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On the Existence of a Riemannian Manifolds at a Given Connection

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Abstract

We give necessary and sufficient conditions for a linear connection without torsion to come from a Riemannian metric. The nullity space and the image space of the curvature are involved in the formulation of the results.

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Introduction

A Riemannian manifold defines a unique linear connection without torsion and conservative. By studying the properties of the space of nullity and images of the curvatures, we reduce the resolution of the inverse problem to a system of linear equations with some properties of the curvatures quite simple to verify.

The formalism used is that of [1] and [2] and the formulas are those of [3], the method following [4].

Canonical Connection of a Riemannian Manifold

Let M be a n dimensional differentiable paracompact manifold and of class $\mathcal{C}^\infty$, J is the natural tangent structure of the tangent bundle: TM ⟶ M. The derivation d_J [5] is defined by $d_J=[i_J, d]$.

Definition 1

We call the Riemannian manifold, [1], the couple (M, E):

- M a differentiable manifold
- E a function of $\mathcal{TM}=TM-\{0\}$ in $\mathbb{R}^+$, with E(0)=0, $\mathcal{C}^\infty$ on $\mathcal{TM}$, $\mathcal{C}^2$ on the null section, and homogeneous of degree two such that dd_JE has a maximum rank

The application E is called energy function, the fundamental scalar 2-form $\Omega=dd_JE$ defines a spray S [6]:

$$i_S dd_J E=-dE, \tag{2.1}$$

the derivation i_s being the inner product with respect to S. The vector 1-form $\Gamma=[J, S]$ is called canonical connection. The connection Γ thus defined is an almost product structure: $\Gamma^2=I$, I is the vector 1-form identity. By asking

$$h=\frac{1}{2}(I+\Gamma)\, and\, v=\frac{1}{2}(I-\Gamma),$$

h is the horizontal projector, projector of the proper subspace at the eigenvalue +1, v the vertical projector corresponding to the eigenvalue-1. The curvature of Γ is defined by

$$R=\frac{1}{2}[h,h] \tag{2.2}$$

The vector 2-form R is also called the Nijenhuis tensor of h. The scalar 2-form allows to define a metric g on the vertical bundle by

g(JX, JY) = Ω (JX, Y)

for all $X,Y\in\chi(TM)$, where $\chi(TM)$ denotes the set of vector fields on TM.

There exists, [2], one and only one metric lift D of the canonical connection such that:

1. DJ=0;
2. DC=v, (C being the Liouville canonical field on TM);
3. DΓ=0;
4. Dg=0.

The D connection is called the Cartan connection. We have

$$D_{JX}JY=[J,JY]X, \quad D_{hX}JY=[h,JY]X. \tag{2.3}$$

With the linear connection D, we associate a curvature

$$\mathfrak{R}(X,Y)Z=D_{hX}D_{hY}JZ-D_{hY}D_{hX}JZ-D_{[hX,hY]}JZ,$$

for all $X,Y,Z\in\chi(TM)$. The relation between the curvatures $\mathfrak{R}$ and *R* is [2]:

$$\mathfrak{R}(X,Y)Z=J[Z,R(X,Y)]-[JZ,R(X,Y)]+R([JZ,X],Y)+R(X,[JZ,Y]). \tag{2.4}$$

In particular,

$$\mathfrak{R}(X,Y)S=-R(X,Y). \tag{2.5}$$

In natural local coordinates on an open set U of M, $(x^i, y^i)\in TU$, i, j ∈ { 1,...,n}. The energy function is written [1] p.330

$$E=\frac{1}{2}g_{ij}(x^1,...,x^n)y^iy^j$$

where $g_{ij}(x^1,..., x^n)$ are symmetric positive functions such that the matrix $(g_{ij}(x^1,..., x^n)_{i,j})$ is invertible. And the relation $i_S dd_J E=-dE$ gives the spray S

$$S=y^i\frac{\partial}{\partial x^i}-2G^i(x^1,...,x^n,y^1,...,y^n)\frac{\partial}{\partial y^i}$$

with

$$G_k=\frac{1}{2}y^iy^j\gamma_{ikj}$$

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where

$$\gamma_{ikj}=\frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right).$$

By asking

$$\gamma_{ij}^k=g^{kl}\gamma_{ilj}$$ i, j, k, l∈ {1,...,n},

we have

$$G^k=\frac{1}{2}y^iy^j\gamma_{ij}^k.$$

We note $\Gamma_i^j(x,y)=y^l\Gamma_{il}^j(x)$, the horizontal projector is written:

$$h\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial x^i}-\Gamma_i^j\frac{\partial}{\partial y^j},\quad h\left(\frac{\partial}{\partial y^j}\right)=0.$$

