clear that a commutative Moufang loop is right semimedial.

(5) From $x \cdot yz = f(x)y \cdot xz$ we have $x \cdot yz = y \cdot xz$. From the last identity by $z = 1$ we obtain $x \cdot y = y \cdot x$. Therefore we can rewrite identity $x \cdot yz = y \cdot xz$ in the form $yz \cdot x = y \cdot zx$.

Case (6) is proved in the similar way to Case (5). □

Commutative Moufang loop in which any element has the order 3 is called 3-CML.

Remark 2.21. Center $C(Q, +)$ of a CML $(Q, +)$ is a normal Abelian subgroup of $(Q, +)$ and it coincides with the left nucleus of $(Q, +)$ [12, 24].

Lemma 2.22. *In a commutative Moufang loop the map $\delta : x \mapsto 3x$ is central endomorphism [12, 24].*

Proof. In a CML $(Q, +)$ we have $n(x + y) = nx + ny$ for any natural number n since by Moufang theorem [12, 24, 76] CML is diassociative (any two elements generate an associative subgroup). Therefore the map δ is an endomorphism. See [65] for many details on commutative diassociative loops.

The proof of centrality of the endomorphism δ is standard [83, 12, 24, 56] and we omit it. □

A quasigroup $(Q, \cdot)$ with identities $xy = yx$, $x \cdot xy = y$, $x \cdot yz = xy \cdot xz$ is called a distributive Steiner quasigroup [12, 15].

Theorem 2.23. *(1) Every commutative Moufang loop $(Q, +)$ has the following structure:*

$$(Q, +) \cong (A, \oplus) \times (B, +)$$

where $(A, \oplus)$ is an Abelian group and there exists a number m such that $|s^m(A, \oplus)| = 1$; $(B, +)$ is an isotope of a distributive quasigroup $(B, \star)$, $x + y = s(x \star y)$ for all $x, y \in B$, $s \in \mathrm{Aut}(B, +)$, $s \in \mathrm{Aut}(B, \star)$.

(2) $C(Q, +) \cong (A, \oplus) \times C(B, +)$.

(3) $(Q, +)/C(Q, +) \cong (B, +)/C(B, +) \cong (D, +)$ is 3-CML in which the endomorphism s is permutation I such that $Ix = -x$.

(4) Quasigroup $(D, \star)$, $x \star y = -x - y$, $x, y \in (D, +)$, is a distributive Steiner quasigroup.

Proof. (1) The existence of decomposition of $(Q, +)$ into two factors follows from Theorem 2.17.

From Corollary 2.19 it follows that any equivalence class $\bar{a} \equiv H_j$ of the normal congruence $\operatorname{Ker} s_j$ containing an idempotent element $0 \in Q$ is an unipotent loop $(H_j, \cdot)$ isotopic to an Abelian group with isotopy of the form $(\varphi, -\varphi, \varepsilon)$, where $\varphi \in \mathrm{Aut}(H_j, \cdot)$ for all suitable values of j.

Since $(H_j, \cdot)$ is a commutative loop, we have that $x \cdot 0 = \varphi x - \varphi 0 = \varphi x = x$, $\varphi = \varepsilon$, $0 \cdot x = \varphi 0 - \varphi x = -\varphi x = x$, $-\varphi = \varepsilon$. Thus $x \cdot y = x + y$ for all $x, y \in (H_j, \cdot)$.

We notice in a commutative Moufang loop $(Q, +)$ the map s^i takes the form 2^i, i.e., $s^i(x) = 2^i(x)$. Then in the loop $(A, \oplus)$ any nonzero element has the order 2^i or infinite order.

If an element x of the loop $(A, \oplus)$ has a finite order, then $x \in C(A, \oplus)$, where $C(A, \oplus)$ is a center of $(A, \oplus)$ since $G.C.D.(2^i, 3) = 1$.

If an element x of the loop $(A, \oplus)$ has infinite order, then by Lemma 2.22 $3x \in C(A, \oplus)$, $\langle x \rangle \cong \langle 3x \rangle$.

Therefore $(A, \oplus) \cong 3(A, \oplus) \subseteq C(A, \oplus)$, $(A, \oplus)$ is an Abelian group.

From Theorem 2.17(1), (2) it follows that $(B, +)$ is an isotope of left and right distributive quasigroup. Therefore, $(B, +)$ is an isotope of distributive quasigroup.

(2) From Lemma 1.74 it follows that $C(Q, +) \cong C(A, \oplus) \times C(B, +)$. Therefore $C(Q, +) \cong (A, \oplus) \times C(B, +)$ since $C(A, \oplus) = (A, \oplus)$.

(3) The fact that $(Q, +)/C(Q, +)$ is 3-CML is well known and it follows from Lemma 2.22. Isomorphism $(Q, +)/C(Q, +) \cong ((A, \oplus) \times (B, +))/((A, \oplus) \times C(B, +))$ follows from Cases (1) and (2).

Isomorphism

$$\big((A, \oplus) \times (B, +)\big)/\big((A, \oplus) \times C(B, +)\big) \cong (B, +)/C(B, +)$$

follows from the Second Isomorphism Theorem (see [27, p. 51], for group case see [47]) and the fact that $(A, \oplus) \cap C(B, +) = \{(0, 0)\}$, i.e., $|(A, \oplus) \cap C(B, +)| = 1$.

(4) It is clear that in 3-CML $(D, +)$ the map s takes the form $s(x) = 2x = -x = Ix$. Moreover, $I^{-1} = I$. It is easy to see that the quasigroup $(D, \star)$ is a distributive Steiner quasigroup. $\square$

Corollary 2.24. *If in CML $(Q, +)$ the endomorphism s has finite order m, then (i) any nonzero element of the group $(A, \oplus)$ has the order 2^i, $1 \leqslant i \leqslant m$; (ii)* $\mathrm{Aut}(Q, +) \cong \mathrm{Aut}(A, \oplus) \times \mathrm{Aut}(B, +)$.

Proof. (i). It is easy to see.

(ii). Let $(Q, +)$ be a commutative Moufang loop, $\alpha \in \mathrm{Aut}(Q, +)$. Then the order of an element x coincides with the order of element $\alpha(x)$. Indeed, if $nx = 0$, then $n(\alpha x) = \alpha(nx) = 0$.

The loops $(A, \oplus)$ and $(B, +)$ have elements of different orders. Indeed, the orders of elements of the loop $(A, \oplus)$ are powers of the number 2 and orders of the elements of the loop $(B, +)$ are some odd numbers or, possibly, ∞.

Therefore loops $(A, \oplus)$ and $(B, +)$ are invariant relative to any automorphism of the loop $(Q, +)$. Then $\mathrm{Aut}(Q, +) \cong \mathrm{Aut}(A, \oplus) \times \mathrm{Aut}(B, +)$. $\square$

3 The structure

Theorems 2.7 and 2.17 give us a possibility to receive some information on left and right F-, SM-, E-quasigroups and some combinations of these classes.

3.1 Simple left and right F-, E-, and SM-quasigroups

Simple quasigroups of some classes of finite left distributive quasigroups are described in [39]. The structure and properties of F-quasigroups are described in [55, 56].

We give Ježek-Kepka Theorem [46] in the following form [101, 103].

Theorem 3.1. *If a medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \alpha x + \beta y + a$ over an Abelian group $(Q, +)$ is simple, then*

(1) the group $(Q, +)$ is the additive group of a finite Galois field $\mathrm{GF}(p^k)$*;*

(2) the group $\langle \alpha, \beta \rangle$ is the multiplicative group of the field $\mathrm{GF}(p^k)$ *in the case $k > 1$, the group $\langle \alpha, \beta \rangle$ is any subgroup of the group* $\mathrm{Aut}(Z_p, +)$ *in the case $k = 1$;*

(3) the quasigroup $(Q, \cdot)$ in the case $|Q| > 1$ can be quasigroup from one of the following disjoint quasigroup classes:

(a) $\alpha + \beta = \varepsilon, a = 0$; in this case the quasigroup $(Q, \cdot)$ is an idempotent quasigroup;

(b) $\alpha + \beta = \varepsilon$ and $a \neq 0$; in this case the quasigroup $(Q, \cdot)$ does not have any idempotent element, the quasigroup $(Q, \cdot)$ is isomorphic to the quasigroup $(Q, *)$ with the form $x * y = \alpha x + \beta y + 1$ over the same Abelian group $(Q, +)$;

(c) $\alpha + \beta \neq \varepsilon$; in this case the quasigroup $(Q, \cdot)$ has exactly one idempotent element, the quasigroup $(Q, \cdot)$ is isomorphic to the quasigroup $(Q, \circ)$ of the form $x \circ y = \alpha x + \beta y$ over the group $(Q, +)$.

Theorem 3.2. *If a simple distributive quasigroup $(Q, \circ)$ is isotopic to finitely generated commutative Moufang loop $(Q, +)$, then $(Q, \circ)$ is a finite medial distributive quasigroup [94, 95, 96].*

Proof. It is known [24] that any finitely generated CML $(Q, +)$ has a nonidentity center $C(Q, +)$ (for short C).

We check that center of CML $(Q, +)$ is invariant (is a characteristic subloop) relative to any automorphism of the loop $(Q, +)$ and the quasigroup $(Q, \circ)$.

Indeed, if $\varphi \in \mathrm{Aut}(Q, +)$, $a \in C(Q, +)$ (see Remark 2.21), then we have $\varphi(a + (x + y)) = \varphi((a+x)+y) = \varphi a+(\varphi x+\varphi y) = (\varphi a+\varphi x)+\varphi y$. Thus $\varphi a \in C(Q, +)$, $\varphi C(Q, +) \subseteq C(Q, +)$.

For any distributive quasigroup $(Q, \circ)$ of the form $x \circ y = \varphi x + \psi y$ we have

$$\mathrm{Aut}(Q, \circ) \cong M(Q, +) \leftthreetimes (\mathbb{C}/I),$$

where I is the group of inner permutations of commutative Moufang loop $(Q, +)$,

$$\mathbb{C} = \{\omega \in \mathrm{Aut}(Q, +) \mid \omega\varphi = \varphi\omega\}$$

Therefore any automorphism of $(Q, \circ)$ has the form $L^+_a \alpha$, where $\alpha \in \mathrm{Aut}(Q, +)$ [97].

The center C defines normal congruence θ of the loop $(Q, +)$ in the following way: $x\theta y \Leftrightarrow x + C = y + C$. We give a little part of this standard proof: $(x + a) + C = (y + a) + C \Leftrightarrow (x + C) + a = (y + C) + a \Leftrightarrow x + C = y + C$. In fact C is coset class of θ containing zero element of $(Q, +)$.

The congruence θ is admissible relative to any permutation of the form $L^+_a \alpha$, where $\alpha \in \mathrm{Aut}(Q, +)$, since θ is central congruence. Therefore, θ is congruence in the quasigroup $(Q, \circ)$.

Since $(Q, \circ)$ is simple quasigroup and θ cannot be diagonal congruence, then $\theta = Q \times Q$, $C(Q, +) = (Q, +)$, $(Q, \circ)$ is medial. From Theorem 3.1 it follows that $(Q, \circ)$ is finite. □

We notice it is possible to prove Theorem 3.2 using Theorem 1.63 [96].

Lemma 3.3. *If simple quasigroup $(Q,\cdot)$ is isotope either of the form $(f,\varepsilon,\varepsilon)$, or of the form $(\varepsilon,e,\varepsilon)$, or of the form $(\varepsilon,\varepsilon,s)$ of a distributive quasigroup $(Q,\circ)$, where $f,e,s\in \mathrm{Aut}(Q,\circ)$ and $(Q,\circ)$ is isotopic to finitely generated commutative Moufang loop $(Q,+)$, then $(Q,\cdot)$ is finite medial quasigroup.*

Proof. Since $(Q,+)$ is finitely generated, then $|\,C(Q,+)\,|>1$ [24]. From the proof of Theorem 3.2 it follows that $C(Q,+)$ is invariant relative to any automorphism of $(Q,\circ)$.

Therefore necessary condition of simplicity of $(Q,\cdot)$ is the fact that $C(Q,+)=(Q,+)$. Then $(Q,\circ)$ is medial.

Prove that $(Q,\cdot)$ is medial, if $x\cdot y=fx\circ y$. We have $xy\cdot uv=f(fx\circ y)\circ(fu\circ v)=(f^2x\circ fu)\circ(fy\circ v)=(xu)\cdot(yv)$ [78].

Prove that $(Q,\cdot)$ is medial, if $x\cdot y=x\circ ey$. We have $xy\cdot uv=(x\circ ey)\circ(eu\circ e^2v)=(x\circ eu)\circ(ey\circ e^2v)=(xu)\cdot(yv)$ [78].

Prove that $(Q,\cdot)$ is medial, if $x\cdot y=s(x\circ y)$. We have $xy\cdot uv=(s^2x\circ s^2y)\circ(s^2u\circ s^2v)=(s^2x\circ s^2u)\circ(s^2y\circ s^2v)=(xu)\cdot(yv)$ [78]. $\square$

We can obtain some information on simple left and right F-, E-, and SM-quasigroups.

Theorem 3.4. *(1) Left F-quasigroup $(Q,\cdot)$ is simple if and only if it lies in one from the following quasigroup classes:*

(i) $(Q,\cdot)$ is a right loop of the form $x\cdot y=x+\psi y$, where $\psi\in \mathrm{Aut}(Q,+)$ and the group $(Q,+)$ is ψ-simple;

(ii) $(Q,\cdot)$ has the form $x\cdot y=x\circ\psi y$, where $\psi\in\mathrm{Aut}(Q,\circ)$ and $(Q,\circ)$ is ψ-simple left distributive quasigroup.

(2) Right F-quasigroup $(Q,\cdot)$ is simple if and only if it lies in one from the following quasigroup classes:

(i) $(Q,\cdot)$ is a left loop of the form $x\cdot y=\varphi x+y$, where $\varphi\in\mathrm{Aut}(Q,+)$ and the group $(Q,+)$ is φ-simple;

(ii) $(Q,\cdot)$ has the form $x\cdot y=\varphi x\circ y$, where $\varphi\in\mathrm{Aut}(Q,\circ)$ and $(Q,\circ)$ is φ-simple left distributive quasigroup.

(3) Left SM-quasigroup $(Q,\cdot)$ is simple if and only if it lies in one from the following quasigroup classes:

(i) $(Q,\cdot)$ is a unipotent quasigroup of the form $x\circ y=-\varphi x+\varphi y$, $(Q,+)$ is a group, $\varphi\in\mathrm{Aut}(Q,+)$ and the group $(Q,+)$ is φ-simple;

(ii) $(Q,\cdot)$ has the form $x\cdot y=\varphi(x\circ y)$, where $\varphi\in\mathrm{Aut}(Q,\circ)$ and $(Q,\circ)$ is φ-simple left distributive quasigroup.

(4) Right SM-quasigroup $(Q,\cdot)$ is simple if and only if it lies in one from the following quasigroup classes:

(i) $(Q,\cdot)$ is a unipotent quasigroup of the form $x\circ y=\varphi x-\varphi y$, $(Q,+)$ is a group, $\varphi\in\mathrm{Aut}(Q,+)$ and the group $(Q,+)$ is φ-simple;

(ii) $(Q,\cdot)$ has the form $x\cdot y=\varphi(x\circ y)$, where $\varphi\in\mathrm{Aut}(Q,\circ)$ and $(Q,\circ)$ is φ-simple right distributive quasigroup.

(5) Left E-quasigroup $(Q,\cdot)$ is simple if and only if it lies in one from the following quasigroup classes:

(i) $(Q,\cdot)$ is a left loop of the form $x \cdot y = \alpha x + y$, $\alpha 0 = 0$, and $(Q,+)$ is α-simple Abelian group;

(ii) $(Q,\cdot)$ has the form $x \cdot y = \varphi x \circ y$, where $\varphi \in \mathrm{Aut}(Q,\circ)$ and $(Q,\circ)$ is φ-simple left distributive quasigroup.

(6) Right E-quasigroup $(Q,\cdot)$ is simple if and only if it lies in one from the following quasigroup classes:

(i) $(Q,\cdot)$ is a right loop of the form $x \cdot y = x + \beta y$, $\beta 0 = 0$, and $(Q,+)$ is β-simple Abelian group;

(ii) $(Q,\cdot)$ has the form $x \cdot y = x \circ \psi y$, where $\psi \in \mathrm{Aut}(Q,\circ)$ and $(Q,\circ)$ is ψ-simple right distributive quasigroup.

Proof. (1) Suppose that $(Q,\cdot)$ is simple left F-quasigroup. From Theorem 2.7 it follows that $(Q,\cdot)$ can be a quasigroup with a unique idempotent element or an isotope of a left distributive quasigroup.

