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On the Fundamental Theorem in Arithmetic Progression of Primes

Jiang CX*

Institute for Basic Research, Palm Harbor, P. O. Box 3924, Beijing 100854, P. R. China

Abstract

Using Jiang function we prove the fundamental theorem in arithmetic progression of primes. The primes contain only $k<P_g+1$ long arithmetic progression, but the primes have no $k>P_g+1$ long arithmetic progressions theorem.

Keywords: Arithmetic; Lie theory; Fundamental theorem; Progression; Asymptotic formula

Theorem

The fundamental theorem in arithmetic progression of primes.

We define the arithmetic progression of primes [1-3].

$$P_{i+1}=P_1+\omega_g i, i=0,1,2,\ldots,k-1, \tag{1}$$

Where $\omega_g=\prod_{2\le P\le P_g} P$ is called a common difference, P_g is called g-th prime.

We have Jiang function [1-3]:

$$\prod_{i=1}(q+\omega_g i)\equiv 0 (\bmod P), \tag{2}$$

$X(P)$ denotes the number of solutions for the following congruence:

$$\prod_{i=1}^{k-1}(q+\omega_g i)\equiv 0 (\bmod P), \tag{3}$$

Where $q=1,2,\ldots,P-1$.

If $P|\omega_g$, then $X(P)=0$;$X(P)=k-1$ otherwise. From eqn (3) we have:

$$J_2(\omega)=\prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-k). \tag{4}$$

If $k=P_{g+1}$ then $J_2(P_{g+1})=0$, $J_2(\omega)=0$, there exist finite primes P_1 such that $P_2,\ldots,P_k$ are primes. If $k<P_{g+1}$ then $J_2()\neq 0$, there exist infinitely many primes P_1 such that $P_2,\ldots,P_k$ are primes. The primes contain only $k<P_{g+1}$ long arithmetic progression, but the primes have no $k>P_{g+1}$ long arithmetic progression geometry. We have the best asymptotic formula [1-3]:

$$\pi_k(N,2)=\left|\{P_1+\omega_g i=\text{prime}, 0\le i\le k-1, P_1\le N\}\right|$$

$$=\frac{J_2(\omega)\omega^{k-1}}{\varphi^k(\omega)}\frac{N}{\log^k N}(1+o(1)), \tag{5}$$

Where $\omega=\prod_{2\le P}P$, $\varphi(\omega)=\prod_{2\le P}(P-1)$, ω is called primorial, $\varphi(\omega)$ Euler function.

Suppose $k=P_{g+1}-1$. From eqn (1) we have:

$$P_{i+1}=P_1+\omega_g i, i=0,1,2,\ldots,P_{g+1}2. \tag{6}$$

From eqn (4) we have [1,2]:

$$J_2(\omega)=\prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-P_{g+1}+1)\to\infty \text{ as } \omega\to\infty \tag{7}$$

We prove that there exist infinitely many primes P_1 such that $P_2,\cdots,P_{P_{g+1}-1}$ are primes for all P_{g+1}.

From eqn (5) we have:

$$\pi_{P_{g+1}-1}(N,2)=$$

$$\prod_{2\le P\le P_g}\left(\frac{P}{P-1}\right)^{P_{g+1}-2}\prod_{P_{g+1}\le P}=\frac{P^{P_{g+1}-2}(P-P_{g+1}+1)}{(P-1)^{P_{g+1}-1}}\frac{N}{(\log N)^{P_{g+1}-1}}(1+o(1)). \tag{8}$$

From eqn (8) we are able to find the smallest solutions $\pi_{P_{g+1}-1}(N,2)>1$ for large P_{g+1}.

Theorem is foundation for arithmetic progression of primes.