The vertical projector becomes

$$v\left(\frac{\partial}{\partial x^i}\right)=\Gamma_i^j\frac{\partial}{\partial y^j},\quad v\left(\frac{\partial}{\partial y^i}\right)=\frac{\partial}{\partial y^i}.$$

The curvature R= $\frac{1}{2}$[h, h] is then

$$R=\frac{1}{2}R_{ij}^k dx^i\wedge dx^j\otimes\frac{\partial}{\partial y^k}$$

with

$$R_{ij}^k=\frac{\partial\Gamma_i^k}{\partial x^j}-\frac{\partial\Gamma_j^k}{\partial x^i}+\Gamma_i^l\frac{\partial\Gamma_j^k}{\partial y^l}-\Gamma_j^l\frac{\partial\Gamma_i^k}{\partial y^l},\ i,j,k,l\in\{1,\dots,n\}.$$

Proposition 1

Let E be an energy function, Γ a connection such that Γ=[J,S]. The following two relationships are equivalent:

i. $i_S dd_J E=-dE$;

ii. $d_h E=0$.

Proof: We notice that if we have the relation (i), we get

$$i_S i_S dd_J E=0=L_S E. \tag{2.6}$$

We can write successively, taking into account that E is homogeneous of degree 2 and that JS=C,

$$i_S dd_J E=(L_S-di_S)d_J E=L_S d_J E-2dE=-dE$$

or

$$L_S d_J E=dE. \tag{2.7}$$

From the formula $L_S d_J-d_J L_S=d_{[S,J]}$, we have

$$L_S d_J E=d_{[S,J]}E+d_J L_S E=(2d_h E-dE)+d_J L_S E.$$

According to the two relations (2.6) and (2.7), we find

$$d_h E=0.$$

Conversely, if we have $d_h E=0$, by definition $h=\frac{[J,S]+I}{2}$, we can write

$$d_{[J,S]}E+dE=0.$$

By noting that $hS=S$ and $L_S E=0$, we can go up the calculation and one obtains

$$i_S dd_J E=-dE$$

Properties of the Curvature $\mathfrak{R}$ of a Riemannian Manifold

From the properties of the connection stated above, we obtain the following classic results: for all $X,Y,Z,T\in\chi(TM)$

$$g(\mathfrak{R}(X,Y)Z,JT)=-g(\mathfrak{R}(X,Y)T,JZ); \tag{3.1}$$

$$\mathfrak{R}(X,Y)Z+\mathfrak{R}(Y,Z)X+\mathfrak{R}(Z,X)Y=0; \tag{3.2}$$

$$g(\mathfrak{R}(T,X)Y,JZ)+g(\mathfrak{R}(T,Y)Z,JX)+g(\mathfrak{R}(T,Z)X,JY)=0; \tag{3.3}$$

$$g(\mathfrak{R}(X,Y)Z,JT)=g(\mathfrak{R}(Z,T)X,JY). \tag{3.4}$$

Properties of the Horizontal Nullity Space of Curvatures

From the properties of the curvatures given above and by posing:

$$N_R=\{X\in\chi(TM)\ such\ that\ R(X,Y)=0,\forall Y\in\chi(TM)\},$$
$$N_{\mathfrak{R}}=\{X\in\chi(TM)\ such\ that\ \mathfrak{R}(X,Y)Z=0,\forall Y,Z\in\chi(TM)\};$$

we have

$$JX\perp ImR^{\circ}\Leftrightarrow X\in KerR^{\circ}\left(R^{\circ}=i_S R\right),X\in\chi(TM); \tag{4.1}$$

$$JX\perp ImR\Leftrightarrow\mathfrak{R}(S,X)Y=0,X\in\chi(TM),\forall Y\in\chi(TM); \tag{4.2}$$

$$X\in N_R\Leftrightarrow JX\perp\mathfrak{R}(S,Y)Z,X\in\chi(TM),\forall Y,Z\in\chi(TM); \tag{4.3}$$

$$X\in N_{\mathfrak{R}}\Leftrightarrow JX\perp\mathfrak{R}(Y,Z)T,X\in\chi(TM),\forall Y,Z,T\in\chi(TM); \tag{4.4}$$

$$X\in N_{\mathfrak{R}}\Leftrightarrow\mathfrak{R}(Y,Z)X=0,\forall Y,Z\in\chi(TM). \tag{4.5}$$

Properties of the Nullity Space and the Curvature Image Space

Let H=Imh the horizontal space, V=Imv the vertical space and ImR (respectively Im$\mathfrak{R}$) the module on *F*(TM) generated by the image of the curvature R (respectively $\mathfrak{R}$).