By Theorem 1.51 the endomorphism e defines the corresponding normal congruence $\mathrm{Ker}\, e$. Since $(Q,\cdot)$ is simple, then this congruence is the diagonal $\hat{Q} = \{(q,q)\,|q \in Q\}$ or the universal congruence $Q \times Q$.

From Theorem 2.7 it follows that in simple left F-quasigroup the map e is zero endomorphism or a permutation.

Structure of left F-quasigroups in the case when e is zero endomorphism follows from Lemma 2.2.

Structure of left F-quasigroups in the case when e is an automorphism follows from Lemma 2.3. Additional properties of quasigroup $(Q,\circ)$ follow from Lemma 1.62.

Conversely, using Corollary 1.70 we can say that left F-quasigroups from these quasigroup classes are simple.

Cases (2)–(6) are proved in a similar way. □

Remark 3.5. Left F-quasigroup $(Z,\cdot)$, where $x \cdot y = -x + y$, $(Z,+)$ is the infinite cyclic group, (Example 2.6) is not simple. Indeed, in this quasigroup the endomorphism e is not a permutation (a bijection) of the set Z or a zero endomorphism.

We can also apply Theorem 3.4(3), since $(Z,\cdot)$ is a left SM-quasigroup, and so on.

3.2 F-quasigroups

Simple F-quasigroups isotopic to groups (FG-quasigroups) are described in [56]. The authors prove that any simple FG-quasigroup is a simple group or a simple medial quasigroups. We notice that simple medial quasigroups are described in [46]. See also [101, 103]. Conditions when a group isotope is a left (right) F-quasigroup are there in [66, 113].

The following examples demonstrate that in an F-, E-, SM-quasigroup the order of map e does not coincide with the order of map f, i.e., there exists some independence of the orders of maps e, f, and s.

Example 3.6. By $(Z_3,+)$ we denote the cyclic group of order 3 and we take $Z_3 = \{0, 1, 2\}$. Groupoid $(Z_3,\cdot)$, where $x \cdot y = x - y$, is a medial E-, F-, SM-quasigroup and $e^{\cdot}(Z_3) = s^{\cdot}(Z_3) = \{0\}$, $f^{\cdot}(Z_3) = Z_3$.

Example 3.7. By $(Z_6,+)$ we denote the cyclic group of order 6 and we take $Z_6 = \{0,1,2,3,4,5\}$. Groupoid $(Z_6,\cdot)$, where $x\cdot y = x - y$, is a medial E-, F-, SM-quasigroup and $e^{\cdot}(Z_6) = s^{\cdot}(Z_6) = \{0\}$, $f^{\cdot}(Z_6) = \{0,\ 2,\ 4\}$.

The following lemmas give connections between the maps e and f in F-quasigroups.

Lemma 3.8. *(1) Endomorphism e of an F-quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $e(x) = 0$ for all $x \in Q$ if and only if $x\cdot y = x + \psi y$, $(Q,+)$ is a group, $\psi \in \mathrm{Aut}(Q,+)$, $(Q,\cdot)$ contains unique idempotent element 0, $x + fy = fy + x$ for all $x, y \in Q$.*

(2) Endomorphism f of an F-quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $f(x) = 0$ for all $x \in Q$ if and only if $x\cdot y = \varphi x + y$, $(Q,+)$ is a group, $\varphi \in \mathrm{Aut}(Q,+)$, $(Q,\cdot)$ contains unique idempotent element 0, $x + ey = ey + x$ for all $x, y \in Q$.

Proof. (1) From Lemma 2.2(1) it follows that $(Q,\cdot)$ is a right loop, isotope of a group $(Q,+)$ of the form $x\cdot y = x + \psi y$, where $\psi \in \mathrm{Aut}(Q,+)$.

If $a\cdot a = a$, then $a + \psi a = a$, $\psi a = 0$, $a = 0$.

If we rewrite right F-quasigroup equality in terms of the operation $+$, then we obtain $x + \psi y + \psi z = x + \psi f(z) + \psi y + \psi^2 z$, $\psi y + \psi z = \psi f(z) + \psi y + \psi^2 z$. If we take $y = 0$ in the last equality, then $\psi z = \psi f(z) + \psi^2 z$. Therefore $\psi y + \psi f(z) + \psi^2 z = \psi f(z) + \psi y + \psi^2 z$, $\psi y + \psi f(z) = \psi f(z) + \psi y$, $y + f(z) = f(z) + y$.

Conversely, from $x\cdot y = x + \psi y$ we have $x\cdot e(x) = x + \psi e(x) = x$, $e(x) = 0$ for all $x \in Q$.

(2) This case is proved in a similar way to Case (1). $\square$

Lemma 3.9. *(1) If endomorphism e of an F-quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $e(x) = 0$ for all $x \in Q$, then*

(i) $f(x) = x - \psi x$, $f \in End(Q,+)$;

(ii) $f(Q,+) \subseteq C(Q,+)$;

(iii) $(H,+) \trianglelefteq (Q,+)$, $f(Q,+) \trianglelefteq (Q,+)$, $(Q,+)/(H,+) \cong f(Q,+)$, *where $(H,+)$ is equivalence class of the congruence* $\mathrm{Ker}\, f$ *containing identity element of $(Q,+)$;*

(iv) $f(Q,\cdot)$ *is a medial F-quasigroup; $(H,\cdot) = (H,+)$ is a group; $(\bar{a},\cdot)$, where $\bar{a}$ is equivalence class of the normal congruence* $\mathrm{Ker}\, f_j$ *containing an idempotent element $a \in Q$, $i \geqslant 1$, is an Abelian group.*

(2) If endomorphism f of an F-quasigroup $(Q,\cdot)$ is zero endomorphism, i.e., $f(x) = 0$ for all $x \in Q$, then

(i) $e(x) = -\varphi x + x$, $e \in End(Q,+)$;

(ii) $e(Q,+) \subseteq C(Q,+)$;

(iii) $(H,+) \trianglelefteq (Q,+)$, $e(Q,+) \trianglelefteq (Q,+)$, $(Q,+)/(H,+) \cong e(Q,+)$, *where $(H,+)$ is equivalence class of the congruence* $\mathrm{Ker}\, e$ *containing identity element of $(Q,+)$;*

(iv) $e(Q,\cdot)$ *is a medial F-quasigroup; $(H,\cdot) = (H,+)$ is a group; $(\bar{a},\cdot)$, where $\bar{a}$ is equivalence class of the normal congruence* $\mathrm{Ker}\, e_j$ *containing an idempotent element $a \in Q$, $i \geqslant 1$, is an Abelian group.*

Proof. (1) (i) From Lemma 3.8(1) we have $f(x)\cdot x = f(x) + \psi x = x$, $f(x) = x - \psi x$. We can rewrite equality $f(x\cdot y) = f(x)\cdot f(y)$ in the form $f(x + \psi y) = f(x) + \psi f(y)$. If $x = y = 0$, then we have $f(0) = 0$. If $x = 0$, then $f\psi(y) = \psi f(y)$. Therefore

$$f(x + \psi y) = f(x) + f\psi(y) \tag{3.1}$$

(ii) If we apply to equality (3.1) the equality $f(z) = z - \psi z$, then we obtain $x + \psi y - \psi(x + \psi y) = x - \psi x + \psi y - \psi^2 y$, $x + \psi y - \psi^2 y - \psi x = x - \psi x + \psi y - \psi^2 y$, $\psi y - \psi^2 y - \psi x = -\psi x + \psi y - \psi^2 y$, $y - \psi y - x = -x + y - \psi y$, $fy - x = -x + fy$, $x + fy = fy + x$, i.e., $f(Q, +) \subseteq C(Q, +)$.

(iii) From definitions and Case (ii) it follows that $(H, +) \trianglelefteq (Q, +)$, $f(Q, +) \trianglelefteq (Q, +)$. The last follows from definition of $(H, +)$.

(iv) $f(Q, \cdot)$ is a medial F-quasigroup since from Case (ii) it follows that $f(Q, +)$ is an Abelian group. Quasigroup $(H, \cdot)$ is a group since in this quasigroup the maps e and f are zero endomorphisms and we can use Case (i).

$(\bar{a}, \cdot) \cong f^i(Q, \cdot)/f^{i+1}(Q, \cdot)$ is an Abelian group since in this quasigroup the maps e and f are zero endomorphisms and $f^i(Q, \cdot)$ is a medial quasigroup for any suitable value of the index i. Moreover, it is well known that a medial quasigroup any its subquasigroup is normal [60]. Then $f^{i+1}(Q, \cdot) \trianglelefteq f^i(Q, \cdot)$.

(2) This case is proved in a similar way to Case (1). □

Corollary 3.10. *Both endomorphisms e and f of an F-quasigroup $(Q, \cdot)$ are zero endomorphisms if and only if $(Q, \cdot)$ is a group.*

Proof. By Lemma 3.8(1) $x \cdot y = x + \psi y$. By Lemma 3.9(1)(i), $f(x) = x - \psi x$. Since $f(x) = 0$ for all $x \in Q$, further we have $\psi = \varepsilon$.

Conversely, it is clear that in any group $e(x) = f(x) = 0$ for all $x \in Q$. □

Example 3.11. By $(Z_4, +)$ we denote the cyclic group of order 4 and we take $Z_4 = \{0, 1, 2, 3\}$. Groupoid $(Z_4, \cdot)$, where $x \cdot y = x + 3y$, is a medial E-, F-, SM-quasigroup, $e^{\cdot}(Z_4) = s^{\cdot}(Z_4) = \{0\}$ and $f^{\cdot}(Z_4) = \{0, 2\} = H$.

Corollary 3.12. *(1) If in F-quasigroup $(Q, \cdot)$ endomorphism e is zero endomorphism and the group $(Q, +)$ has identity center, then $(Q, \cdot) = (Q, +)$.*

(2) If in F-quasigroup $(Q, \cdot)$ endomorphism f is zero endomorphism and the group $(Q, +)$ has identity center, then $(Q, \cdot) = (Q, +)$.

Proof. The proof follows from Lemma 3.9(ii), (iii). □

Corollary 3.13. *(1) If endomorphism e of an F-quasigroup $(Q, \cdot)$ is zero endomorphism, i.e., $e(x) = 0$ for all $x \in Q$, endomorphism f is a permutation of the set Q, then $x \cdot y = x + \psi y$, $(Q, +)$ is an Abelian group, $\psi \in \mathrm{Aut}(Q, +)$ and $(Q, \circ)$, $x \circ y = fx + \psi y$, is a medial distributive quasigroup.*

(2) If endomorphism f of an F-quasigroup $(Q, \cdot)$ is zero endomorphism, i.e., $f(x) = 0$ for all $x \in Q$, endomorphism e is a permutation of the set Q, then $x \cdot y = \varphi x + y$, $(Q, +)$ is an Abelian group, $\varphi \in \mathrm{Aut}(Q, +)$ and $(Q, \circ)$, $x \circ y = \varphi x + ey$, is a medial distributive quasigroup.

Proof. The proof follows from Lemma 3.9. It is a quasigroup folklore that idempotent medial quasigroup is distributive [91, 92]. □

Remark 3.14. It is easy to see that condition "$(D, \cdot)$ is a medial F-quasigroup of the form $x \cdot y = x + \psi y$ such that $(D, \circ)$, $x \circ y = fx + \psi y$, is a medial distributive quasigroup" in Corollary 3.13 is equivalent to the condition that the automorphism ψ of the group $(D, +)$ is complete (Definition 1.32).

Lemma 3.15. *(1) If endomorphism e of an F-quasigroup $(Q,\cdot)$ is a permutation of the set Q, i.e., e is an automorphism of $(Q,\cdot)$, then $(Q,\circ)$, $x\circ y = x\cdot e(y)$, is a left distributive quasigroup which satisfies the equality $(x\circ y)\circ z = (x\circ fz)\circ(y\circ e^{-1}z)$, for all $x,y,z\in Q$.*

(2) If endomorphism f of an F-quasigroup $(Q,\cdot)$ is a permutation of the set Q, i.e., f is an automorphism of $(Q,\cdot)$, then $(Q,\circ)$, $x\circ y = f(x)\cdot y$, is a right distributive quasigroup which satisfies the equality $x\circ(y\circ z) = (f^{-1}x\circ y)\circ(ex\circ z)$, for all $x,y,z\in Q$.

Proof. (1) The fact that $(Q,\circ)$, $x\circ y = x\cdot e(y)$, is a left distributive quasigroup follows from Lemma 2.3. If we rewrite right F-quasigroup equality in terms of the operation $\circ$, then $(x\circ e^{-1}y)\circ e^{-1}z = (x\circ e^{-1}fz)\circ(e^{-1}y\circ e^{-2}z)$. If we replace $e^{-1}y$ by y, $e^{-1}z$ by z and take into consideration that $e^{-1}f = fe^{-1}$, then we obtain the equality $(x\circ y)\circ z = (x\circ fz)\circ(y\circ e^{-1}z)$.

(2) The proof is similar to Case (1). $\square$

Corollary 3.16. *(1) If endomorphism e of an F-quasigroup $(Q,\cdot)$ is identity permutation of the set Q, then $(Q,\cdot)$ is a distributive quasigroup.*

(2) If endomorphism f of an F-quasigroup $(Q,\cdot)$ is identity permutation of the set Q, then $(Q,\cdot)$ is a distributive quasigroup.

Proof. (1) If $fx\cdot x = x$, then $fx\circ e^{-1}x = x$. Further proof follows from Lemma 3.15. Indeed from $fx\circ e^{-1}x = x$ it follows $fx\circ x = x$, $fx = x$, since $(Q,\circ)$ is idempotent quasigroup. Then $f = \varepsilon$.

(2) The proof is similar to Case (1). $\square$

The following proof belongs to the OTTER 3.3 [73]. We also have used much of Phillips' article [85]. Here we give the adopted (humanized) form of this proof.

Theorem 3.17. *If in a left distributive quasigroup $(Q,\circ)$ the equality*

$$(x\circ y)\circ z = (x\circ fz)\circ(y\circ ez) \tag{3.2}$$

is fulfilled for all $x,y,z\in Q$, where f,e are the maps of Q, then the following equality is fulfilled in $(Q,\circ)$: $(x\circ y)\circ fz = (x\circ fz)\circ(y\circ fz)$.

Proof. If we pass in equality (3.2) to operation $/$, then we obtain

$$\big((x\circ y)\circ z\big)/\big(y\circ e(z)\big) = x\circ fz \tag{3.3}$$

From equality (3.2) by $x=y$ we obtain $x\circ z = (x\circ fz)\circ(x\circ ez)$ and using left distributivity we have $x\circ z = x\circ(fz\circ e(z))$,

$$z = fz\circ e(z), \quad e(z) = fz\setminus z \tag{3.4}$$

If we change in equality (3.3) the expression $e(z)$ using equality (3.4), then we obtain

$$\big((x\circ y)\circ z\big)/\big(y\circ(fz\setminus z)\big) = x\circ fz \tag{3.5}$$

We make the following replacements in (3.5): $x\to x/z$, $y\to z$, $z\to y$. Then we obtain $(x\circ y)\circ z \to ((x/z)\circ z)\circ y = x\circ y$ and the following equality is fulfilled:

$$(x\circ y)/\big(z\circ\big(f(y)\backslash y\big)\big) = (x/z)\circ f(y) \tag{3.6}$$

Using the operation $/$ we can rewrite left distributive identity in the following form:

$$\big(x\circ(y\circ z)\big)/(x\circ z) = x\circ y \tag{3.7}$$

If we change in identity (3.7) $(y \circ z)$ by y, then variable y passes in y/z. Indeed, if $y \circ z = t$, then $y = t/z$. Therefore, we have

$$(x \circ y)/(x \circ z) = x \circ (y/z) \tag{3.8}$$

From equality (3.2) using left distributivity to the right-hand side of this equality we obtain $(x \circ y) \circ z = ((x \circ fz) \circ y) \circ ((x \circ fz) \circ ez)$. After applying of the operation / to the last equality we obtain

$$\big((x \circ y) \circ z\big)/\big(\big(x \circ f(z)\big) \circ e(z)\big) = \big(x \circ f(z)\big) \circ y \tag{3.9}$$

After substitution of (3.4) in (3.9) we obtain

$$\big((x \circ y) \circ z\big)/\big(\big(x \circ f(z)\big) \circ (fz \setminus z)\big) = \big(x \circ f(z)\big) \circ y \tag{3.10}$$

Now we show the most unexpected OTTER's step. We apply the left-hand side of equality (3.6) to the left-hand side equality (3.10). In this case expression $((x \circ y) \circ z)$ from (3.10) plays the role of $(x \circ y)$, $(x \circ f(z))$ the role of z, and $(fz \setminus z)$ the role of $(f(y)\backslash y)$.

