On the Fundamental Theorem in Arithmetic Progression of Primes

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Abstract
Using Jiang function we prove the fundamental theorem in arithmetic progression of primes. The primes contain only \( k < P_i + 1 \) long arithmetic progression, but the primes have no \( k > P_i + 1 \) long arithmetic progressions theorem.

Keywords: Arithmetic; Lie theory; Fundamental theorem; Progression; Asymptotic formula

Theorem
The fundamental theorem in arithmetic progression of primes.

We define the arithmetic progression of primes [1-3].

\[
P_{i+1} = P_i + \omega_k, k=0, 1, 2, \ldots, \omega_{P_i} - 1,
\]

(1)

Where \( \omega_k = \prod_{P \leq k} P \) is called a common difference, \( P_{i+1} \) is called \( g \)-th prime.

We have Jiang function [1-3]:

\[
\omega_k \equiv (q + \omega_k) \equiv 0 \pmod{P},
\]

(2)

\( X(P) \) denotes the number of solutions for the following congruence:

\[
\prod_{i=1}^{k-1} (q + \omega_k) \equiv 0 \pmod{P}.
\]

(3)

Where \( q=1, 2, \ldots, P-1 \).

If \( P \mid \omega_k \), then \( X(P)=0, X(P)=k-1 \) otherwise. From eqn (3) we have:

\[
J_2(\omega) = \prod_{P \leq \omega_k} (P - 1) \prod_{P \leq -k} (P - k).
\]

(4)

If \( k= P_i \), then \( J_2(P_i) = 0, J_2(\omega) = 0 \), there exist finite primes \( P_i \) such that \( P_{i+1}, \ldots, P_{P_i} \) are primes. If \( k < P_i \), then \( J_2(\omega) \neq 0 \), there exist infinitely many primes \( P_i \) such that \( P_{i+1}, \ldots, P_{P_i} \) are primes. The primes contain only \( k < P_i \) long arithmetic progression, but the primes have no \( k > P_i \) long arithmetic progression geometry. We have the best asymptotic formula [1-3]:

\[
\pi_{\omega_k}(N, 2) = \frac{J_2(\omega) \omega_k^{-1}}{\phi(\omega)} \frac{N}{\log^2 N} (1 + o(1)).
\]

(5)

Where \( \omega = \prod_{P \leq \omega_k} P, \phi(\omega) = \prod_{P \leq -k} (P - 1), \omega \) is called primorial, \( \phi(\omega) \) Euler function.

Suppose \( k= P_i - 1 \). From eqn (1) we have:

\[
P_{i+1} = P_i + \omega_k, j=0, 1, 2, \ldots, P_i - 2.
\]

(6)

From eqn (4) we have [1-2]:

\[
J_2(\omega) = \prod_{P \leq \omega_k} (P - 1) \prod_{P \leq -P_i + 1} (P - P_{P_i}) \rightarrow \infty \quad \text{as} \quad \omega \rightarrow \infty
\]

(7)

We prove that there exist infinitely many primes \( P_i \) such that \( P_{i+1}, \ldots, P_{P_i} \) are primes for all \( P_i \).

From eqn (5) we have:

\[
\pi_{\omega_k-1}(N, 2) = \frac{\prod_{P \leq \omega_k} (P - 1) \prod_{P \leq -P_i + 1} (P - P_{P_i})}{(P - 1) \omega_k^{-1}} \frac{N}{\log^2 N} (1 + o(1)).
\]

(8)

From eqn (8) we are able to find the smallest solutions \( \pi_{\omega_k-1}(N, 2) > 1 \) for large \( P_{P_i} \).

Theorem is foundation for arithmetic progression of primes.