Proposition 2

On a Riemannian manifold (M, E) the following properties are verified:

1) $\mathrm{Im}\,\mathfrak{R}$ and $JN_{\mathfrak{R}}$ are orthogonal and $\mathrm{Im}\,\mathfrak{R}\oplus JN_{\mathfrak{R}}=V$;

2) ImR and JN_R+{C} are orthogonal; ImR $\oplus$ {JN_R+{C}}=V;

3) $N_R=N_{\mathfrak{R}}$

4) At the neighborhood of a point z of TM, the subspaces $N_R\cap H, N_{\mathfrak{R}}\cap H, N_R, N_{\mathfrak{R}}, hN_R\oplus JN_R$, ImR and $\mathrm{Im}\,\mathfrak{R}$ are involutive

Proof: The property 1) follows from the relation (4.4).

According to the relation (4.2), JX $\perp$ ImR if and only if $\mathfrak{R}(S,X)Y=0,\forall Y\in\chi(TM)$. From the relation (2.4), we have

$$\mathfrak{R}(S,X)Y=J[Y,R^{\circ}(X)]-[JY,R^{\circ}(X)]+R([JY,S],X)+R^{\circ}([JY,X])=0.$$

Taking into account the relation (4.1), the relation $J[JY,S]=JY,\forall Y\in\chi(TM)$ and that the curvature R is semi-basic, we get

$$R(X,Y)=R^{\circ}[JY,X],\forall Y\in\chi(TM) \tag{5.1}$$

As R is alternating bilinear on the vector fields of $\chi(TM)$ considered as $\mathcal{F}$(TM)-module, the relation is not possible unless X=S or, if X ∈ N_R, and if X is generated by projectable vector fields in the nullity space of R. The converse is immediate according to the relations (4.2), (4.3) and (3.3).

The property $N_R=N_{\mathfrak{R}}$ follows from the link between $\mathfrak{R}$ and R given by the relation (2.4) and from the above remark that N_R is generated by projectable vector fields in N_R. At the neighborhood of a point z of TM, the subspaces $N_R\cap$ H, $N_{\mathfrak{R}}\cap H$, are involutive. These are well known results see [7].

Let us now show that $hN_R \oplus JN_R$ is involutive. You just have to check on the generators of hN_R and JN_R. Since hN_R is generated by projectable vector fields in hN_R and, hN_R is involutive, then JN_R is generated by the commutative generators in JN_R. JN_R is therefore involutive. It is the same for [X, JY] with $X \in hN_R$ and $Y \in JN_R$. Let K, L $\in$ H such that JK and JL be orthogonal to $JN_{\mathfrak{R}}$. As we have $D_{JZ}g=0$ for all Z $\in$ H, and taking into account $J[JZ, X] \in JN_{\mathfrak{R}}$ for all $X \in N_{\mathfrak{R}}$, we have

g(J[JK, L], JX)=0, g(J[K, JL], JX)=0,

that is to say, according to the nullity of the Nijenhuis tensor of J, we have

[JK, JL]=J[JK, L]+J[K, JL] $\in JN_{\mathfrak{R}}$

This proves that $\mathrm{Im}\,\mathfrak{R}$ is involutive.

The space ImR is also involutive, this follows from the assertion 3) and from the following property:

$$g([JY, JZ], C) = 0, \textit{for all } JY, JZ \in \mathrm{Im}\, R.$$

Proposition 3

Let X be a projectable vector field. The following two relationships are equivalent:

(i) [hX, J]=0

(ii) [JX, h]=0

Proof: For a connection Γ=[J,S], the torsion is zero, that is,

[J,h](X,Y)=0

for all $X, Y \in \chi(TM)$. By developing the above relation, we have,

[JX, hY]+[hX, JY]+Jh[X,Y]+hJ[X,Y]-J[hX, Y]-J[X, hY]-h[JX,Y]-h[X, JY]=0

as we have hJ=0 and, h[X,JY]=0, because X being projectable vector fields by hypothesis, we have

[JX, h]Y+[hX, J]Y=J[X, h]Y

The vector field X being projectable and Jh=J, the term J[X, h] is null. We obtain

[JX, h]Y+[hX, J]Y = 0

Hence the equivalence of the relations (i) and (ii).