Therefore we obtain

$$\big((x \circ y)/\big(x \circ f(z)\big)\big) \circ f(z) = (x \circ fz) \circ y \tag{3.11}$$

After application to the left-hand side of equality (3.11) equality (3.8) we have

$$\big(x \circ (y/fz)\big) \circ f(z) = (x \circ fz) \circ y \tag{3.12}$$

If we change in equality (3.12) (y/fz) by y, then variable y passes in $y \circ fz$. Therefore $(x \circ y) \circ fz = (x \circ fz) \circ (y \circ fz)$. □

Corollary 3.18. *If in a left distributive quasigroup $(Q, \circ)$ the equality*

$$(x \circ y) \circ z = (x \circ fz) \circ (y \circ ez)$$

is fulfilled for all $x, y, z \in Q$, where e is a map, f is a permutation of the set Q, then $(Q, \circ)$ is a distributive quasigroup.

Proof. The proof follows from Theorem 3.17. □

Theorem 3.19. *If in F-quasigroup $(Q, \cdot)$ endomorphisms e and f are permutations of the set Q, then $(Q, \cdot)$ is isotope of the form $x \cdot y = x \circ e^{-1}y$ of a distributive quasigroup $(Q, \circ)$.*

Proof. Quasigroup $(Q, \circ)$ of the form $x \circ y = x \cdot e(y)$ is a left distributive quasigroup (Lemma 2.3) in which the equality $(x \circ y) \circ z = (x \circ fz) \circ (y \circ e^{-1}z)$, is true (Lemma 3.15). By Corollary 3.18 $(Q, \circ)$ is distributive. □

Theorem 3.20. *An F-quasigroup $(Q, \cdot)$ is simple if and only if $(Q, \cdot)$ lies in one from the following quasigroup classes:*

(i) $(Q, \cdot)$ is a simple group in the case when the maps e and f are zero endomorphisms;

(ii) $(Q, \cdot)$ has the form $x \cdot y = x + \psi y$, where $(Q, +)$ is a ψ-simple Abelian group, $\psi \in \mathrm{Aut}(Q, +)$, in the case when the map e is a zero endomorphism and the map f is a permutation; in this case $e = -\psi$, $fx + \psi x = x$ for all $x \in Q$;

(iii) $(Q, \cdot)$ has the form $x \cdot y = \varphi x + y$, where $(Q, +)$ is a φ-simple Abelian group, $\varphi \in \mathrm{Aut}(Q, +)$, in the case when the map f is a zero endomorphism and the map e is a permutation; in this case $f = -\varphi, \varphi x + ex = x$ for all $x \in Q$;

(iv) $(Q, \cdot)$ has the form $x \cdot y = x \circ \psi y$, where $(Q, \circ)$ is a ψ-simple distributive quasigroup $\psi \in \mathrm{Aut}(Q, \circ)$, in the case when the maps e and f are permutations; in this case $e = \psi^{-1}$, $fx \circ \psi x = x$ for all $x \in Q$.

Proof. ($\Rightarrow$) (i) It is clear that in this case left and right F-quasigroup equalities are transformed in the identity of associativity.

(ii) From Lemma 3.9 (iii) and the fact that the map f is a permutation of the set Q it follows that $(Q,+)$ is an Abelian group.

(iii) This case is similar to Case (ii).

(iv) By Belousov result [15] (see Lemma 2.3 of this paper) if the endomorphism e of a left F-quasigroup $(Q,\cdot)$ is a permutation of the set Q, then quasigroup $(Q,\cdot)$ has the form $x\cdot y = x\circ\psi y$, where $(Q,\circ)$ is a left distributive quasigroup and $\psi\in \mathrm{Aut}(Q,\circ)$, $\psi\in\mathrm{Aut}(Q,\cdot)$. The right distributivity of $(Q,\circ)$ follows from Theorem 3.19.

($\Leftarrow$) Using Corollary 1.70 we can say that F-quasigroups from these quasigroup classes are simple. $\square$

Remark 3.21. There exists a possibility to formulate Theorem 3.20(iv) in the following form.

$(iv)^*$ $(Q,\cdot)$ has the form $x\cdot y=\varphi x\circ y$, where $(Q,\circ)$ is a φ-simple distributive quasigroup, in the case when the maps e and f are permutations; in this case $f=\varphi^{-1}, \varphi x\circ ex = x$ for all $x\in Q$.

Corollary 3.22. *Finite simple F-quasigroup $(Q,\cdot)$ is a simple group or a simple medial quasigroup.*

Proof. Theorem 3.20(i) demonstrates that simple F-quasigroup can be a simple group.

Taking into consideration Toyoda Theorem (Theorem 1.30) we see that Theorem 3.20(ii), (iii) provide that simple F-quasigroups can be simple medial quasigroups.

We will prove that in Theorem 3.20(iv) we also obtain medial quasigroups.

The quasigroup $(Q,\cdot)$ is isotopic to distributive quasigroup $(Q,\circ)$, quasigroup $(Q,\circ)$ is isotopic to CML $(Q,+)$. Therefore $(Q,\cdot)$ is isotopic to the $(Q,+)$ and we can apply Lemma 3.3. $\square$

Taking into consideration Lemma 1.27 we can say that some properties of finite simple medial F-quasigroups are described in Theorem 3.1.

Using the results obtained in this section we can add information on the structure of F-quasigroups [56].

Theorem 3.23. *Any finite F-quasigroup $(Q,\cdot)$ has the following structure:*

$$(Q,\cdot)\cong (A,\circ)\times(B,\cdot)$$

where $(A,\circ)$ is a quasigroup with a unique idempotent element; $(B,\cdot)$ is isotope of a left distributive quasigroup $(B,\star)$, $x\cdot y = x\star\psi y$, $\psi\in\mathrm{Aut}(B,\cdot)$, $\psi\in\mathrm{Aut}(B,\star)$. In the quasigroups $(A,\circ)$ and $(B,\cdot)$ there exist the following chains:

$$A\supset e(A)\supset\cdots\supset e^{m-1}(A)\supset e^m(A)=0,\quad B\supset f(B)\supset\cdots\supset f^r(B)=f^{r+1}(B)$$

where

(1) Let D_i be an equivalence class of the normal congruence $\mathrm{Ker}\, e_i$ containing an idempotent element $a\in A$, $i\geqslant 0$. Then

(a) $(D_i,\circ)$ is linear right loop of the form $x\circ y = x+\psi y$, where $\psi\in\mathrm{Aut}(D_i,+)$;

(b) $\mathrm{Ker}(f|_{(D_i,\circ)})$ is a group;

(c) if $j \geqslant 1$, then $\mathrm{Ker}(f_j|_{(D_i,\circ)})$ *is an Abelian group;*

(d) if f is a permutation of $f^l(D_i,\circ)$, then $f^l(D_i,\circ)$ is a medial right loop of the form $x \circ y = x + \psi y$, where ψ is a complete automorphism of the group $f^l(D_i,+)$;

(e) $(D_i,\circ) \cong (E_i,+) \times f^l(D_i,\circ)$, where $(E_i,+)$ is a linear right loop, an extension of an Abelian group by Abelian groups and by a group.

(2) Let H_j be an equivalence class of the normal congruence $\mathrm{Ker}\, f_j$ *containing an idempotent element $b \in B$, $j \geqslant 0$. Then*

(a) $(H_0,\cdot)$ is a linear left loop of the form $x \cdot y = \varphi x + y$;

(b) $f(B,\cdot)$ is isotope of a distributive quasigroup $f(B,\star)$ of the form $x \cdot y = x \star e^{-1}y$;

(c) if $0 < j < r$, then $(H_j,\cdot)$ is medial left loop of the form $x \cdot y = \varphi x + y$, where $(H_j,+)$ is an Abelian group, $\varphi \in$ $\mathrm{Aut}(H_j,+)$ *and $(H_j,\star)$, $x \star y = \varphi x + ey$, is a medial distributive quasigroup;*

(d) $(B,\cdot) \cong (G,+) \times f^r(B,\cdot)$, where $(G,+)$ has a unique idempotent element, is an extension of an Abelian group by Abelian groups and by a linear left loop $(H_0,\cdot)$, $f^r(B,\cdot)$ is a distributive quasigroup.

Proof. From Theorem 2.7(1) it follows that F-quasigroup $(Q,\cdot)$ is isomorphic to the direct product of quasigroups $(A,\circ)$ and $(B,\cdot)$.

In F-quasigroup $(A,\circ)$ the chain

$$A \supset e(A) \supset e^2(A) \supset \cdots \supset e^{m-1}(A) \supset e^m(A) = e^{m+1}(A) = 0$$

becomes stable on a number m, where 0 is idempotent element.

Case (1)(a). If $0 \leqslant j < m$, then by Lemma 3.8 any quasigroup $(D_j,\circ)$ is a right loop, isotope of a group $(D_j,+)$ of the form $(D_j,\circ) = (D_j,+)(\varepsilon,\psi,\varepsilon)$, where $\psi \in \mathrm{Aut}(D_j,+)$.

Case (1)(b). “Behaviour” of the map f in the right loop $(D_j,\circ)$ is described by Lemma 3.9. If f is zero endomorphism, then $(D_j,\circ)$ is a group in case $j = 0$ (Lemma 3.9(i)) and it is an Abelian group in the case $j > 0$ (Lemma 3.9(ii)).

If f is a nonzero endomorphism of $(D_j,\circ)$, then information on the structure of $(D_j,\circ)$ follows from Lemma 3.9 and Corollary 3.13.

Case (1)(c). The proof follows from Lemma 3.9(ii)–(iv) and the fact that in the quasigroup $\mathrm{Ker}(f_j|_{(D_j,\circ)})$ the maps e and f are zero endomorphisms.

Case (1)(d). The proof follows from Corollary 3.13(1).

Case (1)(e). The proof follows from results of the previous cases of this theorem and Theorem 2.7(2).

Using Lemma 3.15 we can state that F-quasigroup $(B,\cdot)$ is isotopic to left distributive quasigroup $(B,\star)$, where $x \star y = x \cdot e(y)$.

In order to have more detailed information on the structure of the quasigroup $e^m(Q,\cdot)$ we study the following chain:

$$B \supset f(B) \supset \cdots \supset f^r(B) = f^{r+1}(B)$$

which becomes stable on a number r.

Case (2)(a). The proof follows from Corollary 3.13(2).

Case (2)(b). The proof follows from Theorem 3.17.

Case (2)(c). Since f is zero endomorphism of quasigroup $(H_j,\cdot)$, $e_j|_{H_j}$ is a permutation of the set H_j, then by Corollary 3.13 quasigroup $(H_j,\cdot)$ has the form $x \cdot y = \varphi x + y$, where

$(H_j, +)$ is an Abelian group, $\varphi \in \mathrm{Aut}(H_j, +)$ and $(H_j, \circ)$, $x \circ y = \varphi x + ey$, is a medial distributive quasigroup.

Case (2)(d). The existence of direct decomposition follows from Theorem 2.7(2). □

We notice that information on the structure of finite medial quasigroups is there in [103].

3.3 E-quasigroups

We recall a quasigroup $(Q, \cdot)$ is trimedial if and only if $(Q, \cdot)$ is an E-quasigroup [64]. Any trimedial quasigroup is isotopic to CML [50]. The structure of trimedial quasigroups has been studied in [20, 54, 107, 59]. Here slightly other point of view on the structure of trimedial quasigroups is presented.

Lemma 3.24. *(1) If endomorphism f of an E-quasigroup $(Q, \cdot)$ is zero endomorphism, i.e., $f(x) = 0$ for all $x \in Q$, then $x \cdot y = \varphi x + y$, $(Q, +)$ is an Abelian group, $\varphi \in \mathrm{Aut}(Q, +)$.*

(2) If endomorphism e of an E-quasigroup $(Q, \cdot)$ is zero endomorphism, i.e., $e(x) = 0$ for all $x \in Q$, then $x \cdot y = x + \psi y$, $(Q, +)$ is an Abelian group, $\psi \in \mathrm{Aut}(Q, +)$.

Proof. (1) From Theorem 2.14(3) it follows that $(Q, \cdot)$ is a left loop, $x \cdot y = \alpha x + y$, $(Q, +)$ is an Abelian group, $\alpha \in S_Q$, $\alpha 0 = 0$.

Further we have $x \cdot e(x) = \alpha x + e(x) = x$, $\alpha x = x - e(x) = (\varepsilon - e)x$. Therefore α is an endomorphism of $(Q, +)$, moreover, it is an automorphism of $(Q, +)$, since α is a permutation of the set Q.

(2) The proof of Case (2) is similar to the proof of Case (1). □

Corollary 3.25. *If endomorphisms f and e of an E-quasigroup $(Q, \cdot)$ are zero endomorphisms, i.e., $f(x) = e(x) = 0$ for all $x \in Q$, then $x \cdot y = x + y$, $(Q, +)$ is an Abelian group.*

Proof. From equality $\alpha x + e(x) = x$ of Lemma 3.24 we have $\alpha x = x$, $\alpha = \varepsilon$. □

Corollary 3.26. *(1) If endomorphism f of an E-quasigroup $(Q, \cdot)$ is zero endomorphism and endomorphism e is a permutation of the set Q, then $x \cdot y = \varphi x + y$, $(Q, +)$ is an Abelian group, $\varphi \in \mathrm{Aut}(Q, +)$ and $(Q, \circ)$, $x \circ y = \varphi x + ey$, is a medial distributive quasigroup.*

(2) If endomorphism e of an E-quasigroup $(Q, \cdot)$ is zero endomorphism and endomorphism f is a permutation of the set Q, then $x \cdot y = x + \psi y$, $(Q, +)$ is an Abelian group, $\psi \in \mathrm{Aut}(Q, +)$ and $(Q, \circ)$, $x \circ y = fx + \psi y$, is a medial distributive quasigroup.

Proof. (1) From Lemma 3.24 it follows that in this case $(Q, \cdot)$ has the form $x \cdot y = \varphi x + y$ over Abelian group $(Q, +)$. Then $x \cdot e(x) = \varphi x + e(x) = x$, $e(x) = x - \varphi x$, $e(0) = 0$. We can rewrite equality $e(x \cdot y) = e(x) \cdot e(y)$ in the form $e(\varphi x + y) = \varphi e(x) + e(y)$. By $y = 0$ we have $e\varphi(x) = \varphi e(x)$. Then $e(\varphi x + y) = e\varphi x + ey$, the map e is an endomorphism of $(Q, +)$. Moreover, the map e is an automorphism of $(Q, +)$.

From Toyoda Theorem and equality $e(x) = x - \varphi x$ it follows that quasigroup $(Q, \circ)$ is medial idempotent. It is well known that a medial idempotent quasigroup is distributive.

Case (2) is proved in a similar way to Case (1). □

Theorem 3.27. *If the endomorphisms f and e of an E-quasigroup $(Q, \cdot)$ are permutations of the set Q, then quasigroup $(Q, \circ)$ of the form $x \circ y = f(x) \cdot y$ is a distributive quasigroup and $f, e \in \mathrm{Aut}(Q, \circ)$.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.19.

By Lemma 2.15 $(Q, \cdot)$ is isotope of the form $x \cdot y = f^{-1}x \circ y$ of a left distributive quasigroup $(Q, \circ)$ and $f \in \mathrm{Aut}(Q, \circ)$.

Moreover, by Lemma 2.15 $(Q, \cdot)$ is isotope of the form $x \cdot y = x \diamond e^{-1}y$ of a right distributive quasigroup and $e \in \mathrm{Aut}(Q, \diamond)$. Therefore $f^{-1}x \circ y = x \diamond e^{-1}y$, $x \circ y = fx \diamond e^{-1}y$.

Automorphisms e, f of the quasigroup $(Q, \cdot)$ lie in $\mathrm{Aut}(Q, \circ)$ (Lemma 1.24 or [72, Corollary 12]). We recall, $ef = fe$ (Lemma 1.60).

Now we need to rewrite right distributive identity in terms of operation $\circ$. We have

$$f\big(fx \circ e^{-1}y\big) \circ e^{-1}z = f\big(fx \circ e^{-1}z\big) \circ e^{-1}\big(fy \circ e^{-1}z\big)$$
$$\big(f^2x \circ fe^{-1}y\big) \circ e^{-1}z = \big(f^2x \circ fe^{-1}z\big) \circ \big(e^{-1}fy \circ e^{-2}z\big)$$

If in the last equality we change element f^2x by element x, element $fe^{-1}y = e^{-1}fy$ by element y, element $e^{-1}z$ by element z, then we obtain

$$(x \circ y) \circ z = (x \circ fz) \circ \big(y \circ e^{-1}z\big)$$

In order to finish this proof we will apply Corollary 3.18. □

Corollary 3.28. *An E-quasigroup $(Q, \cdot)$ is simple if and only if this quasigroup lies in one from the following quasigroup classes:*

- *(i) $(Q, \cdot)$ is a simple Abelian group in the case when the maps e and f are zero endomorphisms;*
- *(ii) $(Q, \cdot)$ is a simple medial quasigroup of the form $x \cdot y = \varphi x + y$ in the case when the map f is a zero endomorphism and the map e is a permutation;*
- *(iii) $(Q, \cdot)$ is a simple medial quasigroup of the form $x \cdot y = x + \psi y$ in the case when the map e is a zero endomorphism and the map f is a permutation;*
- *(iv) $(Q, \cdot)$ has the form $x \cdot y = x \circ \psi y$, where $(Q, \circ)$ is a ψ-simple distributive quasigroup, $\psi \in \mathrm{Aut}(Q, \circ)$, in the case when the maps e and f are permutations.*

Proof. ($\Rightarrow$) (i) The proof follows from Corollary 3.25. (ii) The proof follows from Lemma 3.24(1). (iii) The proof follows from Lemma 3.24(2). (iv) The proof is similar to the proof of Theorem 3.20(iv).