Example 1: Suppose $P_1=2$, $\omega_1=2$, $P_2=3$. From eqn (6) we have the twin primes theorem:

$$P_2=P_1+2. \tag{9}$$

From eqn (7) we have:

$$J_2(\omega)=\prod_{3\le P}(P-2)\to\infty \text{ as } \omega\to\infty, \tag{10}$$

We prove that there exist infinitely many primes P_1 such that P_2 are primes. From eqn (8) we have the best asymptotic formula [4-6]:

$$\pi_2(N,2)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N}{\log^2 N}(1+o(1)). \tag{11}$$

Twin prime theorem is the first theorem in arithmetic progression of primes.

Example 2: Suppose $P_2=3$, $\omega_2=6$, $P_3=5$. From eqn (6) we have:

$$P_{i+1}=P_1+6i, i=0,1,2,3. \tag{12}$$

From eqn (7) we have:

$$J_2(\omega)=2\prod_{5\le P}(P-4)\to\infty \text{ as } \omega\to\infty, \tag{13}$$

We prove that there exist infinite many primes P_1 such that P_2, P_3 and P_4 are primes. From eqn (8) we have the best generalized asymptotic formula:

$$\pi_4(N,2)=27\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N}{\log^4 N}(1+o(1)). \tag{14}$$

Example 3: Suppose $P_9=23$, $\omega_9=223092870$, $P_{10}=29$. From eqn (6) we have:

***Corresponding author:** Jiang CX, Institute for Basic Research, Palm Harbor, P. O. Box 3924, Beijing 100854, P. R. China, Tel: +1-727-688 3992; E-mail: jcxuan@sina.com

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$P_{i+1}=P_1+223092870i,\ i=0,1,2,\ldots,27.$ (15)

From eqn (7) we have:

$$J_2(\omega)=36495360\prod_{29\le P}(P-28)\to\infty \text{ as } \omega\to\infty, \quad (16)$$

We prove that there exist infinitely many primes P_1 such that $P_2,\ldots,P_{28}$ are primes. From eqn (8) we have the best asymptotic formula [7]:

$$\pi_{28}(N,2)=\prod_{2\le P\le 23}\left(\frac{P}{P-1}\right)^{27}\prod_{29\le P}\frac{P^{27}(P-28)}{(P-1)^{28}}\frac{N}{\log^{28}N}(1+o(1)). \quad (17)$$

From eqn (17) we are able to find the smallest solutions $\pi_{28}(N_0,2)>1$.

On May 17, 2008, Wroblewski and Raanan Chermoni found the first known case of 25 primes:

$61711054912832631+366384\times\omega_{23}\times n$, for $n=0$ to 24.

Theorem can help in finding for 26, 27, 28,…, primes in arithmetic progressions of primes.

Corollary 1: Arithmetic progression with two prime variables.

Suppose $\omega_g=d$. From eqn (1) we have:

$P_1, P_2=P_1+d, P_3=P_1+2d,\ldots,P_k=P_1+(k-1)d,\ (P_1,d)=1$ (18)

From eqn (18) we obtain the arithmetic progression with two prime variables: P_1 and P_2,

$P_3=2P_2-P_1,\ P_j=(j-1)P_2-(j-2)P_1,\ 3\le j\le k<P_{g+1}.$ (19)

We have Jiang function [3]:

$$J_3(\omega)=\prod_{3\le P}[(P-1)^2-X(P)], \quad (20)$$

$X(P)$ denotes the number of solutions for the following congruence matrices:

$$\prod_{j=3}^{k}[(j-1)q_2-(j-2)q_1]\equiv 0 (\text{mod } P), \quad (21)$$

where $q_1=1,2,\ldots,P-1$; $q2=1,2,\ldots,P-1$.

From eqn (21) we have:

$$J_3(\omega)=\prod_{3\le P\le k}(P-1)\prod_{k<P}(P-1)(P-k+1)\to\infty \text{ as } \omega\to\infty. \quad (22)$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3,\ldots,P_k$ are primes for $3\le k<P_{g+1}$.