Proposition 4

If the rank of the horizontal nullity space hN_R of the curvature R is constant, there is a local base of hN_R verifying the proposition 3.

Proof: We will solve the equation

[hX, J]=0

The vector fields which annihilate the tangent structure by the Lie derivative is well known [8]. The equation to be solved is written in natural local coordinates on an open set U $\subset$ M, $(x^i, y^j) \in$ TU, i, j $\in$ {1,..., n}

$$X^i(x)\frac{\partial}{\partial x^i} - X^i(x)\Gamma_i^j\frac{\partial}{\partial y^j} = X^i(x)\frac{\partial}{\partial x^i} + y^i\frac{\partial X^j(x)}{\partial x^i}\frac{\partial}{\partial y^j}$$

that is to say:

$$y^i\frac{\partial X^j(x)}{\partial x^i} = -X^i(x)\Gamma_i^j.$$

As we have $y^l\Gamma_{il}^j(x) = \Gamma_i^j(x,y)$, and $\Gamma_{il}^j = \Gamma_{li}^j$, the previous equation is equivalent to

$$\frac{\partial X^j(x)}{\partial x^i} = -X^l(x)\Gamma_{li}^j.$$

The condition for compatibility of the equation, according to the Frobenius theorem, is

$$X^l\left(\frac{\partial \Gamma_{li}^j}{\partial x^k} - \frac{\partial \Gamma_{lk}^j}{\partial x^i} + \Gamma_{li}^s\Gamma_{sk}^j - \Gamma_{lk}^s\Gamma_{si}^j\right) = 0, \quad i,j,k,l,s \in \{1,...,n\}.$$

This condition is none other than X $\in hN_R$. Hence the result.

Proposition 5

On a Riemannian manifold (M, E), the space H $\oplus$ ImR is involutive if the rank of the space hN_R is constant.

Proof: It is clear that for all $X, Y \in \chi(TM)$,

[hX, hY]=h[hX,hY]+v[hX, hY]=h[hX, hY]+R(X,Y)

So [hX, hY] $\in$ H $\oplus$ ImR.

According to the proposition 2, it suffices to show that v[hX, R(Y,Z)] is orthogonal to JN_R+{C}. We have

$$D_{hX}g(R(Y,Z),C) - g(D_{hX}R(Y,Z),C) - g(R(Y,Z),D_{hX}C) = 0.$$

The relation (2.3) gives

$$D_{hX}R(Y,Z) = v[hX,R(Y,Z)] \text{and } D_{hX}C = [h,C]X.$$

The homogeneity of h leads to [h, C]=0, we get

$$g(v[hX,R(Y,Z)],C) = 0.$$

Let T$\in hN_R$, then we can write

$$D_{hX}g(R(Y,Z),JT) - g(D_{hX}R(Y,Z),JT) - g(R(Y,Z),D_{hX}JT) = 0.$$

As we have $g(R(Y,Z),JT) = 0, D_{hX}R(Y,Z) = v[hX,R(Y,Z)]$ and $D_{hX}JT = [h,JT]X$, we obtain

$$g(v[hX,R(Y,Z)],JT) + g(R(Y,Z),v[h,JT]X) = 0.$$

According to the proposition 4, we have v[h, JT]X $\in JN_R$, therefore

$$g(v[hX,R(Y,Z)],JT) = 0.$$

Hence the result.

Proposition 6

On a Riemannian manifold (M, E), the space H $\oplus$ ImR $\oplus$ JN_R is involutive if the rank of the space hN_R is constant.

Proof: The relation (2.4) is written:

$\mathfrak{R}$(X, Y)Z=J[Z, R(X, Y)]-[JZ, R(X, Y)]+R([JZ, X], Y)+R(X, [JZ, Y]).

If we have Z $\in hN_R$, we get $\mathfrak{R}$(X, Y)Z=0 according to the relation (4.5). As the elements of hN_R are generated by projectable vector fields in hN_R, the relation above shows that [JZ, R(X,Y)] $\in$ ImR $\oplus$ JN_R, for all $X, Y \in \chi(TM)$. The proposition 5 shows that H $\oplus$ ImR is involutive. We will now show that v[JZ, hX] $\in JN_R$. According to the proposition 4, we can find a base of hN_R verifying the proposition 3. This proves that v[JZ, hX] $\in$ JN_R.