($\Leftarrow$) It is clear that any quasigroup from these quasigroup classes is simple E-quasigroup. □

Corollary 3.29. *Finite simple E-quasigroup $(Q, \cdot)$ is a simple medial quasigroup.*

Proof. The proof follows from Corollary 3.28 and is similar to the proof of Corollary 3.22. We can use Lemma 3.3. □

Taking into consideration Corollary 3.29 we can say that properties of finite simple E-quasigroups are described by Theorem 3.1.

Lemma 3.30. *(1) If endomorphism f of an E-quasigroup $(Q, \cdot)$ is zero endomorphism, then $(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$, where $(A, \circ)$ a medial E-quasigroup of the form $x \cdot y = \varphi x + y$ and there exists a number m such that $|e^m(A, \circ)| = 1$, $(B, \cdot)$ is a medial E-quasigroup of the form $x \cdot y = \varphi x + y$ such that $(B, \star)$, $x \star y = \varphi x + ey$, is a medial distributive quasigroup.*

(2) If endomorphism e of an E-quasigroup $(Q, \cdot)$ is zero endomorphism, then $(Q, \cdot) \cong (A, +) \times (B, \cdot)$, where $(A, \circ)$ is a medial E-quasigroup of the form $x \cdot y = x + \psi y$ and there exists a number m such that $|f^m(A, \circ)| = 1$, $(B, \cdot)$ is a medial E-quasigroup of the form $x \cdot y = x + \psi y$ such that $(B, \star)$, $x \star y = fx + \psi y$, is a medial distributive quasigroup.

Proof. (1) By Theorem 2.17(4) any right E-quasigroup $(Q,\cdot)$ has the structure $(Q,\cdot) \cong (A,\circ)\times(B,\cdot)$, where $(A,\circ)$ is a quasigroup with a unique idempotent element and there exists a number m such that $|e^m(A,\circ)| = 1$; $(B,\cdot)$ is an isotope of a right distributive quasigroup $(B,\star)$, $x\cdot y = x\star e^{-1}(y)$ for all $x,y\in B$, $e\in \mathrm{Aut}(B,\cdot)$, $e\in \mathrm{Aut}(B,\star)$.

From Lemma 3.24 it follows that $(Q,\cdot)$ has the form $x\cdot y = \varphi x + y$ over an Abelian group $(Q,+)$.

We recall that $e = \varepsilon - \varphi$, $e\varphi = \varphi e$ (Corollary 3.26). From equalities $x\cdot y = \varphi x + y$ and $x\cdot y = x\star e^{-1}(y)$ we have $x\star y = \varphi x + ey$. Then $(B,\star)$ is medial, idempotent, therefore it is distributive.

(2) The proof is similar to Case (1). □

Remark 3.31. If $m = 1$, then $(A,\circ)$ is an Abelian group (Corollary 3.25).

If $m = 2$, then $(A,\circ)$ is an extension of an Abelian group by an Abelian group. If, in addition, the conditions of Lemma 1.81 are fulfilled, then $(A,\circ)$ is an Abelian group.

If the number m is finite and the conditions of Lemma 1.81 are fulfilled, then after application of Lemma 1.81 $(m-1)$ times we obtain that $(A,\circ)$ is an Abelian group.

Now we have a possibility to give in more details information on the structure of finite E-quasigroups. The proof of the following theorem in many details is similar to the proof of Theorem 3.23.

Let D_i be an equivalence class of the normal congruence $\mathrm{Ker}\, e_i$ containing an idempotent element $a\in A$, $i\geqslant 0$. Let H_j be an equivalence class of the normal congruence $\mathrm{Ker}\, f_j$ containing an idempotent element, $j\geqslant 0$.

Theorem 3.32. *In any finite E-quasigroup $(Q,\cdot)$ there exist the following finite chains:*

$$Q\supset e(Q)\supset\cdots\supset e^{m-1}(Q)\supset e^m(Q) = e^{m+1}(Q)$$
$$e^m(Q)\supset fe^m(Q)\supset\cdots\supset f^re^m(Q) = f^{r+1}e^m(Q)$$

where

(1) if $i<m$, then $(D_i,\cdot)\cong(H_i,+)\times(G_i,\cdot)$, where right loop $(H_i,+)$ is an extension of an Abelian group by Abelian groups, $(G_i,\cdot)$ is a medial E-quasigroup of the form $x\cdot y = x+\psi y$ such that ψ is complete automorphism of the group $(G_i,+)$;

(2) if $i=m$, then $(e^mQ,\cdot)$ is isotope of right distributive quasigroup $(e^mQ,\circ)$, where $x\circ y = x\cdot ey$;

(a) if $j<r$, then $(H_j,\cdot)$ is medial left loop, $(H_j,\cdot)$ has the form $x\cdot y = \varphi x + y$, where $(H_j,+)$ is an Abelian group, $\varphi\in\mathrm{Aut}(H_j,+)$ and $(H_j,\circ)$, $x\circ y = \varphi x + ey$, is a medial distributive quasigroup;

(b) if $j=r$, then $(f^re^mQ,\cdot)$ is isotope of the form $x\circ y = f(x)\cdot y$ of a distributive quasigroup $(f^re^mQ,\circ)$.

Proof. It is clear that in E-quasigroup $(Q,\cdot)$ chain (2.2)

$$Q\supset e(Q)\supset e^2(Q)\supset\cdots\supset e^{m-1}(Q)\supset e^m(Q) = e^{m+1}(Q)$$

becomes stable.

(1) $(i<m)$. By Lemma 3.24(2) any quasigroup $(D_i,\cdot)$ is a medial right loop, isotope of an Abelian group $(D_i,+)$ of the form $(D_i,\cdot) = (D_i,+)(\varepsilon,\psi,\varepsilon)$, where $\psi\in\mathrm{Aut}(D_i,+)$,

for all suitable values of index i, since in the quasigroup $(D_i, \cdot)$ endomorphism e is zero endomorphism.

If f is zero endomorphism, then in this case $(D_i, \cdot)$ is an Abelian group (Corollary 3.25).

If f is a nonzero endomorphism of $(D_i, \cdot)$, then we can use Lemma 3.30(2).

Case (1) $(i = m)$. From Lemma 2.15(4) it follows that E-quasigroup $(e^m Q, \cdot)$ is isotopic to right distributive quasigroup $(e^m Q, \circ)$, where $x \circ y = x \cdot e(y)$.

In order to have more detailed information on the structure of the quasigroup $e^m(Q, \cdot)$ we study the following chain:

$$e^m(Q) \supset f e^m(Q) \supset f^2 e^m(Q) \cdots \supset f^r e^m(Q) = f^{r+1} e^m(Q)$$

Case (2)(a) $(j < r)$. From Lemma 2.15(4) it follows that E-quasigroup $(H_j, \cdot)$ is isotopic to right distributive quasigroup $(H_j, \circ)$, $x \circ y = x \cdot e(y)$.

From Lemma 3.24(1) it follows that $(H_j, \cdot)$ has the form $x \cdot y = \varphi x + y$, where $(H_j, +)$ is an Abelian group, $\varphi \in \operatorname{Aut}(H_j, +)$.

From equalities $x \circ e^{-1} y = x \cdot y$ and $x \cdot y = \varphi x + y$, we have $x \circ e^{-1} y = \varphi x + y$, $x \circ y = \varphi x + ey$. Then right distributive quasigroup $(H_j, \circ)$ is isotopic to Abelian group $(H_j, +)$.

If we rewrite identity $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$ in terms of the operation $+$, then $\varphi^2 x + \varphi e y + e z = \varphi^2 x + \varphi e z + e \varphi y + e^2 z$, $\varphi e y + e z = \varphi e z + e \varphi y + e^2 z$. By $z = 0$ from the last equality it follows that $\varphi e = e \varphi$. Then $(H_j, \circ)$ is a medial quasigroup. Moreover, $(H_j, \circ)$ is a medial distributive quasigroup, since any medial right distributive quasigroup is distributive.

Case (2)(b) $(j = r)$. If e and f are permutations of the set $f^r e^m Q$, then by Theorem 3.27 $(f^r e^m Q, \cdot)$ is isotope of the form $x \circ y = f(x) \cdot y$ of a distributive quasigroup $(f^r e^m Q, \circ)$. $\square$

3.4 SM-quasigroups

We recall that left and right SM-quasigroup is called an SM-quasigroup. The structure theory of SM-quasigroups mainly has been developed by Kepka and Shchukin [52, 51, 106, 6].

If an SM-quasigroup $(Q, \cdot)$ is simple, then the endomorphism s is zero endomorphism or a permutation of the set Q.

If $s(x) = 0$, then from Theorem 2.14 we have the following.

Corollary 3.33. *If the endomorphism s of a semimedial quasigroup $(Q, \cdot)$ is zero endomorphism, i.e., $s(x) = 0$ for all $x \in Q$, then $(Q, \cdot)$ is a medial unipotent quasigroup, $(Q, \cdot) \cong (Q, \circ)$, where $x \circ y = \varphi x - \varphi y$, $(Q, +)$ is an Abelian group, $\varphi \in \operatorname{Aut}(Q, +)$.*

Remark 3.34. By Corollary 3.33 equivalence class D_i of the congruence $\operatorname{Ker} s_i$ containing an idempotent element is a medial unipotent quasigroup $(D_i, \cdot)$ of the form $x \circ y = \varphi x - \varphi y$, where $(D_i, +)$ is an Abelian group, $\varphi \in \operatorname{Aut}(D_i, +)$ for all suitable values of index i.

Information on the structure of medial unipotent quasigroups is there in [103].

If $s(x)$ is a permutation of the set Q, then from Lemma 2.15 we have the following.

Lemma 3.35. *If the endomorphism s of a semimedial quasigroup $(Q, \cdot)$ is a permutation of the set Q, then quasigroup $(Q, \circ)$ of the form $x \circ y = s^{-1}(x \cdot y)$ is a distributive quasigroup and $s \in \operatorname{Aut}(Q, \circ)$.*

Corollary 3.36. *Any semimedial quasigroup $(Q, \cdot)$ has the structure $(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$, where $(A, \circ)$ is a quasigroup with a unique idempotent element and there exists a number m such that $|s^m(A, \circ)| = 1$; $(B, \cdot)$ is an isotope of a distributive quasigroup $(B, \star)$, $x \cdot y = s(x \star y)$ for all $x, y \in B$, $s \in \operatorname{Aut}(B, \cdot)$, $s \in \operatorname{Aut}(B, \star)$.*

Proof. The proof follows from Theorem 2.17(3), (4). □

Corollary 3.37. *An SM-quasigroup $(Q, \cdot)$ is simple if and only if it lies in one from the following quasigroup classes:*

(i) $(Q, \cdot)$ *is a medial unipotent quasigroup of the form* $x \circ y = \varphi x - \varphi y$, $(Q, +)$ *is Abelian group,* $\varphi \in \mathrm{Aut}(Q, +)$ *and the group* $(Q, +)$ *is* φ*-simple;*
(ii) $(Q, \cdot)$ *has the form* $x \cdot y = \varphi(x \circ y)$, *where* $\varphi \in \mathrm{Aut}(Q, \circ)$ *and* $(Q, \circ)$ *is* φ*-simple distributive quasigroup.*

Proof. The proof follows from Theorem 3.4(3), (4). □

The similar result on properties of simple SM-quasigroups is there in [106, Corollary 4.13].

Corollary 3.38. *Any finite simple semimedial quasigroup* $(Q, \cdot)$ *is a simple medial quasigroup [106].*

Proof. Conditions of Lemma 3.3 are fulfilled and we can apply it. □

3.5 Simple left FESM-quasigroups

Kinyon and Phillips have defined left FESM-quasigroups in [64].

Definition 3.39. A quasigroup $(Q, \cdot)$ which simultaneously is left F-, E-, and SM-quasigroup we will name *left FESM-quasigroup.*

From Definition 3.39 it follows that in FESM-quasigroup the maps e, f, s are its endomorphisms.

Lemma 3.40. *(1) If endomorphism* e *of a left FESM-quasigroup* $(Q, \cdot)$ *is zero endomorphism, then* $(Q, \cdot)$ *is a medial right loop,* $x \cdot y = x + \psi y$, $(Q, +)$ *is an Abelian group,* $\psi \in \mathrm{Aut}(Q, +)$, $\psi^2 = \varepsilon$, $\psi s = s, \psi f = f\psi = -f$.

(2) If endomorphism f *of an FESM-quasigroup* $(Q, \cdot)$ *is zero endomorphism, then* $(Q, \cdot)$ *is a medial left loop, i.e.,* $x \cdot y = \varphi x + y$, $(Q, +)$ *is an Abelian group,* $\varphi \in \mathrm{Aut}(Q, +)$, $\varphi^2 = \varepsilon$, $\varphi s = s, \varphi e = e\varphi = -e$.

(3) If endomorphism s *of a left FESM-quasigroup* $(Q, \cdot)$ *is zero endomorphism, then* $(Q, \cdot)$ *is medial unipotent quasigroup of the form* $x \cdot y = \varphi x - \varphi y$, *where* $(Q, +)$ *is an Abelian group,* $\varphi \in \mathrm{Aut}(Q, +)$, $\varphi f = f\varphi$, $\varphi e = e\varphi$.

Proof. (1) From Lemma 2.2 it follows that $(Q, \cdot)$ has the form $x \cdot y = x + \psi y$, where $(Q, +)$ is a group, $\psi \in \mathrm{Aut}(Q, +)$.

Then $s(x) = x \cdot x = x + \psi x$. Since s is an endomorphism of $(Q, \cdot)$, further we have $s(x \cdot y) = x + y + \psi x + \psi y$, $sx \cdot sy = sx + \psi sy = x + \psi x + \psi y + \psi^2 y$. Then $x + y + \psi x + \psi y = x + \psi x + \psi y + \psi^2 y$, $y + \psi x + \psi y = \psi x + \psi y + \psi^2 y$. By $x = 0$ we have $y + \psi y = \psi(y + \psi y)$, $sy = \psi sy$. Then $y + \psi x + \psi y = \psi x + \psi y + \psi^2 y = \psi x + \psi sy = \psi x + sy = \psi x + y + \psi y$. Therefore $y + \psi x + \psi y = \psi x + y + \psi y$, $y + \psi x = \psi x + y$, the group $(Q, +)$ is commutative. From equality $y + \psi x + \psi y = \psi x + \psi y + \psi^2 y$ we obtain $y = \psi^2 y$, $\psi^2 = \varepsilon$.

Further we have $f(x) \cdot x = fx + \psi x = x$, $fx = x - \psi x$, $\psi f x = \psi x - x = -fx$, $f\psi x = \psi x - x$.

(2) From Theorem 2.14(3) it follows that $(Q, \cdot)$ is a left loop, $x \cdot y = \varphi x + y$, $(Q, +)$ is an Abelian group, $\varphi \in S_Q$, $\varphi 0 = 0$.

Further we have $x \cdot e(x) = \varphi x + e(x) = x$, $\varphi x = x - e(x) = (\varepsilon - e)x$. Therefore φ is an endomorphism of $(Q, +)$, moreover, it is an automorphism of $(Q, +)$, since φ is a permutation of the set Q.

Then $sx = x{\cdot}x = \varphi x{+}x$, $s(x{\cdot}y) = \varphi x{+}\varphi y{+}x{+}y = sx{\cdot}sy = \varphi sx{+}sy = \varphi^2x{+}\varphi x{+}\varphi y{+}y$. From equality $\varphi x{+}\varphi y{+}x{+}y = \varphi^2x{+}\varphi x{+}\varphi y{+}y$ we obtain $\varphi^2 = \varepsilon$. Then $\varphi sx = \varphi(\varphi x{+}x) = sx$.

Further, $x \cdot ex = \varphi x + ex = x$. Then $ex = x - \varphi x$, $\varphi ex = \varphi x - x = -ex$, $e\varphi x = \varphi x - x$.

(3) From Theorem 2.14(1) it follows that $(Q, \cdot)$ is unipotent quasigroup of the form $x \cdot y = -\varphi x + \varphi y$, where $(Q, +)$ is a group, $\varphi \in \mathrm{Aut}(Q, +)$.