We have the best asymptotic formula [8]:

$$\pi_{k-1}(N,3)=|\{(j1)P_2(j2)P_1=\text{prime},\ 3\le j\le k, P_1, P_2\le N\}|$$

$$=\frac{J_3(\omega)\omega^{k-2}}{\varphi^k(\omega)}\frac{N^2}{\log^k N}(1+o(1)), \quad (23)$$

From eqn (23) we have the best asymptotic formula:

$$\pi_{k-1}(N,3)=\prod_{2\le P\le k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k<P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \quad (24)$$

From eqn (24) we are able to find the smallest solution $\pi_{k-1}(N_0,3)>1$ for large $k<P_{g+1}$.

Example 4. Suppose $k=3$ and $P_{g+1}>3$. From eqn (19) we have:

$P_3=2P_2-P_1.$ (25)

From eqn (22) we have:

$$J_3(\omega)=\prod_{3\le P}(P-1)(P-2)\to\infty \text{ as } \omega\to\infty, \quad (26)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 are primes. From eqn (24) we have the best asymptotic formula:

$$\pi_2(N,3)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N^2}{\log^3 N}(1+o(1))=1.32032\frac{N^2}{\log^3 N}(1+o(1)). \quad (27)$$

Example 5: Suppose $k=4$ and $P_{g+1}>4$. From eqn (19) we have:

$P_3=2P_2-P_1,\ P_4=3P_2-2P_1.$ (28)

From eqn (22) we have:

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-3)\to\infty \text{ as } \omega\to\infty, \quad (29)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 and P_4 are primes. From eqn (24) we have the best asymptotic formula:

$$\pi_3(N,3)=\frac{9}{2}\prod_{5\le P}\frac{P^2(P-3)}{(P-1)^3}\frac{N^2}{\log^4 N}(1+o(1)). \quad (30)$$

Example 6: Suppose $k=5$ and $P_{g+1}>5$. From eqn (19) we have:

$P_3=2P_2-P_1,\ P_4=3P_2-2P_1,\ P_5=4P_2-3P_1.$ (31)

From eqn (22) we have:

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-4)\to\infty \text{ as } \omega\to\infty, \quad (32)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3, P_4 and P_5 are primes. From eqn (24) we have the best asymptotic formula:

$$\pi_4(N,3)=\frac{27}{2}\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N^2}{\log^5 N}(1+o(1)). \quad (33)$$

Corollary 2: Arithmetic progression with three prime variables.

From eqn (18) we obtain the arithmetic progression with three prime Lie Theory variables: P_1, P_2 and P_3.

$P_4=P_3+P_2-P_1,\ P_j=P_3+(j-3)P_2-(j-3)P_1, 4\le j\le k<P_{g+1}$ (34)

We have Jiang function:

$$J_4(\omega)=\prod_{3\le P}((P-1)^3-X(P)), \quad (35)$$

$X(P)$denotes the number of solutions for the following congruence:

$$\prod_{j=4}^{k}(q_3+(j-3)q_2-(j-3)q_1)\equiv 0(\text{mod } P), \quad (36)$$

Where $q_i=1,2,\ldots,P-1$, $i=1,2,3$.

Example 7: Suppose $k=4$ and $P_{g+1}>4$. From eqn (34) we have:

$P_4=P_3+P_2-P_1.$ (37)

From eqns (35) and (36) we have:

$$J_4(\omega)=\prod_{3\le P}(P-1)(P^2-3P+3)\to\infty \text{ as } \omega\to\infty, \quad (38)$$

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that P_4 are primes. We have the best asymptotic formula:

$$\pi_2(N,4)=2\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\frac{N^3}{\log^4 N}(1+o(1)). \quad (39)$$

For $k\ge5$ from eqns (35) and (36) we have Jiang function:

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2\times\prod_{(k-1)\le P}(P-1)[(P-1)^2-(P-2)(k-3)]\to\infty \text{ as } \omega\to\infty. \quad (40)$$

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that $P_4,\ldots,P_k$ are primes for $5\le k<P_{g+1}$.