Hence the result.

Proposition 7

Let Γ be a linear connection without torsion satisfying the following relationships:

1) there is an energy function $E_\circ$ which satisfies $d_R E_\circ = 0$;

2) the space H $\oplus$ ImR is involutive;

3) the vertical space V decomposes into V = ImR $\oplus$ {JN_R +{C}} and the rank of JN_R is constant, then the 1-form $d_v E_\circ$ is completely integrable

Proof: The kernel of $d_v E_\circ$ is H the horizontal space of Γ since we have $v \circ h = 0$, ImR by hypothesis, and the vertical vector fields such that $L_{JX}E_\circ = 0$. We will show that these vector fields are involutive.

We can write

[hX, hY]=h[hX, hY]+v[hX, hY]=h[hX, hY]+R(X, Y)

So

[hX, hY] $\in$H $\oplus$ ImR

Let $v_\circ$ be the vertical projector of the linear connection given by $E_\circ, d_v E_\circ$ and $d_{v_\circ} E_\circ$ are identical on the vertical space JH since $vJ = v_\circ J = J$. The difference between $\Gamma_\circ$ linear connections coming from the energy function $E_\circ$ and Γ is in the horizontal space $\operatorname{Im} h_\circ$ and Imh, the two connections have the same curvature. It remains to calculate v[JY, hX] for all $X, Y \in \chi(TM)$. If JY $\in$ ImR, v[JY, hX] $\in$ ImR by hypothesis. If JY $\in$ JN_R, according to the proposition 4, v[JY, hX] $\in$ JN_R and that H $\oplus$ ImR $\oplus$ JN_R is also involutive according to the proposition 6. The scalar 1-form $d_v E_\circ$ is completely integrable like the scalar 1-form $d_{v_\circ} E_\circ$.

Theorem 1

Let Γ be a linear connection without torsion such that the rank of the curvature R is constant. The connection Γ comes from an energy function if and only if

1) the distribution H $\oplus$ ImR is involutive and the vertical space V=ImR $\oplus$ {JN_R+{C}} with hN_R is generated by projectable vector fields in hN_R

2) there is an energy function $E_\circ$ such that $d_R E_\circ = 0$.

Then, there exist a constant φ function on the bundles such that $e^{\varphi(x)} E_\circ$ is the energy function of Γ

Proof: A necessary and sufficient condition for a connection Γ to come from an energy function E is according to the proposition 1,

$d_h E=0$

This results in

$d_h d_h E=0=2d_R E$

The conditions are therefore necessary according to the previous studies. Conversely, let $E_\circ$ be an energy function such that $d_R E_\circ = 0$. We will show that with the hypothesis, there exist a constant φ function on the bundles such that

$$d_h(e^\varphi E_\circ) = 0.$$

The equation is equivalent to

$$d\varphi = -\frac{1}{E_\circ} d_h E_\circ.$$

The condition of integrability of such an equation is

$$d\left(\frac{1}{E_\circ}\right) \wedge d_h E_\circ + \frac{1}{E_\circ} dd_h E_\circ = 0,$$

that is to say

$$dd_h E_\circ = \frac{dE_\circ}{E_\circ} \wedge d_h E_\circ.$$

According to the proposition 7, $d_v E_\circ$ is completely integrable. We have, according to the Frobenius theorem,

$$dd_v E_\circ \wedge d_v E_\circ = 0,$$

By applying the inner product i_C to the above equality, we obtain

$$dd_v E_\circ = \frac{dE_\circ}{E_\circ} \wedge d_v E_\circ,$$

that is to say

$$dd_h E_\circ = \frac{dE_\circ}{E_\circ} \wedge d_h E_\circ.$$

This is the condition of integrability sought.

Corollary

Let M be a differentiable, paracompact, connected manifold of dimension n $\geq$ 2, Γ a linear connection without torsion such that the rank of the curvature is locally constant over M. Then, Γ is a canonical connection of a Riemannian manifold if and only if at the neighborhood of any point where the rank of the curvature R is constant;

1) The distribution H $\oplus$ ImR is involutive and V=ImR $\oplus$ {JN_R +{C}} with hN_R is generated by projectable vector fields in hN_R

2) There is an energy function $E_\circ$ such that $d_R E_\circ = 0$

Proof: It follows from the theorem 1. Using a partition of the unit, we glue together the local metrics to have a global metric.