Since f is an endomorphism of quasigroup $(Q, \cdot)$, we have $f(x \cdot y) = e(x) \cdot f(y)$, $f(-\varphi x + \varphi y) = -\varphi f(x) + \varphi f(y)$. If $y = 0$, then $f(-\varphi) = -\varphi f$. If $x = 0$, then $f\varphi = \varphi f$. Then f is an endomorphism of the group $(Q, +)$. Similarly, $e(-\varphi) = -\varphi e$, $e\varphi = \varphi e$, e is an endomorphism of the group $(Q, +)$.

From $x \cdot ex = x$ we have $-\varphi x + e\varphi x = x$, $e\varphi x = \varphi x + x$, $ex = x + \varphi^{-1}x$. Then

$$\begin{aligned} &e(x \cdot y) = x \cdot y + \varphi^{-1}(x \cdot y) = -\varphi x + \varphi y - x + y \\ &ex \cdot ey = -\varphi(x + \varphi^{-1}x) + \varphi(y + \varphi^{-1}y) = -\varphi x - x + \varphi y + y \end{aligned} \tag{3.13}$$

Comparing the right sides of equalities (3.13) we obtain that $(Q, +)$ is a commutative group. □

Lemma 3.41. *If endomorphisms e, f, and s of a left FESM-quasigroup $(Q, \cdot)$ are permutations of the set Q, then quasigroup $(Q, \circ)$ of the form $x \circ y = x \cdot e(y)$ is a left distributive quasigroup and $e, f, s \in \mathrm{Aut}(Q, \circ)$.*

Proof. By Lemma 2.3 endomorphism e of a left F-quasigroup $(Q, \cdot)$ is a permutation of the set Q if and only if quasigroup $(Q, \circ)$ of the form $x{\circ}y = x{\cdot}e(y)$ is a left distributive quasigroup and $e \in \mathrm{Aut}(Q, \circ)$ [15]. Then $x \cdot y = x \circ e^{-1}y$, $s(x) = x \circ e^{-1}x$, $f(x) \cdot x = fx \circ e^{-1}x = x$.

The fact that $e, f, s \in \mathrm{Aut}(Q, \circ)$ follows from Lemma 1.24(2). □

Theorem 3.42. *If $(Q, \cdot)$ is a simple left FESM-quasigroup, then*

(i) $(Q, \cdot)$ is simple medial quasigroup in the case when at least one from the maps e, f, and s is zero endomorphism;

(ii) $(Q, \cdot)$ has the form $x \cdot y = x \circ \psi y$, where $(Q, \circ)$ is a ψ-simple left distributive quasigroup, $\psi \in \mathrm{Aut}(Q, \circ)$, in the case when the maps e, f and s are permutations; in this case $e = \psi^{-1}$, $fx \circ \psi x = x$, $s(x) = x \circ \psi x$ for all $x \in Q$.

Proof. It is possible to use Lemma 3.40 for the proof of Case (i) and Lemma 3.41 for the proof of Case (ii). □

Example 3.43. By $(Z_7, +)$ we denote cyclic group of order 7 and we take $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$.

Quasigroup $(Z_7, \circ)$, where $x \circ y = x + 6y = x - y$, is simple medial FESM-quasigroup in which the maps e and s are zero endomorphisms, the map f is a permutation of the set Z_7 ($f(x) = 2x$ for all $x \in Z_7$).

Quasigroup $(Z_7, \cdot)$, where $x \cdot y = 2x + 3y$, is simple medial FESM-quasigroup in which endomorphisms e, f, s are permutations of the set Z_7.

4 Loop isotopes

In this section we give some results on the loops and left loops which are isotopic to left F-, SM-, E-, and FESM-quasigroups.

We recall that any F-quasigroup is isotopic to a Moufang loop [55, 57], any SM-quasigroup is isotopic to a commutative Moufang loop [51]. Since any E-quasigroup is an SM-quasigroup [52, 64], then any E-quasigroup also is isotopic to a commutative Moufang loop.

4.1 Left F-quasigroups

Taking into consideration Theorems 2.7 and 2.17, Lemma 1.79, and Corollary 1.80 we can study loop isotopes of the factors of direct decompositions of left and right F- and E-quasigroups.

Theorem 4.1. *(1) A left F-quasigroup $(Q, \cdot)$ is isotopic to the direct product of a group $(A, +)$ and a left S-loop $(B, \diamond)$, i.e., $(Q, \cdot) \sim (A, +) \times (B, \diamond)$.*

(2) A right F-quasigroup $(Q, \cdot)$ is isotopic to the direct product of a group $(A, +)$ and right S-loop $(B, \diamond)$, i.e., $(Q, \cdot) \sim (A, +) \times (B, \diamond)$.

Proof. (1) By Theorem 2.7(1) any left F-quasigroup $(Q, \cdot)$ has the structure $(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$, where $(A, \circ)$ is a quasigroup with a unique idempotent element; $(B, \cdot)$ is isotope of a left distributive quasigroup $(B, \star)$, $x \cdot y = x \star \psi y$ for all $x, y \in B$, $\psi \in \mathrm{Aut}(B, \cdot)$, $\psi \in \mathrm{Aut}(B, \star)$.

By Corollary 1.80, if a quasigroup Q is the direct product of quasigroups A and B, then there exists an isotopy $T = (T_1, T_2)$ of Q such that $Q\,T \cong AT_1 \times BT_2$ is a loop.

Therefore we have a possibility to divide our proof into two steps.

Step 1. Denote a unique idempotent element of $(A, \circ)$ by 0. We notice that $e^{\circ}\,0 = 0$. Indeed, from $(e^{\circ})^m A = 0$ we have $(e^{\circ})^{m+1} A = e^{\circ}\,0 = 0$.

From left F-equality $x \circ (y \circ z) = (x \circ y) \circ (e^{\circ}(x) \circ z)$ by $x = 0$ we have $0 \circ (y \circ z) = (0 \circ y) \circ (0 \circ z)$. Then $L_0 \in \mathrm{Aut}(A, \circ)$.

Consider isotope $(A, \oplus)$ of the quasigroup $(A, \circ)$: $x \oplus y = x \circ L_0^{-1} y$. We notice that $(A, \oplus)$ is a left loop. Indeed, $0 \oplus y = 0 \circ L_0^{-1} y = y$. Further we have $x \circ y = x \oplus L_0\, y$, $x \oplus e^{\oplus} x = x = x \circ L_0^{-1} e^{\oplus} x = x \circ e^{\circ} x$, $e^{\oplus}(x) = L_0 e^{\circ}(x)$, $e^{\oplus}(0) = L_0 e^{\circ}(0) = 0 \circ 0 = 0$.

Prove that $L_0 \in \mathrm{Aut}(A, \oplus)$. From equality $L_0(x \circ y) = L_0 x \circ L_0 y$ we have $L_0(x \circ y) = L_0(x \oplus L_0 y)$, $L_0 x \circ L_0 y = L_0 x \oplus L_0^2 y$, $L_0(x \oplus L_0 y) = L_0 x \oplus L_0^2 y$.

If we pass in the left F-equality to the operation $\oplus$, then we obtain $x \oplus (L_0 y \oplus L_0^2 z) = (x \oplus L_0 y) \oplus (L_0 e^{\circ}(x) \oplus L_0^2 z)$. If we change $L_0 y$ by y, $L_0^2 z$ by z, then we obtain

$$x \oplus (y \oplus z) = (x \oplus y) \oplus \big(L_0 e^{\circ}(x) \oplus z\big) = (x \oplus y) \oplus (e^{\oplus}(x) \oplus z) \tag{4.1}$$

Then $(A, \oplus)$ is a left F-quasigroup with the left identity element. For short below in this theorem we will use denotation e instead of $e^{\oplus}$.

Further we pass from the operation $\oplus$ to the operation $+$: $x + y = R_0^{-1} x \oplus y$, $x \oplus y = (x \oplus 0) + y$. Then $x + y = (x/0) \oplus y$, where $x/y = z$ if and only if $z \oplus y = x$. We notice, $R_0^{-1}\, 0 = 0$, since $R_0 0 = 0$, $0 \oplus 0 = 0$, $0 = 0$.

It is well known [83, 12, 15] that $(A, +)$ is a loop. Indeed, $0 + y = R_0^{-1}\, 0 \oplus y = 0 \oplus y = y$; $x + 0 = R_0^{-1} x \oplus 0 = R_0 R_0^{-1} x = x$.

We express the map $e(x)$ in terms of the operation $+$. We have $x \oplus e(x) = x$. Then $(x \oplus 0) + e(x) = x$, $e(x) = (x \oplus 0) \backslash\backslash x$, where $x \backslash\backslash y = z$ if and only if $x + z = y$.

If we denote the map $R_0^{\oplus}$ by α, then $x \oplus y = \alpha x + y$, $e(x) = \alpha x \backslash\backslash x$. We can rewrite (4.1) in terms of the loop operation $+$ as follows:

$$\alpha x + (\alpha y + z) = \alpha(\alpha x + y) + (\alpha e(x) + z) \tag{4.2}$$

From $e(x \oplus y) = ex \oplus ey$ we have $e(\alpha x + y) = \alpha e(x) + e(y)$. By $y = 0$ from the last equality we have

$$e\alpha = \alpha e \tag{4.3}$$

Therefore $e(\alpha x + y) = e\alpha(x) + e(y)$, e is a normal endomorphism of $(A, +)$. Changing αx by x and taking into consideration (4.3) we obtain from equality (4.2) the following equality:

$$x + (\alpha y + z) = \alpha(x + y) + (ex + z) \tag{4.4}$$

Next part of the proof was obtained using Prover 9 which is developed by Professor W. McCune [75].

If we put in equality (4.4) $y = z = 0$, then $\alpha x + ex = x$, or, equivalently,

$$\alpha x = x//ex. \tag{4.5}$$

If we put in equality (4.4) $y = 0$, then

$$\alpha x + (ex + z) = x + z \tag{4.6}$$

If we apply equality (4.5) to equality (4.6), then

$$(x//ex) + (ex + z) = x + z \tag{4.7}$$

If we apply equality (4.5) to equality (4.4), then

$$x + \big((y//ey) + z\big) = \big((x + y)//e(x + y)\big) + (ex + z) \tag{4.8}$$

If we change in equality (4.7) x by $x + y$, then

$$\big((x + y)//e(x + y)\big) + \big(e(x + y) + z\big) = (x + y) + z \tag{4.9}$$

Taking into consideration Lemma 2.5 and Theorem 2.7 we can say that there exists a minimal number n (finite or infinite) such that $e^n(a) = 0$ for any $a \in A$.

If we change in equality (4.4) x by $e^{n-1}x$, then

$$e^{n-1}x + (\alpha y + z) = \alpha\big(e^{n-1}x + y\big) + z \tag{4.10}$$

If we change in (4.10) αx by $x//ex$ (equality 4.5), then

$$e^{n-1}x + \big((y//ey) + z\big) = \big((e^{n-1}x + y)//ey\big) + z \tag{4.11}$$

We change in equality (4.9) x by $e^{n-1}x$. Then

$$\big((e^{n-1}x + y)//ey\big) + (ey + z) = \big(e^{n-1}x + y\big) + z \tag{4.12}$$

We rewrite the left-hand side of equality (4.12) as follows:

$$\big((e^{n-1}x + y)//ey\big) + (ey + z) \overset{4.11}{=} e^{n-1}x + \big((y//ey) + (ey + z)\big) \overset{4.7}{=} e^{n-1}x + (y + z) \tag{4.13}$$

From (4.12) and (4.13) we have

$$e^{n-1}x + (y + z) = \big(e^{n-1}x + y\big) + z \tag{4.14}$$

We change in equality (4.9) x by $e^{n-2}x$, then

$$\big((e^{n-2}x + y)//e\big(e^{n-2}x + y\big)\big) + \big(e\big(e^{n-2}x + y\big) + z\big) = \big(e^{n-2}x + y\big) + z \tag{4.15}$$

We rewrite the left-hand side of equality (4.15) as follows:

$$\begin{aligned}
&\big((e^{n-2}x + y)//e\big(e^{n-2}x + y\big)\big) + \big(e\big(e^{n-2}x + y\big) + z\big) \\
&\quad \overset{4.14}{=} \big((e^{n-2}x + y)//e\big(e^{n-2}x + y\big)\big) + \big(e^{n-1}x + (ey + z)\big) \\
&\quad \overset{4.8}{=} e^{n-2}x + \big((y//ey) + (ey + z)\big) \overset{4.7}{=} e^{n-2}x + (y + z)
\end{aligned} \tag{4.16}$$

From (4.15) and (4.16) we have

$$e^{n-2}x + (y + z) = \big(e^{n-2}x + y\big) + z \tag{4.17}$$

Begin Cycle

We change in equality (4.9) x by $e^{n-3}x$. Then

$$\big((e^{n-3}x + y)//e\big(e^{n-3}x + y\big)\big) + \big(e\big(e^{n-3}x + y\big) + z\big) = \big(e^{n-3}x + y\big) + z \tag{4.18}$$

We rewrite the left-hand side of equality (4.18) as follows:

$$\begin{aligned}
&\big((e^{n-3}x + y)//e\big(e^{n-3}x + y\big)\big) + \big(e\big(e^{n-3}x + y\big) + z\big) \\
&\quad \overset{4.17}{=} \big((e^{n-3}x + y)//e\big(e^{n-3}x + y\big)\big) + \big(e^{n-2}x + (ey + z)\big) \\
&\quad \overset{4.8}{=} e^{n-3}x + \big((y//ey) + (ey + z)\big) \overset{4.7}{=} e^{n-3}x + (y + z)
\end{aligned} \tag{4.19}$$

From (4.18) and (4.19) we have

$$e^{n-3}x + (y + z) = \big(e^{n-3}x + y\big) + z$$

End Cycle

Therefore

$$e^{n-i}x + (y + z) = \big(e^{n-i}x + y\big) + z$$

for any natural number i. If the number n is finite, then repeating **Cycle** necessary number of times we will obtain that $x + (y + z) = (x + y) + z$ for all $x, y, z \in A$.

Since n is a fixed number (maybe and an infinite), then $\lim_{i\to\infty}(n - i) \to 0$, where $i \in \mathbb{N}$. We can apply **Cycle** necessary number of times to obtain associativity. Indeed, suppose that λ is a minimal number such that

$$e^{\lambda}x + (y + z) = \big(e^{\lambda}x + y\big) + z \tag{4.20}$$

and there exist $a, b, c \in A$ such that

$$e^{\lambda-1}a + (b + c) \neq \big(e^{\lambda-1}a + b\big) + c$$

But from the other side, if we apply **Cycle** to equality (4.20), then we obtain that

$$e^{\lambda-1}x + (y+z) = \big(e^{\lambda-1}x + y\big) + z$$

for all $x, y, z \in A$, i.e., λ is not a minimal number with declared properties.

Therefore our supposition is not true and

$$e^{\lambda}x + (y+z) = \big(e^{\lambda}x + y\big) + z$$

for all suitable λ and all $x, y, z \in A$.

Step 2. From Theorems 2.7 and 1.33 it follows that

$$(B,\diamond) = (B,\cdot)(\varepsilon,\psi,\varepsilon)\big(\big(R^{\star}_a\big)^{-1}, \big(L^{\star}_a\big)^{-1}, \varepsilon\big) = (B,\cdot)\big(\big(R^{\star}_a\big)^{-1}, \psi\big(L^{\star}_a\big)^{-1}, \varepsilon\big)$$

is a left S-loop.

(2) This case is proved similarly to Case (1). □

Corollary 4.2. *A loop $(Q,*)$, which is the direct product of a group $(A,+)$ and a left S-loop $(B,\diamond)$, is a left special loop.*

Proof. Indeed, any group is left special. Any left S-loop also is a special loop (see [81, p. 61]). Therefore $(Q,*)$ is a left special loop. □

Lemma 4.3. *The fulfilment of equality (4.4) in the group $(A,+)$ is equivalent to the fact that the triple $T_x = (\alpha L_x \alpha^{-1}, \varepsilon, L_{\alpha x})(\varepsilon, L_{e(x)}, L_{e(x)})$ is an autotopy of $(A,+)$ for all $x \in A$.*

Proof. From (4.4) by $y = 0$ we have

$$x + z = \alpha x + (ex + z) \tag{4.21}$$

i.e., $L_x = L_{\alpha x} L_{e(x)}$.

