Journal of
Applied & Computational Mathematics

Azkour, J Appl Computat Math 2018, 7:4
DOI: 10.4172/2168-9679.1000420

Commentary Open Access

On the Dirichlet Eta Function

Azkour M*

Ministry of National Education, Vocational Training, Higher Education and Scientific Research, Morocco

Abstract

This paper gives a proof of the following result: if $\eta(\rho)=0$ and $\Re(\rho)>0$, then:

$$\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\eta_k(\rho)}=2.$$

η is the *Dirichlet* eta function defined for $\Re(s)>0$ by $\eta(s)=\sum_{n=1}^{\infty}\mathcal{X}_n(s)$ with $\chi_n(s)=(-1)^{n-1}n^{-s}$ and $\eta_k(s)=\sum_{n=1}^{k}\mathcal{X}_n(s)$.

Keywords: *Dirichlet* eta function; Riemann zeta function; Complex variable

Introduction

We know that the *Riemann* zeta function ζ is the analytic function of the complex variable *s*, defined in the half-plane $\Re(s)>1$ by Sarnak [1]

$$\zeta(s)=\sum n^{-s}=\prod\left(1-p^{-s}\right)^{-1} \tag{1}$$

where the series $\sum n^{-s}$ is absolutely convergent for $\Re(s)>1$ and the product $\prod(1-p^{-s})^{-1}$ extends over all the prime numbers $p\in P=\{2,3,5,7,\ldots\}$ and as shown by *Riemann*, ζ can be continued analytically to $\mathbb{C}\setminus\{1\}$ as a meromorphic function and has a simple pole at $s=1$ with residue 1 [2]. We also know that ζ is defined for any complex number $s\neq 1$ and having $\Re(s)>0$ by

$$\zeta(s)=(1-2^{1-s})^{-1}\eta(s) \tag{2}$$

where η is the *Dirichlet* eta function which is defined in the half-plane $\Re(s)>0$ by [3]

$$\eta(s)=\sum(-1)^{n-1}n^{-s} \tag{3}$$

so, noticing that $\forall s\in\mathbb{C}$: $(1-2^{1-s})^{-1}\neq 0$, we deduce that [4]:

"if $\zeta(s)=0$ and $\Re(s)>0$, then $\eta(s)=0$".

On the other hand, we know that $\zeta(s)$ is related to $\zeta(1-s)$ by the *Riemann* functional equation [5]

$$\wedge(s)=\wedge(1-s) \tag{4}$$

where $\Lambda(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and Γ is the *Euler* gamma function. So, by the two in eqns. (2) and (4), we obtain:

$$"\forall 0<\Re(s)<1:\frac{\eta(1-s)}{\eta(s)}=\frac{f(s)}{f(1-s)};f(s)=\pi^{-s/2}\Gamma(s/2)\left(1-2^{s}\right)"$$

And we have $\Gamma(s)$ does not vanish for any *s* in $\mathbb{C}$ and has an infinity of simple poles with residue $(-1)^n/n!$ at $s=-n$ where $n=0,1,2,3,\ldots$ etc.,

So

$$\lim_{s\to\rho}\frac{f(s)}{f(1-s)}=0\Rightarrow\rho=3,5,7,9\ldots$$

And

$$\lim_{s\to\rho}\frac{f(s)}{f(1-s)}=\tilde{\infty}\Rightarrow\ldots,-8,-6,-4,-2=\rho$$

Then

$$"\forall 0<\Re(\rho)<1:\lim_{s\to\rho}\frac{f(s)}{f(1-s)}\neq 0,\tilde{\infty}$$

that is to say for every complex number ρ with $0<\Re(\rho)<1$, we have:

$$\lim_{s\to\rho}\frac{\eta(s)}{\eta(1-s)}\neq 0,\tilde{\infty}$$

this means that: "if $\eta(\rho)=0$ and $0<\Re(\rho)<1$, *then* $\eta(1-\rho)=0$".