The search for an energy function E of Γ leads to the search for an energy function $E_\circ$ such that $d_R E_\circ = 0$. In natural local coordinates, the curvature R is written:

$$R = \frac{1}{2} y^l R^k_{l,ij}(x) dx^i \wedge dx^j \otimes \frac{\partial}{\partial y^k}, i, j, k, l \in \{1, \ldots, n\}.$$

An energy function is written

$$E_\circ = \frac{1}{2} g^\circ_{ij} y^i y^j.$$

Thus, the relation $d_R E_\circ = 0$ is equivalent to the following system of equations:

$$\begin{cases} g^\circ_{kl} R^k_{l,ij} = 0 \\ g^\circ_{kl} R^k_{r,ij} = -g^\circ_{kr} R^k_{l,ij} \ with\, l \neq r. \end{cases}$$

In the matrix form, the system is written:

$$\begin{pmatrix} g^\circ_{11} \cdots g^\circ_{n1} \\ \vdots \ \ddots \ \vdots \\ g^\circ_{1n} \cdots g^\circ_{nn} \end{pmatrix} \begin{pmatrix} R^1_{1,ij} \cdots R^1_{n,ij} \\ \vdots \ \ddots \ \vdots \\ R^n_{1,ij} \cdots R^n_{n,ij} \end{pmatrix} = \begin{pmatrix} 0 & & \\ & \ddots \ -{}^t A & \\ A & \ddots & \\ & & 0 \end{pmatrix}$$

Remark: Solving the inverse problem leads to the system of linear equations, the other conditions are easy to verify, which is quite easy, in practice. In addition, the trace of $R^k_{k,ij}$ is zero.

On an Isotropic Finslerian Manifold [9]

An isotropic finslerian manifold is a manifold [10] with a connection such that

$$R^\circ = 2EKJ - Kd_J E \otimes C,$$

$R^\circ = i_s R, K$ denotes the sectional curvature.

If K $\neq$ 0, the manifold is of regular curvature, that is to say, the dimension of ImR is equal to n-1. Our theorem remains valid. In the case of a finslerian manifold of dimension 2, the theorem is written.

Theorem 2 [11]

Let M be a connected paracompact differentiable manifold of dimension 2, Γ a connection without torsion with non-zero curvature over a dense subset of $\mathcal{T}$M=TM-{0}.

Γ comes from a finslerian structure if and only if, at the neighborhood of any point where the curvature is non-zero,

i. H $\oplus$ ImR is involutive

ii. The contracted of the curvature $\tilde{R}$ is such $d\tilde{R}$ is of maximum rank and the function $i_S \tilde{R}$ positive

To illustrate the results, we will give some examples.

Example 1

Let the manifolds M=$\mathbb{R}^3$ have coordinates (x^1, x^2, x^3) $\in \mathbb{R}^3$, (y^1, y^2, y^3) on the bundle of the tangent space T$\mathbb{R}^3$, Γ the connection [J, S] with

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3} - 2((y^1)^2 e^{x^3} + y^2 y^3)\frac{\partial}{\partial y^1}.$$

The coefficients Γ_i^j of Γ being $\Gamma_i^j = \frac{\partial G^j}{\partial y^i}$, the non-zero coefficients are

$$\Gamma_1^1 = 2y^1 e^{x^3}, \Gamma_2^1 = y^3, \Gamma_3^1 = y^2.$$

A base of the horizontal space of Γ is written

$$\frac{\partial}{\partial x^1} - 2y^1 e^{x^3}\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^2} - y^3\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^3} - y^2\frac{\partial}{\partial y^1}$$

The curvature R is written:

$$R = e^{x^3} dx^1 \wedge ((y^1 - y^2)dx^3 - y^3 dx^2) \otimes \frac{\partial}{\partial y^1}.$$

The horizontal nullity space is generated by

$$(y^1 - y^2)\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial x^3} - [(y^2)^2 + (y^1 - y^2)y^3]\frac{\partial}{\partial y^1}.$$

The horizontal nullity space is not generated by projectable vector fields in hN_R, N_R is not involutive. This linear connection according to the proposition 2 cannot come from an energy function.