If we change in (4.4) y by $\alpha^{-1}y$, then

$$x + (y+z) = \alpha\big(x + \alpha^{-1}y\big) + (ex + z) \tag{4.22}$$

Equality (4.22) means that the group $(A,+)$ has an autotopy of the form

$$T_x = \big(\alpha L_x \alpha^{-1}, L_{e(x)}, L_x\big)$$

for all $x \in A$. Taking into consideration that $L_x = L_{\alpha x} L_{e(x)}$, we can rewrite T_x in the form

$$T_x = \big(\alpha L_x \alpha^{-1}, L_{e(x)}, L_{\alpha x} L_{e(x)}\big) = \big(\alpha L_x \alpha^{-1}, \varepsilon, L_{\alpha x}\big)\big(\varepsilon, L_{e(x)}, L_{e(x)}\big)$$ □

Corollary 4.4. *If the group $(A,+)$ has the property $[L_d, \alpha^{-1}] \in LM(A,+)$ for all $d \in A$, then*

(i) $e(A,+) \trianglelefteq C(A,+) \trianglelefteq (A,+)$;

(ii) $\alpha \in \mathrm{Aut}(A,+)$;

(ii) $\alpha|_{(\mathrm{Ker}\, e,+)} = \varepsilon$.

Proof. (i) It is well known that any autotopy of a group $(A,+)$ has the form $(L_a\delta, R_b\delta, L_aR_b\delta)$, where L_a is a left translation of the group $(A,+)$, R_b is a right translation of this group, δ is an automorphism of this group [15].

Therefore if the triple T_d is an autotopy of the loop $(A,+)$, then we have

$$\alpha L_d\alpha^{-1} = L_a\delta, L_{e(d)} = R_b\delta, L_d = L_aR_b\delta \tag{4.23}$$

Then $L_{e(d)}0 = R_b\delta 0$, $e(d) = b$. From $L_d 0 = L_aR_b\delta 0$ we have $d = a + b$, $d = a + e(d)$. But $d = \alpha d + e(d)$. Therefore, $a = \alpha d$.

We can rewrite equalities (4.23) in the form

$$\alpha L_d\alpha^{-1} = L_{\alpha d}\delta, L_{e(d)} = R_{e(d)}\delta, L_d = L_{\alpha d}R_{e(d)}\delta \tag{4.24}$$

Then

$$\delta = R_{-e(d)}L_{e(d)} = L_{e(d)}R_{-e(d)}, \quad \alpha L_d\alpha^{-1} = L_{\alpha d}L_{e(d)}R_{-e(d)} = L_dR_{-e(d)}$$
$$L_{-d}\alpha L_d\alpha^{-1} = R_{-e(d)}, \quad \left[L_d, \alpha^{-1}\right] = R_{-e(d)}$$

We notice that all permutations of the form $\{R_{-e(d)} \mid d \in A\}$ form a subgroup H' of the group $RM(A,+)$, since e is an endomorphism of the group $(A,+)$.

By our assumption $H' \subseteq LM(A,+)$. Then

$$H' \subseteq RM(A,+) \cap LM(A,+)$$

But $LM\langle A,\alpha\rangle \cap RM\langle A,\alpha\rangle \subseteq C\langle A,\alpha\rangle$ [43, 90]. Therefore $R_{-e(d)} = L_{-e(d)}$ for all $d \in A$, $e(A) \subseteq C(A)$.

(ii) From (i) it follows that the triple $(\varepsilon, L_{e(b)}, L_{e(b)})$ is an autotopy of $(A,+)$. Indeed, equality $y + (e(b) + z) = e(b) + (y + z)$ is true for all $b, y, z \in A$ since $e(b) \in C(A)$.

Then the triple $(\alpha L_b\alpha^{-1}, \varepsilon, L_{\alpha b})$ is an autotopy of $(A,+)$, i.e., $\alpha L_b\alpha^{-1}y + z = L_{\alpha b}(y+z)$. By $z = 0$ we have $\alpha L_b\alpha^{-1}y = L_{\alpha b}y$. Then the triple $(L_{\alpha b}, \varepsilon, L_{\alpha b})$ is a loop autotopy.

The equality $\alpha L_b\alpha^{-1} = L_{\alpha b}$ means that $\alpha b + y = \alpha(b + \alpha^{-1}y)$ for all $b, y \in A$. If we change y by αy, then $\alpha b + \alpha y = \alpha(b + y)$ for all $b, y \in A$, $\alpha \in \mathrm{Aut}(A,+)$.

(iii) From equality $\alpha x + e(x) = x$ by $e(x) = 0$ we have $\alpha x = x$. □

Remark 4.5. Conditions $[L_d, \alpha^{-1}] \in LM(A,+)$ for all $d \in A$ and $\alpha \in \mathrm{Aut}(A,+)$ are equivalent.

Corollary 4.6. *If $e(x) = 0$ for all $x \in A$, then $\alpha \in \mathrm{Aut}(A,+)$.*

Proof. In this case equality (4.22) takes the form $(\alpha L_x\alpha^{-1}, \varepsilon, L_{\alpha x})$. If autotopy of such form true in a loop, then $\alpha L_x\alpha^{-1} = L_{\alpha x}$, $\alpha L_x = L_{\alpha x}\alpha$. □

Sokhatsky has proved the following theorem (see [113, Theorem 17]).

Theorem 4.7. *A group isotope $(Q,\cdot)$ with the form $x \cdot y = \alpha x + a + \beta y$ is a left F-quasigroup if and only if β is an automorphism of the group $(Q,+)$, β commutes with α and α satisfies the identity $\alpha(x + y) = x + \alpha y - x + \alpha x$.*

Example 4.8. Dihedral group $(D_8,+)$ with the Cayley table

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	6	7	4	5
2	2	6	7	1	0	3	5	4
3	3	4	5	0	1	2	7	6
4	4	3	0	5	7	6	1	2
5	5	7	6	4	3	0	2	1
6	6	2	1	7	5	4	0	3
7	7	5	4	6	2	1	3	0

has an endomorphism

$$e = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 3 & 0 & 3 & 3 & 0 & 0 \end{pmatrix}, \quad e^2 = 0$$

and permutation $\alpha = (1\,2)(4\,5)$ such that $\alpha \notin \mathrm{Aut}(D_8)$. Using this permutation and taking into consideration Theorem 4.7 we may construct left F-quasigroups $(D_8, \cdot)$ and $(D_8, *)$ with the forms $x \cdot y = \alpha x + a + y$ and $x * y = \alpha x + a + \beta y$, where $\beta = (15)(24)$. These quasigroups are right-linear group isotopes but they are not left linear quasigroups ($\alpha \notin \mathrm{Aut}(D_8,+)$). This example was constructed using Mace 4 [74].

Corollary 4.9. *A left special loop $(Q,\oplus)$ is isotope of a left F-quasigroup $(Q,\cdot)$ if and only if $(Q,\oplus)$ is isotopic to the direct product of a group $(A,+)$ and a left S-loop $(B,\diamond)$.*

Proof. If a left special loop $(Q,\oplus)$ is an isotope of a left F-quasigroup $(Q,\cdot)$, then from Theorem 4.1 it follows that $(Q,\oplus)$ is isotopic to a loop $(Q,*)$ which is the direct product of a group $(A,+)$ and a left S-loop $(B,\diamond)$.

Conversely, suppose that a left special loop $(Q,\oplus)$ is an isotope a loop $(Q,*)$ which is the direct product of a group $(A,+)$ and a left S-loop $(B,\diamond)$. It is easy to see that isotopic image of group $(A,+)$ of the form $(\varepsilon,\psi,\varepsilon)$, where $\psi \in \mathrm{Aut}(A,+)$, is a left F-quasigroup.

From Theorem 1.33 we have that isotopic image of the loop $(B,\diamond)$ of the form $(\alpha,\psi^{\diamond},\varepsilon)$, where $\psi^{\diamond}$ is complete automorphism of $(B,\diamond)$, is a left distributive quasigroup $(B,\circ)$. By Lemma 2.3 (see also [15]) isotope of the form $x \cdot y = x \cdot \psi^{\circ} y$, where $\psi^{\circ} \in \mathrm{Aut}(B,\circ)$, is a left F-quasigroup. Hence, among isotopic images of the left special loop $(Q,\oplus)$ there exists a left F-quasigroup. $\square$

Corollary 4.9 gives an answer to Belousov 1a Problem [12].

Corollary 4.10. *If $(Q,*)$ is a left M-loop which is isotopic to a left F-quasigroup $(Q,\cdot)$, then $(Q,*)$ is isotopic to the direct product of a group and LP-isotope of a left S-loop.*

Proof. By Theorem 4.1 any left F-quasigroup $(Q,\cdot)$ is LP-isotopic to a loop $(Q,\oplus)$ which is the direct product of a group $(A,\oplus)$ and left S-loop $(B,\diamond)$.

By Theorem 1.35 any loop which is isotopic to a left F-quasigroup is a left M-loop.

Up to isomorphism $(Q,*)$ is an LP-isotope of $(Q,\cdot)$. Then the loops $(Q,*)$ and $(Q,\oplus)$ are isotopic with an isotopy $(\alpha,\beta,\varepsilon)$. Moreover, they are LP-isotopic (see [15, Lemma 1.1]).

From the proof of Lemma 1.79 it follows that LP-isotopic image of a loop that is a direct product of two subloops also is isomorphic to the direct product of some subloops.

By Albert Theorem (Theorem 1.36) LP-isotopic image of a group is a group. $\square$

4.2 F-quasigroups

Theorem 4.11. *Any F-quasigroup $(Q,\cdot)$ is isotopic to the direct product of a group $(A,\oplus)\times(G,+)$ and a commutative Moufang loop $(K,\diamond)$, i.e., $(Q,\cdot)\sim(A,\oplus)\times(G,+)\times(K,\diamond)$.*

Proof. By Theorem 2.7(1) any left F-quasigroup $(Q,\cdot)$ has the structure $(Q,\cdot)\cong(A,\circ)\times(B,\cdot)$, where $(A,\circ)$ is a quasigroup with a unique idempotent element; $(B,\cdot)$ is isotope of a left distributive quasigroup $(B,\star)$, $x\cdot y = x\star\psi y$ for all $x,y\in B$, $\psi\in \operatorname{Aut}(B,\cdot)$, $\psi\in\operatorname{Aut}(B,\star)$.

By Theorem 2.7(2) the quasigroup $(B,\cdot)$ has the structure $(B,\cdot)\cong(G,\circ)\times(K,\cdot)$, where $(G,\circ)$ is a quasigroup with a unique idempotent element; $(K,\cdot)$ is isotope of a right distributive quasigroup $(K,\star)$, $x\cdot y=\varphi x\star y$ for all $x,y\in K$, $\varphi\in\operatorname{Aut}(K,\cdot)$, $\varphi\in\operatorname{Aut}(K,\star)$.

By Theorem 4.1(1), the quasigroup $(A,\circ)$ is a group isotope. By Theorem 4.1(2), the quasigroup $(G,\circ)$ is a group isotope.

In the quasigroup $(K,\cdot)$ the endomorphisms e and f are permutations of the set K and by Theorem 3.19 $(K,\cdot)$ is isotope of a distributive quasigroup. Then by Belousov Theorem (Theorem 1.31) quasigroup $(K,\cdot)$ is isotope of a CML $(K,\diamond)$. Therefore $(Q,\cdot)\sim(A,\oplus)\times(G,+)\times(K,\diamond)$. $\square$

Theorem 4.12. *Any loop $(Q,*)$ that is isotopic to an F-quasigroup $(Q,\cdot)$ is isomorphic to the direct product of a group and a Moufang loop [55, 57].*

Proof. By Theorem 4.11 an F-quasigroup $(Q,\cdot)$ is isotopic to a loop $(Q,+)\cong(A,+)\times(B,+)$ which is the direct product of a group and a commutative Moufang loop. Then any left translation L of $(Q,+)$ is possible to be present as a pair (L_1,L_2), where L_1 is a left translation of the loop $(A,+)$, L_2 is a left translation of the loop $(B,+)$.

From Lemma 1.62 it follows that any LP-isotope of the loop $(Q,+)$ is the direct product of its subloops.

By generalized Albert Theorem LP-isotope of a group is a group. Any LP-isotope of a commutative Moufang loop is a Moufang loop [12]. $\square$

Corollary 4.13. *If $(Q,*)$ is an M-loop which is isotopic to an F-quasigroup, then $(Q,*)$ is a Moufang loop.*

Proof. The proof follows from Theorem 4.12. It is well known that any group is a Moufang loop. $\square$

4.3 Left SM-quasigroups

Theorem 4.14. *A left SM-quasigroup $(Q,\cdot)$ is isotopic to the direct product of a group $(A,\oplus)$ and a left S-loop $(B,\diamond)$, i.e., $(Q,\cdot)\sim(A,\oplus)\times(B,\diamond)$.*

Proof. In many details the proof of this theorem repeats the proof of Theorem 4.1.

By Theorem 2.17 any left SM-quasigroup $(Q,\cdot)$ has the structure $(Q,\cdot)\cong(A,\circ)\times(B,\cdot)$, where $(A,\circ)$ is a quasigroup with a unique idempotent element and there exists a number m such that $|s^m(A,\circ)|=1$; $(B,\cdot)$ is an isotope of a left distributive quasigroup $(B,\star)$, $x\cdot y=s(x\star y)$ for all $x,y\in B$, $s\in\operatorname{Aut}(B,\cdot)$, $s\in\operatorname{Aut}(B,\star)$.

By Corollary 1.80, if a quasigroup Q is the direct product of quasigroups A and B, then there exists an isotopy $T=(T_1,T_2)$ of Q such that $Q\,T\cong AT_1\times BT_2$ is a loop.

Therefore we have a possibility to divide our proof into two steps.

Step 1. Denote a unique idempotent element of $(A,\circ)$ by 0. It is easy to check that $s^\circ\,0=0$. Indeed, from $(s^\circ)^m A=0$ we have $(s^\circ)^{m+1}A=s^\circ 0=0$.

(23)-parastrophe of $(A, \circ)$ is left F-quasigroup $(A, \cdot)$ (Lemma 1.85, (5)) such that $|e^m(A, \cdot)| = 1$. Then $(A, \cdot)$ also has a unique idempotent element. By Theorem 4.1 principal isotope of $(A, \cdot)$ is a group $(A, \oplus)$.

We will use multiplication of isostrophies (Definition 1.89, Corollary 1.90, and Lemma 1.91). (23)-parastrophe image of group $(A, \oplus)$ coincides with its isotope of the form $(I, \varepsilon, \varepsilon)$, where $x \oplus Ix = 0$ for all $x \in A$. Indeed, if $x \oplus y = z$, then $x \oplus^{23} z = y$. But $y = Ix \oplus z$. Therefore $x \oplus^{23} z = Ix \oplus z$, i.e., $(\oplus)((23), \varepsilon) = (\oplus)(\varepsilon, (I, \varepsilon, \varepsilon))$. Then $(\oplus) = (\oplus)((23), (I, \varepsilon, \varepsilon))$, since $I^2 = \varepsilon$.

We have

$$(\oplus) = (\circ)\big((23), \varepsilon\big)\big(\varepsilon, (\alpha, \beta, \varepsilon)\big) = (\circ)\big((23), (\alpha, \beta, \varepsilon)\big),$$
$$(\oplus) = (\oplus)\big((23), (I, \varepsilon, \varepsilon)\big) = (\circ)\big((23), (\alpha, \beta, \varepsilon)\big)\big((23), (I, \varepsilon, \varepsilon)\big) = (\circ)\big(\varepsilon, (\alpha I, \varepsilon, \beta)\big).$$

Step 2. The proof of this step is similar to the proof of Step 2 from Theorem 4.1 and we omit them. □

4.4 Left E-quasigroups

Lemma 4.15. *A left E-quasigroup $(Q, \cdot)$ is isotopic to the direct product of a left loop $(A, \oplus)$ with equality $(\delta x \oplus x) \oplus (y \oplus z) = (\delta x \oplus y) \oplus (x \oplus z)$, where δ is an endomorphism of the loop $(A, \oplus)$, and a left S-loop $(B, \diamond)$, i.e., $(Q, \cdot) \sim (A, \oplus) \times (B, \diamond)$.*

Proof. In some details the proof of Lemma 4.15 repeats the proof of Theorem 4.1. By Theorem 2.17 any left E-quasigroup $(Q, \cdot)$ has the structure $(Q, \cdot) \cong (A, \circ) \times (B, \cdot)$, where $(A, \circ)$ is a quasigroup with a unique idempotent element and there exists a number m such that $|f^m(A, \circ)| = 1$; $(B, \cdot)$ is an isotope of a left distributive quasigroup $(B, \star)$, $x \cdot y = f^{-1}(x) \star y$ for all $x, y \in B$, $f \in \mathrm{Aut}(B, \cdot)$, $f \in \mathrm{Aut}(B, \star)$.

By Corollary 1.80, if a quasigroup Q is the direct product of quasigroups A and B, then there exists an isotopy $T = (T_1, T_2)$ of Q such that $Q\,T \cong AT_1 \times BT_2$ is a loop.

Therefore we have a possibility to divide our proof into two steps.