So, for $\Re(s)>0$, we have:

$$\eta(s)=\sum_{n\geq 1}\mathcal{X}_n(s)\text{ with }\mathcal{X}_n(s)=(-1)^{n-1}n^{-s}$$

let's denote:

$$\eta_k(s)=\sum_{n=1}^{k}\mathcal{X}_n(s)\ \textit{for}\ k\geq 1,$$

if $\eta(\rho)=0$ and $\lim_{k\to\infty}\frac{\mathcal{X}_n(\rho)}{\eta_k(\rho)}=\tilde{\infty}$ then,

$$\lim_{k\to\infty}\frac{\mathcal{X}k+1(\rho)}{\eta k+1(\rho)}=\tilde{\infty}$$

and knowing that $\lim_{k\to\infty}\frac{\mathcal{X}k+1(\rho)}{\mathcal{X}k(\rho)}=\lim_{k\to\infty}(-1)\left(\frac{k}{k+1}\right)^{\rho}=-1$ we have

$$\lim_{k\to\infty}\frac{\mathcal{X}k+1(\rho)}{\eta k+1(\rho)}=\lim_{k\to\infty}\frac{\frac{\mathcal{X}k+1(\rho)}{\mathcal{X}k(\rho)}}{\frac{\eta k+1(\rho)}{\mathcal{X}k(\rho)}}=\lim_{k\to\infty}=\frac{\frac{\mathcal{X}k+1(\rho)}{\mathcal{X}k(\rho)}}{\frac{\eta k(\rho)}{\mathcal{X}k(\rho)}+\frac{\mathcal{X}k+1(\rho)}{\mathcal{X}k(\rho)}}=\frac{-1}{\frac{1}{\tilde{\infty}}-1}=\tilde{\infty},$$

this implies $1=\tilde{\infty}$,and this result is absurd, so if $\eta(\rho)=0$, then

$$\lim_{k\to\infty}\frac{\mathcal{X}k(\rho)}{\eta k(\rho)}\neq\tilde{\infty},$$

***Corresponding author:** Azkour M, Ministry of National Education, Vocational Training, Higher Education and Scientific Research, Morocco, Tel: +21671766462; E-mail: mazkour@yahoo.com

Received September 18, 2018; **Accepted** October 03, 2018; **Published** October 10, 2018

Citation: Azkour M (2018) On the Dirichlet Eta Function. J Appl Computat Math 7: 420. doi: 10.4172/2168-9679.1000420

Citation: Azkour M (2018) On the Dirichlet Eta Function. J Appl Computat Math 7: 420. doi: 10.4172/2168-9679.1000420

if $\eta(\rho)$=0 and $\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\eta_k(\rho)}=\lambda\in\mathbb{C}\setminus\{0\}$, then

$$\lim_{k\to\infty}\frac{\mathcal{X}_k+1(\rho)}{\eta_k+1(\rho)}=\lim_{k\to\infty}\frac{\frac{\mathcal{X}_k+1(\rho)}{\mathcal{X}_k(\rho)}}{\frac{\eta_k+1(\rho)}{\mathcal{X}_k(\rho)}}=\lim_{k\to\infty}\frac{\frac{\mathcal{X}_k+1(\rho)}{\mathcal{X}_k(\rho)}}{\frac{\eta_k(\rho)}{\mathcal{X}_k(\rho)}+\frac{\mathcal{X}_k+1(\rho)}{\mathcal{X}_k(\rho)}}=\frac{-1}{\frac{1}{\lambda}-1}=\lambda,$$

that is to say

λ=2,

if $\eta(\rho)$=0 and 0<$\mathfrak{R}(\rho)$<1, then

$$\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\eta_k(\rho)}=0,2,\nexists \text{ and } \lim_{k\to\infty}\frac{\mathcal{X}_k(1-\rho)}{\eta_k(\rho)}=0,2,\nexists\left(e.g\lim_{k\to\infty}e^{ik}=\nexists\right)$$

and

$$\lim_{k\to\infty}\frac{\frac{\mathcal{X}_k(\rho)}{\eta_k(\rho)}}{\frac{\mathcal{X}_k(1-\rho)}{\eta_k(1-\rho)}}=\frac{0,2,\nexists}{0,2,\nexists}=0,1,\gamma\neq1,\tilde{\infty},\nexists=\lim_{k\to\infty}\frac{\eta_k(1-\rho)}{\eta_k(\rho)}\frac{\mathcal{X}_k(\rho)}{\mathcal{X}_k(1-\rho)}$$

for 0<$\mathfrak{R}(\rho)$<1: $\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\mathcal{X}_k(1-\rho)}=0,\tilde{\infty},\nexists$

if $\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\mathcal{X}_k(1-\rho)}=0 \text{ and } \lim_{k\to\infty}\frac{\eta_k(1-\rho)}{\eta_k(\rho)}\neq\tilde{\infty}$, then