Example 2

We take $M = \mathbb{R}^4$ and the energy function is written:

$$E = \frac{1}{2}(e^{x^1}(y^1)^2 + e^{x^2}(y^2)^2 + e^{x^3}(y^3)^2 + e^{2x^1}(y^4)^2).$$

The canonical spray of E is written:

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3} + y^4\frac{\partial}{\partial x^4} - (\frac{1}{2}(y^1)^2 - (y^4)^2 e^{x^1})\frac{\partial}{\partial y^1} - \frac{1}{2}(y^2)^2\frac{\partial}{\partial y^2} - \frac{1}{2}(y^3)^2\frac{\partial}{\partial y^3}.$$

The non-zero coefficients of Γ are

$$\Gamma_1^1 = \frac{1}{2}y^1, \Gamma_4^1 = -y^4 e^{x^1}, \Gamma_2^2 = \frac{1}{2}y^2, \Gamma_3^3 = \frac{1}{2}y^3, \Gamma_1^4 = y^4, \Gamma_4^4 = y^1.$$

The horizontal fields are generated by

$$\frac{\partial}{\partial x^1} - \frac{1}{2}y^1\frac{\partial}{\partial y^1} - y^4\frac{\partial}{\partial y^4}, \frac{\partial}{\partial x^2} - \frac{1}{2}y^2\frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^3} - \frac{1}{2}y^3\frac{\partial}{\partial y^3}, \frac{\partial}{\partial x^4} + y^4 e^{x^1}\frac{\partial}{\partial y^1} - y^1\frac{\partial}{\partial y^4}$$

The horizontal nullity space is generated by

$$\frac{\partial}{\partial x^2} - \frac{1}{2}y^2\frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^3} - \frac{1}{2}y^3\frac{\partial}{\partial y^3}.$$

The two basic elements of the nullity space verifying the proposition 3 are:

$$e^{\frac{-x^2}{2}}(\frac{\partial}{\partial x^2} - \frac{1}{2}y^2\frac{\partial}{\partial y^2}), e^{\frac{-x^3}{2}}(\frac{\partial}{\partial x^3} - \frac{1}{2}y^3\frac{\partial}{\partial y^3}).$$

The non-zero elements of the curvature are:

$$R_{4,14}^1 = -\frac{e^{x^1}}{2}, R_{1,14}^4 = \frac{1}{2}.$$

and $\operatorname{Im} R = \left\langle -y^4 e^{x^1}\frac{\partial}{\partial y^1} + y^1\frac{\partial}{\partial y^4}\right\rangle \mathcal{F}(T\mathbb{R}^4)$.

We see that H $\oplus$ ImR is involutive.

The matrix is written

$$\begin{pmatrix} e^{x^1} & 0 & 0 & 0 \\ 0 & e^{x^2} & 0 & 0 \\ 0 & 0 & e^{x^3} & 0 \\ 0 & 0 & 0 & e^{2x^1} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -\frac{e^{x^1}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\frac{e^{2x^1}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{e^{2x^1}}{2} & 0 & 0 & 0 \end{pmatrix}$$

Example 3

We take M=$\mathbb{R}^4$ and the energy function is written:

$$E = \frac{1}{2}(e^{x^4}(y^1)^2 + e^{x^4}(y^2)^2 + e^{x^3}(y^3)^2 + (y^4)^2).$$

The canonical spray of E is written:

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} + y^3\frac{\partial}{\partial x^3} + y^4\frac{\partial}{\partial x^4} - y^1 y^4\frac{\partial}{\partial y^1} - y^2 y^4\frac{\partial}{\partial y^2} - \frac{1}{2}(y^3)^2\frac{\partial}{\partial y^3} + \frac{e^{x^4}((y^1)^2 + (y^2)^2)}{2}\frac{\partial}{\partial y^4}.$$

The non-zero coefficients of Γ are

$$\Gamma_1^1 = \frac{1}{2}y^4, \Gamma_4^1 = \frac{1}{2}y^1, \Gamma_2^2 = \frac{1}{2}y^4, \Gamma_4^2 = \frac{1}{2}y^2, \Gamma_3^3 = \frac{1}{2}y^3, \Gamma_1^4 = -\frac{e^{x^4}y^1}{2}, \Gamma_2^4 = -\frac{e^{x^4}y^2}{2}.$$