Step 1. We will prove that $(A, \oplus)$ is a left loop. Denote a unique idempotent element of $(A, \circ)$ by 0. It is easy to check that $f^\circ 0 = 0$. Indeed, from $(f^\circ)^m A = 0$ we have $(f^\circ)^{m+1} A = f^\circ 0 = 0$.

From left E-equality $x \circ (y \circ z) = (f^\circ(x) \circ y) \circ (x \circ z)$ by $x = 0$ we have $0 \circ (y \circ z) = (0 \circ y) \circ (0 \circ z)$. Then $L_0 \in \mathrm{Aut}(A, \circ)$.

Consider isotope $(A, \oplus)$ of the quasigroup $(A, \circ)$: $x \oplus y = x \circ L_0^{-1} y$. We notice that $(A, \oplus)$ is a left loop. Indeed, $0 \oplus y = 0 \circ L_0^{-1} y = y$.

Prove that $L_0 \in \mathrm{Aut}(A, \oplus)$. From equality $L_0(x \circ y) = L_0 x \circ L_0 y$ we have $L_0(x \circ y) = L_0(x \oplus L_0 y)$, $L_0 x \circ L_0 y = L_0 x \oplus L_0^2 y$, $L_0(x \oplus L_0 y) = L_0 x \oplus L_0^2 y$.

If we pass in left E-equality to the operation $\oplus$, then we obtain $x \oplus (L_0 y \oplus L_0^2 z) = (f^\circ x \oplus L_0 y) \oplus (L_0 x \oplus L_0^2 z)$. If we change $L_0 y$ by y, $L_0^2 z$ by z, then we obtain

$$x \oplus (y \oplus z) = \big(f^\circ x \oplus y\big) \oplus \big(L_0 x \oplus z\big) \tag{4.25}$$

We notice that $f^\circ x \circ x = x$. Then $f^\circ x \oplus L_0 x = x$. Moreover, from $f^\circ(x \circ y) = f^\circ x \circ f^\circ y$ we have $f^\circ(x \oplus L_0 y) = f^\circ(x) \oplus L_0 f^\circ(y)$. If $x = 0$, then $f^\circ L_0(y) = L_0 f^\circ(y)$, $f^\circ(x \oplus L_0 y) = f^\circ(x) \oplus f^\circ L_0(y)$, f° is an endomorphism of the left loop $(A, \oplus)$.

We can rewrite equality (4.25) in the following form:

$$\big(f^\circ x \oplus L_0 x\big) \oplus (y \oplus z) = \big(f^\circ x \oplus y\big) \oplus \big(L_0 x \oplus z\big) \tag{4.26}$$

If we change in (4.26) x by L_x^{-1}, then we obtain

$$\big(f^{\circ}L_0^{-1}x \oplus x\big) \oplus (y \oplus z) = \big(f^{\circ}L_0^{-1}x \oplus y\big) \oplus (x \oplus z) \tag{4.27}$$

If we denote the map $f^{\circ}L_0^{-1}$ of the set Q by δ, then from (4.27) we have $(\delta x \oplus x) \oplus (y \oplus z) = (\delta x \oplus y) \oplus (x \oplus z)$. The map $\delta = f^{\circ}L_0^{-1}$ is an endomorphism of the left loop $(A, \oplus)$ since f° is an endomorphism and L_0^{-1} an automorphism of $(A, \oplus)$. We notice, $f^{\circ}L_0^{-1}0 = 0$.

Step 2. From Theorems 2.17 and 1.33 it follows that

$$(B, \diamond) = (B, \cdot)(f, \varepsilon, \varepsilon)\big((R_a^{\star})^{-1}, (L_a^{\star})^{-1}, \varepsilon\big) = (B, \cdot)\big(f(R_a^{\star})^{-1}, (L_a^{\star})^{-1}, \varepsilon\big)$$

is a left S-loop. $\square$

Remark 4.16. If we take $f^{\circ}a = 0$, then from $f^{\circ}a \oplus L_0 a = a$ we have $L_0 a = a$. Thus from (4.26) we have $a \oplus (y \oplus z) = y \oplus (a \oplus z)$.

Lemma 4.17. *A left E-quasigroup $(Q, \cdot)$ is isotopic to the direct product of a loop $(A, +)$ with equality $(\delta x + x) + (y + z) = (\delta x + y) + (x + z)$, where δ is an endomorphism of the loop $(A, +)$, and a left S-loop $(B, \diamond)$, i.e., $(Q, \cdot) \sim (A, +) \times (B, \diamond)$.*

Proof. We pass from the operation $\oplus$ to operation $+$: $x + y = R_0^{-1}x \oplus y$, $x \oplus y = (x \oplus 0) + y$. Then $x + y = (x/0) \oplus y$, where $x/y = z$ if and only if $z \oplus y = x$. We notice that $R_0^{-1}\,0 = 0$, since $R_0 0 = 0$, $0 \oplus 0 = 0$.

If we denote the map $R_0^{\oplus}$ by α, then $x \oplus y = \alpha x + y$. We can rewrite (4.27) in terms of the loop operation $+$ as follows:

$$\alpha(\delta\alpha x + x) + (\alpha y + z) = \alpha(\delta\alpha x + y) + (\alpha x + z) \tag{4.28}$$

Prove that $\alpha\delta = \delta\alpha$. Notice that $R_y^{\oplus}x = R^{\circ}_{L_0^{-1}y}x$. Then $R_0^{\oplus} = R_0^{\circ}$. Thus

$$L_0 R_0^{\oplus}x = L_0 R_0 x = 0 \circ (x \circ 0) = (0 \circ x) \circ 0 = R_0^{\oplus} L_0 x$$
$$f^{\circ}R_0^{\oplus}x = f^{\circ}(x \circ 0) = f^{\circ}x \circ 0 = R_0^{\oplus} f^{\circ}x$$

Then δ is an endomorphism of the loop $(A, +)$. Indeed, $\delta(x + y) = \delta(\alpha^{-1}x \oplus y) = \delta\alpha^{-1}x \oplus \delta y = \alpha^{-1}\delta x \oplus \delta y = \delta x + \delta y$.

Equality (4.28) takes the form

$$\alpha(\delta x + x) + (\alpha y + z) = \alpha(\delta x + y) + (x + z) \tag{4.29}$$

If we put in equality (4.29) $x = y$, then $\alpha x = x$, $\alpha = \varepsilon$ and equality (4.29) takes the form

$$(\delta x + x) + (y + z) = (\delta x + y) + (x + z) \tag{4.30}$$

$\square$

Lemma 4.18. *If $\delta x = 0$ for all $x \in A$, then $(A, +)$ is a commutative group.*

Proof. If we put in equality (4.30) $z = 0$, then $x + y = y + x$. Therefore, from $x + (y + z) = y + (x + z)$ we have $(y + z) + x = y + (z + x)$. $\square$

Lemma 4.19. *There exists a number m such that in the loop $(A, +)$ the chain*

$$(A, +) \supset \delta(A, +) \supset \delta^2(A, +) \supset \cdots \supset \delta^m(A, +) = (0, +) \tag{4.31}$$

is stabilized on the element 0.

Proof. From Theorem 2.17 it follows that $(A, \circ)$ is a left E-quasigroup with a unique idempotent element 0 such that the chain

$$(A, \circ) \supset f^{\circ}(A, \circ) \supset \big(f^{\circ}\big)^{2}(A, \circ) \supset \cdots \supset \big(f^{\circ}\big)^{m}(A, \circ) = (0, \circ) \tag{4.32}$$

is stabilized on the element 0.

From Lemmas 4.15 and 4.17 it follows that $(A, +) = (A, \circ)T$, where isotopy T has the form $(R_0^{-1}, L_0^{-1}, \varepsilon)$. Since $0 \in (f^{\circ})^{i}(A, \circ)$, then $((f^{\circ})^{i}(A, \circ))T = (f^{\circ})^{i}(A, +)$ is a subloop of the loop $(A, +)$ (Lemma 1.14) for all suitable values of i.

Thus we obtain that the isotopic image of chain (4.32) is the following chain:

$$(A, +) \supset f^{\circ}(A, +) \supset \big(f^{\circ}\big)^{2}(A, +) \supset \cdots \supset \big(f^{\circ}\big)^{m}(A, +) = (0, +) \tag{4.33}$$

We recall that $\delta = f^{\circ}L_0^{-1}$ and $f^{\circ}L_0^{-1} = L_0^{-1}f^{\circ}$ (Lemma 4.15). Then $\delta^{i} = (f^{\circ})^{i}L_0^{-i}$ and $\delta^{i}(A, +) = (f^{\circ})^{i}L_0^{-i}(A, +)$. It is clear that L_0^{-i} is a bijection of the set A for all suitable values of i.

Thus we can establish the following bijection: $(f^{\circ})^{i}(A, +) \leftrightarrow \delta^{i}(A, +)$. Then $\delta^{i}(A, +) \supset \delta^{i+1}(A, +)$, since $(f^{\circ})^{i}(A, +) \supset (f^{\circ})^{i+1}(A, +)$. Therefore $(f^{\circ})^{m}(A, +) \leftrightarrow \delta^{m}(A, +)$, $\delta^{m}(A, +) = (0, +)$. □

Lemma 4.20. *The loop $(A, +)$ is a commutative group.*

Proof. From Lemma 4.19 it follows that in $(A, +)$ there exists a number m such that $\delta^{m}x = 0$ for all $x \in A$. We have used Prover's 9 help [75]. From (4.30) by $y = 0$ we obtain

$$(\delta x + x) + y = \delta x + (x + y) \tag{4.34}$$

If we change in equality (4.34) y by $y + z$, then we obtain

$$(\delta x + x) + (y + z) = \delta x + \big(x + (y + z)\big) \tag{4.35}$$

From (4.30) by $z = 0$ using (4.34) we have

$$(\delta x + y) + x = \delta x + (x + y) \tag{4.36}$$

If we change in (4.36) y by $\delta x \backslash y$, then

$$\big(\delta x + (\delta x \backslash y)\big) + x = \delta x + \big(x + (\delta x \backslash y)\big) \tag{4.37}$$

But $(\delta x + (\delta x \backslash y)) = y$ (Definition 1.15, equality (1.1)). Therefore

$$\delta x + \big(x + (\delta x \backslash y)\big) = y + x \tag{4.38}$$

If we change in (4.30) x by $\delta^{m-1}x$, then, using condition $\delta^{m}x = 0$, we have

$$\delta^{m-1}x + (y + z) = y + \big(\delta^{m-1}x + z\big) \tag{4.39}$$

Begin Cycle

If we change in equality (4.38) the element x by the element $\delta^{m-2}x$, then we have

$$\delta^{m-1}x + \big(\delta^{m-2}x + \big(\delta^{m-1}x \backslash y\big)\big) = y + \delta^{m-2}x \tag{4.40}$$

If we change in (4.39) z by $\delta^{m-1}x\backslash z$, then, using Definition 1.15, equality (1.1), we obtain

$$\delta^{m-1}x + \big(y + (\delta^{m-1}x\backslash z)\big) = y + z \tag{4.41}$$

If we change in (4.41) y by $\delta^{m-2}x$, z by y and compare (4.41) with (4.40), then we obtain

$$\delta^{m-2}x + y = y + \delta^{m-2}x \tag{4.42}$$

We have $\delta^{m-1}(A) \subseteq \delta^{m-2}(A)$ since δ is an endomorphism of the loop $(A, +)$. Notice from equalities (4.39) and (4.42) it follows that $\delta^{m-1}(A) \subseteq N_l(A)$.

From equality $(y/\delta^{m-2}x) + \delta^{m-2}x = y$ (Definition 1.15, equality (1.2)) using commutativity (4.42) we obtain

$$\delta^{m-2}x + \big(y/\delta^{m-2}x\big) = y \tag{4.43}$$

From equality (4.43) and definition of the operation $\backslash$ we have

$$\delta^{m-2}x\backslash y = y/\delta^{m-2} \tag{4.44}$$

If we change in (4.39) $y + z$ by y, then y pass in y/z and we have

$$\delta^{m-1}x + y = (y/z) + \big(\delta^{m-1}x + z\big) \tag{4.45}$$

Applying to (4.45) the operation $/$ we have

$$\big(\delta^{m-1}x + y\big)/\big(\delta^{m-1}x + z\big) = (y/z) \tag{4.46}$$

Write equality (4.30) in the form

$$(\delta x + y)\backslash\big((\delta x + x) + (y + z)\big) = x + z \tag{4.47}$$

From (4.47) using (4.35) we obtain

$$(\delta x + y)\backslash\big(\delta x + \big(x + (y + z)\big)\big) = x + z \tag{4.48}$$

From equality (4.48) using (4.44) we have

$$\big(\delta x + \big(x + (y + z)\big)\big)/(\delta x + y) = x + z \tag{4.49}$$

If we change in equality (4.49) x by δ^{m-2}, then we obtain

$$\big(\delta^{m-1}x + \big(\delta^{m-2}x + (y + z)\big)\big)/\big(\delta^{m-1}x + y\big) = \delta^{m-2}x + z \tag{4.50}$$

Using equality (4.46) in equality (4.50) we have

$$(\delta^{m-2}x + (y + z))/y = \delta^{m-2}x + z \tag{4.51}$$

Therefore

$$\delta^{m-2}x + (y + z) = \big(\delta^{m-2}x + z\big) + y$$

and

$$\delta^{m-2}x + (y + z) = y + \big(\delta^{m-2}x + z\big) \tag{4.52}$$

End Cycle

Therefore we can change equality (4.39) by the equality (4.52) and start new step of the cycle.

After m steps we obtain that in the loop $(A, +)$ the equality $x + (y + z) = y + (x + z)$ is fulfilled, i.e., $(A, +)$ is an Abelian group. If $m = \infty$, then we can use arguments similar to the arguments from the proof of Theorem 4.1. □

Theorem 4.21. *(1) A left E-quasigroup $(Q, \cdot)$ is isotopic to the direct product of an Abelian group $(A, +)$ and a left S-loop $(B, \diamond)$, i.e., $(Q, \cdot) \sim (A, +) \times (B, \diamond)$.*

(2) A right E-quasigroup $(Q, \cdot)$ is isotopic to the direct product of an Abelian group $(A, +)$ and a right S-loop $(B, \diamond)$, i.e., $(Q, \cdot) \sim (A, +) \times (B, \diamond)$.

Proof. (1) The proof follows from Lemmas 4.17 and 4.18. □

Theorem 4.21 gives an answer to Kinyon-Phillips problems (see [64, Problem 2.8, (1)]).

Corollary 4.22. *A left FESM-quasigroup $(Q, \cdot)$ is isotopic to the direct product of an Abelian group $(A, \oplus)$ and a left S-loop $(B, \diamond)$.*

Proof. We can use Theorem 4.21. □

Corollary 4.22 gives an answer to Kinyon-Phillips problem (see [64, Problem 2.8, (2)]).

We hope in a forthcoming paper we will discuss a generalization of Murdoch theorems about the structure of finite binary and n-ary medial quasigroups [78, 102] on infinite case and medial groupoids.

Acknowledgment

The author thanks MRDA-CRDF (ETGP, Grant no. 1133), Consiliul Suprem pentru Ştiinţă şi Dezvoltare Tehnologică al Republicii Moldova (Grant no. 08.820.08.08 RF) and organizers of the conference LOOPS'07 for financial support. The author also thanks Professor V. I. Arnautov for his helpful comments.

References

[1] J. Aczél, V. D. Belousov, and M. Hosszú. Generalized associativity and bisymmetry on quasigroups. Acta Math. Acad. Sci. Hungar., **11** (1960), 127–136.

[2] A. A. Albert. Quasigroups. I. Trans. Amer. Math. Soc., **54** (1943), 507–519.

[3] A. A. Albert, Quasigroups. II. Trans. Amer. Math. Soc., **55** (1944), 401–419.

[4] A. S. Basarab. A class of LK-loops. Mat. Issled., **120** (1991), 3–7 (in Russian).

[5] G. E. Bates and F. Kiokemeister. A note on homomorphic mappings of quasigroups into multiplicative systems. Bull. Amer. Math. Soc., **54** (1948), 1180–1185.

[6] V. A. Beglaryan and K. K. Shchukin. The structure of tri-Abelian totally symmetric quasigroups. In *"Mathematics, No. 3"*. Erevan. Univ., Erevan, 1985, 82–88 (in Russian).

[7] V. A. Beglaryan. On the theory of homomorphisms in quasigroups. Ph.D. thesis, IM AN MSSR, 1982 (in Russian).

[8] V. D. Belousov. On one class of quasigroups. Učen. Zap. Bel'ck. Gos. Ped. Inst., **5** (1960), 29–46 (in Russian).

[9] V. D. Belousov. The structure of distributive quasigroups. Mat. Sb., **50** (1960), 267–298 (in Russian).

[10] V. D. Belousov. Globally associative systems of quasigroups. Mat. Sb., **55** (1961), 221–236 (in Russian).