$$\lim_{k\to\infty}\frac{\eta_k(1-\rho)}{\eta_k(\rho)}\frac{\mathcal{X}_k(\rho)}{\mathcal{X}_k(1-\rho)}=0=\frac{0}{0,2,\nexists}$$

that is to say if $\eta(\rho)$=0 and $\frac{1}{2}<\mathfrak{R}(\rho)<1 \text{ and } \lim_{k\to\infty}\frac{\eta_k(1-\rho)}{\eta_k(\rho)}\neq\tilde{\infty}$

then

$$\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\eta_k(1-\rho)}=0$$

but, it is very likely that if $\eta(\rho)$=0, then

$$\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\eta_k(\rho)}=2\neq0$$

let $f_{n,\alpha}$ be the function of the variable x defined by

$$f_{n,\alpha}(x)=1-\left(\frac{x}{x+1}\right)^\alpha+\left(\frac{x}{x+2}\right)^\alpha-\left(\frac{x}{x+3}\right)^\alpha+\ldots+(-1)^n\left(\frac{x}{x+n}\right)^\alpha$$

Then

$$\lim_{\substack{n\to\infty\\ n/x\to0}}f_{n,\alpha}(x)=1-1+1-1+\ldots+(-1)^n=\frac{1}{2}(Grandi\text{'series})$$

let's take:

$$\eta_m(s)=\eta_k(s)+\eta_{k,m}(s) \text{ with } \eta_{k,m}(s)=\sum_{n=k+1}^{m}\mathcal{X}_n(s)$$

so, if $\eta(\rho)$=0, then:

$$\lim_{m\to\infty}\eta_m(\rho)=0 \text{ and } \lim_{m\to\infty}\frac{\eta_{k,m}(\rho)}{\eta_k(\rho)}=-1$$

and we have $\forall n\geq0$:

$$\frac{\mathcal{X}_{k+n}(\rho)}{\mathcal{X}_k(\rho)}=(-1)^n\left(\frac{k}{k+n}\right)^\rho$$

so, using $f_{n,\alpha}$, we have:

$$1+\sum_{n=1}^{m}\frac{\mathcal{X}_{k+n}(\rho)}{\mathcal{X}_k(\rho)}=1+\sum_{n=1}^{m}(-1)^n\left(\frac{k}{k+n}\right)^\rho=f_{m,\rho}(k)$$

and

$$\lim_{\substack{m\to\infty\\ m/x\to0}}f_{m,\rho}(x)=1-1+1-1+\ldots+(-1)^m=\frac{1}{2}$$

that is to say

$$1+\lim_{\substack{m\to\infty\\ m/x\to0}}\sum_{n=1}^{m}\frac{\mathcal{X}_{k+n}(\rho)}{\mathcal{X}_k(\rho)}=1+\lim_{\substack{m\to\infty\\ m/x\to0}}\frac{\sum_{n=1}^{m}\mathcal{X}_{k+n}(\rho)}{\mathcal{X}_k(\rho)}=1+\lim_{\substack{m\to\infty\\ m/x\to0}}\frac{\eta_{k,m}(\rho)}{\mathcal{X}_k(\rho)}=\frac{1}{2}$$

then

$$1+\lim_{\substack{m\to\infty\\ m/x\to0}}\frac{\eta_{k,m}(\rho)}{\eta_k(\rho)}\frac{\eta_k(\rho)}{\mathcal{X}_k(\rho)}=1-\lim_{k\to\infty}\frac{\eta_k(\rho)}{\mathcal{X}_k(\rho)}=\frac{1}{2}$$

that is to say

$$\lim_{k\to\infty}\frac{\eta_k(\rho)}{\mathcal{X}_k(\rho)}=\frac{1}{2}$$

So

$$\lim_{k\to\infty}\frac{\mathcal{X}_k(\rho)}{\eta_k(\rho)}=2\,.$$

References

1. Sarnak P (2004) Problems of the Millennium: The Riemann Hypothesis. Princeton University and Courant Institute of Math. Sciences.
2. Borwein P, Choi S, Rooney B, Weirathmueller A (2007) The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike. Springer, Berlin, Heidelberg NewYork, Hong Kong, London, Milan Paris, Tokyo.
3. Jaiswal AK, Khandelwal A (2009) A text book of computer based numerical and statistical techniques, p: 618
4. Iyengar SRK, Jain RK (2009) Numerical Methods, New Age International Publishers, pp: 182-184.
5. Zaliznaik V (2008) Essentials of scientific computing (1stedn.), Numerical Methods in Science and Engineering.