Example 1: Suppose \( P_i=2, \omega_i=2, P_{23}=3 \). From eqn (6) we have the twin primes theorem:

\[
P_i = P_i + 2.
\]

(9)

From eqn (7) we have:

\[
J_2(\omega) = \prod_{P \leq \omega_k} (P - 2) \rightarrow \infty \quad \text{as} \quad \omega \rightarrow \infty
\]

(10)

We prove that there exist infinitely many primes \( P_i \) such that \( P_i \) are primes. From eqn (8) we have the best asymptotic formula [4-6]:

\[
\pi_{\omega_k-1}(N, 2) = \frac{\prod_{P \leq \omega_k} (P - 1) \prod_{P \leq -P_i + 1} (P - P_{P_i})}{(P - 1) \omega_k^{-1}} \frac{N}{\log^2 N} (1 + o(1)).
\]

(11)

Twin prime theorem is the first theorem in arithmetic progression of primes.

Example 2: Suppose \( P_i=3, \omega_i=2, P_{23}=5 \). From eqn (6) we have:

\[
P_i = P_i + 6, i=0, 1, 2, 3.
\]

(12)

From eqn (7) we have:

\[
J_2(\omega) = \prod_{P \leq \omega_k} (P - 4) \rightarrow \infty \quad \text{as} \quad \omega \rightarrow \infty
\]

(13)

We prove that there exist infinitely many primes \( P_i \) such that \( P_i \) are primes. From eqn (8) we have the best generalized asymptotic formula:

\[
\pi_{\omega_k-1}(N, 2) = \frac{\prod_{P \leq \omega_k} (P - 1) \prod_{P \leq -P_i + 1} (P - P_{P_i})}{(P - 1) \omega_k^{-1}} \frac{N}{\log^2 N} (1 + o(1)).
\]

(14)

Example 3: Suppose \( P_i=23, \omega_i=223092870, P_{23}=29 \). From eqn (6) we have:

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From eqn (7) we have:
\[ J_z(\omega) = 36495360 \prod_{P \neq z} (P - 28) \to \infty \text{ as } \omega \to \infty. \]  
(16)

We prove that there exist infinitely many primes \( P \) such that \( P, P^3, \ldots, P^m \) are primes. From eqn (8) we have the best asymptotic formula [7]:
\[ \pi_m(N, 2) = \prod_{P \neq 2, 3} \left( \frac{P}{P - 1} \right)^{\frac{27}{\log N}} \prod_{P \neq 2, 3} \left( \frac{P^2 - (P - 28) N}{(P - 1)^2} \right)^{(1 + o(1))}. \]  
(17)

From eqn (17) we are able to find the smallest solutions \( \pi_m(N, 2) > 1 \).

On May 17, 2008, Wroblewski and Raanan Chermoni found the first known case of 25 primes:
\[ 6171109491283631 + 366384 \times \omega_5, \omega_7, \ldots, \omega_{24} \]  
for \( n = 0 \) to 24.

Theorem can help in finding for 26, 27, 28, \ldots, primes in arithmetic progressions of primes.

**Corollary 1:** Arithmetic progression with two prime variables.

Suppose \( \omega = d \). From eqn (1) we have:
\[ P_j P_k = P_j^{(k-1)} P_k - d, \ldots, P_j = P_j + (k-1) d, \quad (P, d) = 1 \]  
(18)

From eqn (18) we obtain the arithmetic progression with two prime variables: \( P_j, P_k, P_j^{(k-2)}, \ldots, P_j = P_j + (k-1) d \).

We have the best asymptotic function [3]:
\[ J_z(\omega) = \prod_{P \neq (P - 1)} \left( (P - 1) \right)^{-1} \right) = 3 \]  
(22)

We prove that there exist infinitely many primes \( P_j, P_k, P_{j+1}^{(k-2)} \) such that \( P_j, P_k, P_{j+1} \) are primes for \( 3 < k < P_{j+1} \).