[11] V. D. Belousov. Balanced identities on quasigroups. Mat. Sb., **70** (1966), 55–97 (in Russian).

[12] V. D. Belousov. *Foundations of the Theory of Quasigroups and Loops*, Nauka, Moscow, 1967 (in Russian).

[13] V. D. Belousov. The group associated with a quasigroup. Mat. Issled., **4** (1969), 21–39 (in Russian).

[14] V. D. Belousov. n-ary quasigroups. Stiintsa, Kišiněv, 1971 (in Russian).

[15] V. D. Belousov. *Elements of Quasigroup Theory: A Special Course.* Kišiněv State University Printing House, Kišiněv, 1981 (in Russian).

[16] V. D. Belousov and I. A. Florja. On left-distributive quasigroups. Bul. Akad. Štiince RSS Moldoven, (1965), no. 7, 3–13 (in Russian).

[17] V. D. Belousov and V. I. Onoĭ. Loops that are isotopic to left distributive quasigroups. Mat. Issled., **3(25)** (1972), 135–152 (in Russian).

[18] G. B. Belyavskaya. Direct decompositions of quasigroups. Mat. Issled., **95** (1987), 23–38 (in Russian).

[19] G. B. Belyavskaya. Full direct decompositions of quasigroups with an idempotent element. Mat. Issled., **113** (1990), 21–36 (in Russian).

[20] L. Beneteau and T. Kepka. Theoremes de structure dans certains groupoides localement nilpotents. C. R. Acad. Sci. Paris Ser. I Math., **300** (1985), 327–330 (in French).

[21] G. Birkhoff. *Lattice Theory.* Nauka, Moscow, 1984 (in Russian).

[22] R. H. Bruck. Some results in the theory of quasigroups. Trans. Amer. Math. Soc., **55** (1944), 19–52.

[23] R. H. Bruck. Normal endomorphisms. Illinois J. Math., **4** (1960), 38–87.

[24] R. H. Bruck. *A Survey of Binary Systems.* 3rd ed. Springer Verlag, New York, 1971.

[25] R. H. Bruck and L. J. Paige. Loops whose inner mappings are automorphisms. Ann. of Math., **63** (1956), 308–332.

[26] I. I. Burdujan. Certain remarks on the geometry of quasigroups. Nets and Quasigroups, **39** (1976), 40–53 (in Russian).

[27] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*, Springer-Verlag, Berlin, 1981.

[28] A. M. Cheban. The Sushkevich postulates for distributive identities. In *"Algebraic Structures and Geometry"*. Kišiněv, 1991, 152–155 (in Russian).

[29] P. M. Cohn. *Universal Algebra.* Harper & Row, New York, 1965.

[30] W. Dornte. Untersuchungen über einen veralgemeinerten Gruppenbegriff. Math. Z., **29** (1928), 1–19.

[31] J. Duplak. A parastrophic equivalence in quasigroups. Quasigroups Relat. Syst., **7** (2000), 7–14.

[32] T. Evans. Homomorphisms of non-associative systems. J. London Math. Soc. **24** (1949), 254–260.

[33] T. Evans. On multiplicative systems defined by generators and relations. Math. Proc. Camb. Phil. Soc., **47** (1951), 637–649.

[34] I. A. Florja. F-quasigroups with invertibility property. In *"Questions of the Theory of Quasigroups and Loops"*. Kišiněv, 1971, 156–165 (in Russian).

[35] I. A. Florja. A certain class of F-quasigroups with an invertibility property. In *"Studies in the Theory of Quasigroups and Loops"*. Stiintsa, Kišiněv, 1973, 174–186 (in Russian).

[36] I. A. Florja and M. I. Ursul. F-quasigroups with invertibility property (IPF-quasigroups). In *"Questions of the Theory of Quasigroups and Loops"*. Kišiněv, 1971, 145–156 (in Russian).

[37] T. Fujiwara. Note on permutability of congruences on algebraic systems. Proc. Japan Acad., **41** (1965), 822–827.

[38] V. M. Galkin. Finite distributive quasigroups. Mat. Zametki, **24** (1978), 39–41 (in Russian).

[39] V. M. Galkin. Left distributive quasigroups finite order quasigroups. Mat. Issled., **51** (1979), 43–54 (in Russian).

[40] V. M. Galkin. Left distributive quasigroups. D.Sc. Dissertation. Steklov Mathematical Institute, Moscow, 1991 (in Russian).

[41] I. A. Golovko. F-quasigroups with idempotent elements. Mat. Issled., **4** (1969), 137–143 (in Russian).

[42] I. A. Golovko. Loops that are isotopic to F-quasigroups. Bul. Akad. Štiince RSS Moldoven, (1970) 3–13 (in Russian).

[43] M. Hall. *The Theory of Groups.* The Macmillan Company, New York, 1959.

[44] I. N. Herstein, *Abstract Algebra.* 2nd ed. Macmillan Publishing Company, New York, 1990.

[45] J. Ježek. Normal subsets of quasigroups. Comment. Math. Univ. Carolin., **16** (1975), 77–85.

[46] J. Ježek and T. Kepka. Varieties of Abelian quasigroups. Czech. Math. J., **27** (1977), 473–503.

[47] M. I. Kargapolov and M. Yu. Merzlyakov. *Foundations of Group Theory.* Nauka, Moscow, 1977 (in Russian).

[48] A. D. Keedwell and V. A. Shcherbacov. Construction and properties of (r, s, t)-inverse quasigroups, II. Discrete Math., **288** (2004), 61–71.

[49] T. Kepka. Quasigroups which satisfy certain generalized forms of the Abelian identity. Časopis Pěst. Mat., **100** (1975), 46–60.

[50] T. Kepka. Structure of triabelian quasigroups. Comment. Math. Univ. Carolinae, **17** (1976), 229–240.

[51] T. Kepka. A note on WA-quasigroups. Acta Univ. Carolin. Math. Phys., **19** (1978), 61–62.

[52] T. Kepka. Structure of weakly Abelian quasigroups. Czech. Math. J., **28** (1978), 181–188.

[53] T. Kepka. F-quasigroups isotopic to Moufang loops. Czech. Math. J., **29** (1979), 62–83.

[54] T. Kepka, L. Beneteau, and J. Lacaze. Small finite trimedial quasigroups. Comm. Algebra, **14** (1986), 1067–1090.

[55] T. Kepka, M. K. Kinyon, and J. D. Phillips. The structure of F-quasigroups. Preprint arxiv:math/0510298, 2005.

[56] T. Kepka, M. K. Kinyon, and J. D. Phillips. F-quasigroups isotopic to groups. Preprint arxiv:math/601077, 2006.

[57] T. Kepka, M. K. Kinyon, and J. D. Phillips. The structure of F-quasigroups. J. Algebra, **317** (2007), 435–461.

[58] T. Kepka, M. K. Kinyon, and J. D. Phillips. F-quasigroups and generalized modules. Comment. Math. Univ. Carolin., **49** (2008), 249–258.

[59] T. Kepka and P. Nemec. Trimedial quasigroups and generalized modules. Acta Univ. Carolin. Math. Phys., **31** (1990), 3–14.

[60] T. Kepka and P. Němec. T-quasigroups, II. Acta Univ. Carolin. Math. Phys., **12** (1971), 31–49.

[61] M. K. Kinyon and K. Kunen. Power-associative, conjugacy closed loops. Preprint arxiv:math/0507278v3, 2005.

[62] M. K. Kinyon and K. Kunen. Power-associative, conjugacy closed loops. J. Algebra, **304** (2006), 679–711.

[63] M. K. Kinyon and J. D. Phillips. A note on trimedial quasigroups. Quasigroups Relat. Syst., **9** (2002), 65–66.

[64] M. K. Kinyon and J. D. Phillips. Axioms for trimedial quasigroups. Comment. Math. Univ. Carolin., **45** (2004), 287–294.

[65] M. K. Kinyon and P. Vojtechovsky. Primary decompositions in varieties of commutative diassociative loops. Preprint arxiv:math/0702874, 2007.

[66] O. U. Kirnasovsky. Linear isotopes of small order. Quasigroups Relat. Syst., **2** (1995), 51–82.

[67] A. Krapez. Generalized associativity on groupoids. Publ. Inst. Math. (Beograd) (N.S.), **28(42)** (1980), 105–112.

[68] A. G. Kurosh. *Lectures on General Algebra.* Gos. Izdatel'stvo Fiz.-Mat. Literatury, Moscow, 1962 (in Russian).

[69] I. V. Leakh. On transformations of orthogonal systems of operations and algebraic nets. Ph.D. thesis, IM AN RM, Kišiněv, 1986 (in Russian).

[70] A. I. Mal'tsev. On the general theory of algebraic systems. Mat. Sb., **35 (77)** (1954), 3–20 (in Russian).

[71] A. I. Mal'tsev. *Algebraic Systems.* Nauka, Moscow, 1976 (in Russian).

[72] A. Marini and V. A. Shcherbacov. On autotopies and automorphisms of n-ary linear quasigroups. Algebra and Discrete Math., (2004), 51–75.

[73] W. McCune. *OTTER 3.3.* Argonne National Laboratory, www.mcs.anl.gov/AR/otter/, 2004.

[74] W. McCune. *Mace 4.* University of New Mexico, www.cs.unm.edu/ mccune/prover9/, 2007.

[75] W. McCune. *Prover 9*. University of New Mexico, www.cs.unm.edu/ mccune/prover9/, 2007.

[76] R. Moufang. Zur Structur von Alternativ Körpern. Math. Ann., **110** (1935), 416–430.

[77] D. C. Murdoch. Quasigroups which satisfy certain generalized associative laws. Amer. J. Math., **61** (1939), 509–522.

[78] D. C. Murdoch. Structure of Abelian quasigroups. Trans. Amer. Math. Soc., **49** (1941), 392–409.

[79] Gabor P. Nagy. Some remarks on simple Bol loops. Comment. Math. Univ. Carolin., **49** (2008), 259–270.

[80] P. Němec and T. Kepka. T-quasigroups, I. Acta Univ. Carolin. Math. Phys., **12** (1971), 39–49.

[81] V. I. Onoĭ. Left distributive quasigroups and loops, which are isotopic to these quasigroups. Ph.D. thesis, IM AN MSSR, Kišiněv, 1972 (in Russian).

[82] I. I. Parovichenko. *The Theory of Operations over Sets.* Shtiintsa, Kišiněv, 1981.

[83] H. O. Pflugfelder. *Quasigroups and Loops: Introduction.* Heldermann Verlag, Berlin, 1990.

[84] J. D. Phillips. On Moufang A-loops. Comment. Math. Univ. Carolinae, **41** (2000), 371–375.

[85] J. D. Phillips. See Otter digging for algebraic pearls. Quasigroups Relat. Syst., **10** (2003), 95–114.

[86] L. V. Sabinin and Ludmila Sabinina. On the theory of left F-quasigroups. Algebras Groups Geom., **12** (1995), 127–137.

[87] L. V. Sabinin and L. V. Sbitneva. Reductive spaces and left F-quasigroups. In "*Webs and Quasigroups*". Tver Univ. Press, Tver, 1991, 123–128.

[88] L. V. Sabinin. *Smooth Quasigroups and Loops.* Mathematics and Its Applications, **492**, Kluwer Academic Publishers, Dordrecht, 1999.

[89] L. V. Sabinin. *Analytic Quasigroups and Geometry.* Univ. Druzhby Narodov, Moscow, 1991 (in Russian).

[90] V. A. Shcherbacov. Some properties of full associated group of IP-loop. Izvestia AN MSSR. Ser. Fiz.-Techn. i Mat. Nauk, (1984), 51–52 (in Russian).

[91] V. A. Shcherbacov. About one class of medial quasigroups. Mat. Issled., **102** (1988), 111–116 (in Russian).

[92] V. A. Shcherbacov. On left distributive quasigroups isotopic to groups. In *"Proceedings of the XI Conference of Young Scientists of Friendship of Nations University"* (Moscow, 1988). No. 5305-B88, VINITI, 1988, 148–149.

[93] V. A. Shcherbacov. On automorphism groups of quasigroups and linear quasigroups. Reg. in VINITI 04.11.89, No. 6710-B89, Moscow, 1989, 32 pages (in Russian).

[94] V. A. Shcherbacov. On automorphism groups and congruences of quasigroups. Ph.D. thesis, IM AN MSSR, 1991 (in Russian).

[95] V. A. Shcherbacov. On automorphism groups and congruences of quasigroups. Abstract of Ph.D. thesis, Kišiněv, 1991 (in Russian).

[96] V. A. Shcherbacov. On linear quasigroups and their automorphism groups. Mat. Issled., **120** (1991), 104–113 (in Russian).

[97] V. A. Shcherbacov. On automorphism groups of left distributive quasigroups. Bul. Acad. Stiinte Repub. Mold. Mat., (1994), 79–86 (in Russian).

[98] V. A. Shcherbacov. On 20 Belousov's problem. Tech. Report 99.8, IAMI, Milan, Italy, 1999.

[99] V. A. Shcherbacov. Elements of quasigroup theory and some its applications in code theory and cryptology. Retrieved 2003 from www.karlin.mff.cuni.cz/∼drapal/speccurs.pdf.

[100] V. A. Shcherbacov. On Bruck-Belousov problem. Bul. Acad. Stiinte Repub. Mold. Mat., (2005), 123–140.

[101] V. A. Shcherbacov. On simple n-ary medial quasigroups. In *"Proceedings of Conference Computational Commutative and Non-Commutative Algebraic Geometry"*. NATO Science Series: Computer and Systems Sciences 196, IOS Press, 2005, 305–324.

[102] V. A. Shcherbacov. On structure of finite n-ary medial quasigroups and automorphism groups of these quasigroups. Quasigroups Relat. Syst., **13** (2005), 125–156.

[103] V. A. Shcherbacov. On the structure of finite medial quasigroups. Bul. Acad. Stiinte Repub. Mold. Mat., (2005), 11–18.

[104] V. A. Shcherbacov. On definitions of groupoids closely connected with quasigroups. Bul. Acad. Stiinte Repub. Mold. Mat., (2007), 43–54.

[105] V. A. Shcherbacov. On linear and inverse quasigroups and their applications in code theory. D.Sc. Thesis. IMCS of the Academy of Sciences of Moldova, Chişinău, 2008, cnaa.acad.md/files/theses/2008/8175/victor_scerbacov_thesis.pdf.

[106] K. K. Shchukin. *Action of a Group on a Quasigroup.* Kišiněv State University, Kišiněv, 1985 (in Russian).

[107] K. K. Shchukin. On the structure of trimedial quasigroups. In *"Investigations in General Algebra, Geometry, and Their Applications"*. Shtiintsa, Kišiněv, 1986, 148–154.

[108] J. D. H. Smith. *Mal'cev Varieties.* Lecture Notes in Mathematics, **554**, Springer-Verlag, New York, 1976.

[109] J. D. H. Smith. *An Introduction to Quasigroups and Their Representation.* Studies in Advanced Mathematics, Chapman and Hall/CRC, London, 2007.

[110] L. R. Soikis. The special loops. In *"Questions of the Theory of Quasigroups and Loops"*. Kišiněv, 1970, 122–131 (in Russian).

[111] F. N. Sokhatskii. On isotopes of groups, I. Ukraïn. Mat. Zh., **47** (1995), 1387–1398 (in Ukrainian).

[112] F. N. Sokhatskii. On isotopes of groups, II. Ukraïn. Mat. Zh., **47** (1995), 1692–1703 (in Ukrainian).

[113] F.N. Sokhatskii. Some linear conditions and their application to describing group isotopes. Quasigroups Relat. Syst., **6** (1999), 43–59.

[114] F. N. Sokhatskii. Associates and decompositions of multy-place operations. D.Sc. Dissertation. Institute of Mathematics, National Academy of Sciences of Ukraine, 2007 (in Ukrainian).

[115] A. Stein. A conjugacy class as a transversal in a finite group. J. Algebra, **239** (2001), 365–390.

[116] A. K. Suschkewitsch. On a generalization of the associative law. Trans. Amer. Math. Soc., **31** (1929), 204–214.

[117] A. K. Suschkewitsch. *The Theory of Generalized Groups.* DNTVU, Kiev, 1937 (in Russian).

[118] A. Kh. Tabarov. Nuclei, linearity and balanced identities in quasigroups. Ph.D. thesis. IM AN MSSR, 1992 (in Russian).

[119] K. Toyoda. On axioms of linear functions. Proc. Imp. Acad. Tokyo, **17** (1941), 221–227.

[120] Wikipedia, *Homomorphism.* 2007, http://en.wikipedia.org/wiki/Homomorphism.

[121] Wikipedia, *Semidirect product.* 2007, http://en.wikipedia.org/wiki/Semidirect_product.

Received November 08, 2008
Revised January 25, 2009