We have the best asymptotic formula:

$_{k-2}(N, 4) = |\{P_3+(j-3)P_2-(j-3)P_1=\text{prime}, 4 \le j \le k, P_1, P_2, P_3 \le N\}|$

$$= \frac{J_4(\omega)\omega^{k-3}}{\varphi^k(\omega)} \frac{N^3}{\log^k N}(1+o(1)). \quad (41)$$

From eqn (41) we have:

$$\pi_{k-2}(N,4) = \prod_{2\le P<(k-1)} \frac{P^{k-3}}{(P-1)^{k-2}} \prod_{(k-1)\le P} \frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}} \frac{N^3}{\log^k N}(1+o(1)). \quad (42)$$

From eqn (42) we are able to find the smallest solution $\pi_{k-2}(N_0, 4)>1$ for large $k<P_{g+1}$.

Corollary 3: Arithmetic progression with four prime variables.

From eqn (18) we obtain the arithmetic progression of algebra with four prime variables: P_1, P_2, P_3 and P_4

$$P_5=P_4+2P_3-3P_2+P_1,\ P_j=P_4+(j-3)P_3-(j-2)P_2+P_1,\ 5 \le j \le k < P_{g+1} \quad (43)$$

We have Jiang function:

$$J_5(\omega) = \prod_{3\le P}\left[(P-1)^4 - X(P)\right], \quad (44)$$

$X(P)$ denotes the number of solutions for the following congruence:

$$\prod_{j=5}^{k}[q_4+(j-3)q_3-(j-2)q_2+q_1] \equiv 0 \pmod P, \quad (45)$$

Where,

$q_i=1,\ldots,P-1, i=1,2,3,4$

Example 8: Suppose k=5 and $k<P_{g+1}$>5. From eqn (43) we have:

$$P_5=P_4+2P_3-3P_2+P_1. \quad (46)$$

From eqns (44) and (45) we have:

$$J_5(\omega) = 12\prod_{5\le P}(P-1)(P^3-4P^2+6P-4) \to \infty \text{ as } \omega\to\infty. \quad (47)$$

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that P_5 are primes.

We have the best asymptotic formula:

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{\varphi^5(\omega)} \frac{N^4}{\log^5 N}(1+o(1)). \quad (48)$$

Example 9: Suppose k=6 and P_{g+1} >6. From eqn (43) we have:

$$P_5=P_4+2P_3-3P_2+P_1,\ P_6=P_4+3P_3-4P_2+P_1. \quad (49)$$

From eqns (44) and (45) we have:

$$J_5(\omega) = 10\prod_{5\le P}(P-1)(P^3-5P^2+10P-9) \to \infty \text{ as } \omega\to\infty. \quad (50)$$

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that P_5 and P_6 are primes.

We have the best asymptotic formula:

$$\pi_3(N,5) = \frac{J_5(\omega)\omega^2}{\varphi^6(\omega)} \frac{N^4}{\log^6 N}(1+o(1)). \quad (50)$$

For k≥7 from eqns (44) and (45) we have Jiang function:

$$J_5(\omega) = 6\prod_{5\le P\le(k-4)}(P-1)(P^2-3P+3)$$

$$\times \prod_{(k-4)<P}\left\{(P-1)^4-(P-1)^2\left[(P-3)(k-4)+1\right]-(P-1)(2k-9)\right\} \to \infty \text{ as } \omega\to \quad (51)$$

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that $P_5,\ldots,P_k$ are primes.

We have best asymptotic formula:

$\pi_{k-3}(N, 5) = |\{P_4+(j-3)P_3-(j-2)P_2+P_1=\text{prime}, 5 \le j \le k, P_1,\ldots, P_4 \le N\}|$

$$= \frac{J_5(\omega)\omega^{h-4}}{\varphi^k(\omega)} \frac{N^4}{\log^k N}(1+o(1)). \quad (52)$$

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