We have the best asymptotic formula [8]:
\[ \pi_{x}(N, 3) = \frac{(1 + o(1))}{\log^3 N} \]  
(23)

From eqn (23) we have the best asymptotic formula:
\[ \pi_{x}(N, 3) = \prod_{P \neq (P - 1)} \left( \frac{P^2 - (P - 1)^2 + 1}{(P - 1)^2} \right) \frac{N^3}{\log^3 N} \left( 1 + o(1) \right). \]  
(24)

From eqn (24) we are able to find the smallest solution \( \pi_{x}(N, 3) > 1 \) for large \( k < P_{j+1} \).

**Example 4:** Suppose \( k = 3 \) and \( P_{j+1} > 3 \). From eqn (19) we have:
\[ P_j = 2P_j^3 - P_j. \]  
(25)

From eqn (22) we have:
\[ J_z(\omega) = \prod_{P \neq (P - 1)} (P - (P - 2)) = \infty \text{ as } \omega \to \infty. \]  
(26)

We prove that there exist infinitely many primes \( P_j, P_k, P_{j+1} \) such that \( P_j, P_k, P_{j+1} \) are primes. From eqn (24) we have the best asymptotic formula:
\[ \pi_{x}(N, 3) = 2 \prod_{P (P - 1)} \left( 1 + \frac{1}{(P - 1)} \right) \frac{N^3}{\log^3 N} \left( 1 + o(1) \right) \]  
(27)

**Example 5:** Suppose \( k = 4 \) and \( P_{j+1} > 4 \). From eqn (19) we have:
\[ P_j = 2P_j^3 - P_j, P_k = 3P_j^3 - 2P_j. \]  
(28)

From eqn (22) we have:
\[ J_z(\omega) = 2 \prod_{P \neq (P - 1)(P - 3)} \to \infty \text{ as } \omega \to \infty. \]  
(29)

We prove that there exist infinitely many primes \( P_j, P_k, P_{j+1} \) such that \( P_j, P_k, P_{j+1} \) are primes. From eqn (24) we have the best asymptotic formula:
\[ \pi_{x}(N, 3) = \frac{9}{2} \prod_{P \neq (P - 1)} \frac{P^4 - (P - 1)^4}{\log^3 N} \left( 1 + o(1) \right). \]  
(30)

**Example 6:** Suppose \( k = 5 \) and \( P_{j+1} > 5 \). From eqn (19) we have:
\[ P_j = 2P_j^3 - P_j, P_k = 3P_j^3 - 2P_j, P_{j+1} = 4P_j^3 - 3P_j. \]  
(31)

From eqn (22) we have:
\[ J_z(\omega) = 2 \prod_{P \neq (P - 1)(P - 4)} \to \infty \text{ as } \omega \to \infty. \]  
(32)

We prove that there exist infinitely many primes \( P_j, P_k, P_{j+1} \) such that \( P_j, P_k, P_{j+1} \) are primes. From eqn (24) we have the best asymptotic formula:
\[ \pi_{x}(N, 3) = \frac{27}{2} \prod_{P \neq (P - 1)} \frac{P^4 - (P - 1)^4}{\log^3 N} \left( 1 + o(1) \right). \]  
(33)

**Corollary 2:** Arithmetic progression with three prime variables.

From eqn (18) we obtain the arithmetic progression with three prime variables: \( P_j, P_k, P_{j+1} \).

We have the best asymptotic function [3]:
\[ J_z(\omega) = \prod_{P \neq (P - 1)^3} \left( 1 + o(1) \right) \]  
(35)

\[ X(P) \]  

\[ \prod_{j=1}^{\infty} (q_j + (j - 3) q_{j+1} - (j - 3) q_{j+2}) = 0 \mod P \]  
(36)

Where \( q_j = 1, 2, \ldots, P_{j+1} - 1, \quad i = 1, 2, 3. \)

**Example 7:** Suppose \( k = 4 \) and \( P_{j+1} > 4 \). From eqn (34) we have:
\[ P_j = 2P_j^3 - P_j. \]  
(37)

From eqns (35) and (36) we have:
\[ J_z(\omega) = \prod_{P \neq (P - 1)(P - 3)} \to \infty \text{ as } \omega \to \infty. \]  
(38)