The horizontal fields are generated by

$$\frac{\partial}{\partial x^1} - \frac{1}{2}y^4\frac{\partial}{\partial y^1} + \frac{e^{x^4}y^1}{2}\frac{\partial}{\partial y^4}, \frac{\partial}{\partial x^2} - \frac{1}{2}y^4\frac{\partial}{\partial y^2} + \frac{e^{x^4}y^2}{2}\frac{\partial}{\partial y^4}, \frac{\partial}{\partial x^3} - \frac{1}{2}y^3\frac{\partial}{\partial y^3},$$

$$\frac{\partial}{\partial x^4} - \frac{1}{2}y^1\frac{\partial}{\partial y^1} - \frac{1}{2}y^2\frac{\partial}{\partial y^2}.$$

The horizontal nullity space is generated by

$$\frac{\partial}{\partial x^3} - \frac{1}{2}y^3\frac{\partial}{\partial y^3}.$$

The basic element of the nullity space verifying the proposition 3 is:

$$e^{\frac{-x^3}{2}}(\frac{\partial}{\partial x^3} - \frac{1}{2}y^3\frac{\partial}{\partial y^3}).$$

The non-zero elements of the curvature are:

$$R_{2,12}^1 = -\frac{e^{x^4}}{4}, R_{4,14}^1 = -\frac{1}{4}, R_{1,12}^2 = \frac{e^{x^4}}{4},$$

$$R_{4,24}^2 = -\frac{1}{4}, R_{1,14}^4 = \frac{e^{x^4}}{4}, R_{2,24}^4 = \frac{e^{x^4}}{4}$$

And ImR generated by

$$\left\langle e^{x^4}(-y^2\frac{\partial}{\partial y^1} + y^1\frac{\partial}{\partial y^2}), -y^4\frac{\partial}{\partial y^1} + y^1 e^{x^4}\frac{\partial}{\partial y^4}, -y^4\frac{\partial}{\partial y^2} + y^2 e^{x^4}\frac{\partial}{\partial y^4}\right\rangle$$

$\mathcal{F}$(T$\mathbb{R}^4$).

We see that H $\oplus$ ImR is involutive.

The matrix is written

$$(g_{ij}) \times (R_{j,12}^i) = \begin{pmatrix} e^{x^4} & 0 & 0 & 0 \\ 0 & e^{x^4} & 0 & 0 \\ 0 & 0 & e^{x^3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\frac{e^{x^4}}{4} & 0 & 0 \\ \frac{e^{x^4}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{e^{2x^4}}{4} & 0 & 0 \\ \frac{e^{2x^4}}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(g_{ij}) \times (R_{j,14}^i) = \begin{pmatrix} e^{x^4} & 0 & 0 & 0 \\ 0 & e^{x^4} & 0 & 0 \\ 0 & 0 & e^{x^3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{e^{x^4}}{4} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\frac{e^{x^4}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{e^{x^4}}{4} & 0 & 0 & 0 . \end{pmatrix},$$

$$(g_{ij}) \times (R_{j,24}^i) = \begin{pmatrix} e^{x^4} & 0 & 0 & 0 \\ 0 & e^{x^4} & 0 & 0 \\ 0 & 0 & e^{x^3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{e^{x^4}}{4} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{e^{x^4}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{e^{x^4}}{4} & 0 & 0 \end{pmatrix}$$

Conclusion

To recognize a connection without torsion, that is to say,

$\Gamma = [J,S]\, and\, [C,S] = S$

comes from an energy function.

Case 1

The connection is of regular curvature, that is to say, the module generated on $\mathcal{F}$(TM) by ImR is of dimension n-1. The necessary and sufficient condition is written:

i. H ⊕ ImR is involutive

ii. There is an energy function $E_\circ$ such that $d_R E_\circ = 0$

Indeed, according to proposition 1, we have $d_h E=0$, this implies $d_R E=0$. So H ⊕ ImR is the kernel of dE. Then, H ⊕ ImR is involutive, and $d_v E_\circ$ is a completely integrable 1-form.

This result is valid for a Finslerian manifold.

Case 2

dim(ImR)<n-1. The necessary and sufficient condition becomes

i. The horizontal nullity space of the curvature R is generated by projectable vector fields in hN_R, the space H ⊕ ImR is involutive and the vertical space is written V=ImR ⊕ {JN_R+{C}}

ii. There exists an energy function $E_\circ$ such that $d_R E_\circ = 0$

Indeed, it is essential to have the horizontal nullity space of the curvature generated by projectable vector fields in hN_R. This assertion involves that N_R is involutive. With the help of the metric g, we have H ⊕ ImR is involutive, the scalar 1-form $d_v E_\circ$ is then completely integrable.

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