We prove that there exist infinitely many primes \( P_j, P_k, P_{j+1} \) such that \( P_j, P_k, P_{j+1} \) are primes. We have the best asymptotic formula:
\[ \pi_{x}(N, 4) = \frac{2}{3} \prod_{P \neq (P - 1)} \left( 1 + o(1) \right) \]  
(39)

For \( k \geq 5 \) from eqns (35) and (36) we have Jiang function:
\[ J_z(\omega) = \prod_{P \neq (P - 1)(P - 3)} \to \infty \text{ as } \omega \to \infty. \]  
(40)

We prove that there exist infinitely many primes \( P_j, P_k, P_{j+1} \) such that \( P_j, P_k, P_{j+1} \) are primes for \( 5 \leq k < P_{j+1} \).
We have the best asymptotic formula:
\[ \pi_{\leq}(N, 4) = \left\lfloor \frac{P_k + (j-3)P_j - (j-3)P_j}{4} \right\rfloor \text{prime}, \quad 4 \leq j \leq k, P_j, P_{j}, P_k \leq N \]
\[ = \frac{J_4(\omega)\omega^{-3}}{\varphi(\omega)} \log^4 N (1 + o(1)). \]  
(41)

From eqn (41) we have:
\[ \pi_{\leq}(N, 4) = \prod_{P_j \leq x} P_j^3 \left[ (P - 1)^2 - (P - 2)(k - 3) \right] \frac{N^4}{\log^4 N (1 + o(1))}. \]  
(42)

From eqn (42) we are able to find the smallest solution \( \pi_{\leq}(N, 4) > 1 \)
for large \( k < P_{j+1} \).

**Corollary 3:** Arithmetic progression with four prime variables.

From eqn (18) we obtain the arithmetic progression of algebra with four prime variables:
\[ P_1, P_2, P_3, P_4 \]
\[ P_k = P_1 + 2P_2 - 3P_3 + P_4, P_k = P_1 + (j-3)P_j - (j-2)P_j + P_1, 5 \leq j \leq k \leq P_{j+1} \]  
(43)

We have Jiang function:
\[ J_4(\omega) = \frac{J_4(\omega)\omega^{-3}}{\varphi(\omega)} \log^4 N (1 + o(1)). \]  
(44)

\( X(P) \) denotes the number of solutions for the following congruence:
\[ \left\lfloor q_1 + (j - 3)q_j - (j - 2)q_j + q_j \right\rfloor = 0 \text{ (mod } P), \]  
(45)

Where,
\[ q_1 = 1, \ldots, P - 1, j = 1, 2, 3, 4 \]

**Example 8:** Suppose \( k = 5 \) and \( k < P_{j+1} > 5 \). From eqn (43) we have:
\[ P_k = P_1 + 2P_2 - 3P_3 + P_4 \]  
(46)

From eqns (44) and (45) we have:
\[ J_4(\omega) = \frac{J_4(\omega)\omega^{-3}}{\varphi(\omega)} \log^4 N (1 + o(1)). \]  
(48)

**Example 9:** Suppose \( k = 6 \) and \( P_{j+1} > 6 \). From eqn (43) we have:
\[ P_k = P_1 + 2P_2 + 3P_3 + P_4, P_k = P_1 + 3P_2 - 4P_3 + P_4 \]  
(49)

From eqns (44) and (45) we have:
\[ J_4(\omega) = \frac{J_4(\omega)\omega^{-3}}{\varphi(\omega)} \log^4 N (1 + o(1)). \]  
(50)

We prove there exist infinitely many primes \( P_j, P_{j+1}, P_3, P_4 \) such that \( P_j, P_{j+1} \) are primes.

We have the best asymptotic formula:
\[ \pi_{\leq}(N, 5) = \frac{J_4(\omega)\omega^{-3}}{\varphi(\omega)} \log^4 N (1 + o(1)). \]  
(51)

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**References**