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On the Diophantine Equation $1+5x^2=3yn$, n>=3

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Abstract

In this paper we study the Diophantine equation $1+5x^2=3y^n$.

Keywords: Diophantine equation; Multi-Frey variant; Positive integer; Legendre symbol

Introduction

The Diophantine equation $x^2+C=y^n$, in positive integers unknowns x, y and n, has a long story. The first case to have been solved appears to be c=1. In 1850 Victor Lebesgue showed, using a elementary factorization argument, that the only solution is x=0, y=1. Over the next 140 years many equations of the form $x^2+C=y^n$ have been solved using the Lebesgue's elementary trick. In 1993 John Cohn published an exhautive historical survey of this equation which completes the solution for but all 23 values of C in the range $1 \le C \le 100$ [1].

It has been noted recently, that the result of Bilu, Harnot and Voutier can sometimes be applied to equations of the form $x^2+C=y^n$, when instead of C being a fixed integer, C is the product of powers of fixed primes $p_1,\ldots, p_k$.

By comparison, The Diophantine equation $x^2+C=2y^n$ with the same restriction, has been solved partially. For C=1, John Cohn, showed that the only solutions to this equation are x=y=1 and x=239, y=13 and n=4. SZ. Tengely studied the equation $x^2+q^{2m}=2y^p$ where x, y, q, p, m are integers with m>0 and p, q are odd primes and gcd (x,y)=1. He proved that there are only finitely many solutions (m, p, q, x, y) for which y is not a sum of two consecutive squares. He also studied the equation for fixed q and resolved it when q=3. In 2007, Abu Muriefah FS, et al., give a very sharp bound for prime values of the exponent n when $C\equiv 1$(mod 4). When $C\not\equiv 1$(mod 4) they explain how the equation can be solved using the multi-Frey variant of the modular approach. They illustrate their approach by solving completely the equations $x^2+17^{a1}=2y^n$, $x^2+5^{a1}.13^{a2}=2y^n$ and $x^2+3^{a1}.11^{a2}=2y^n$. In 2009, F. Luca, S. Tengely, and A. Togbe give all solutions of that the equation $x^2+C=4y^n$ when gcd (x,y)=1, $C\equiv 3$(mod 4) and $1 \le C \le 100$ [2,3].

The purpose of this paper is to give all solutions of the equation $1+5x^2=3y^n$, for almost values of $n \ge 2$.

Results

Considering the following equation

$$1+5x^2=3y^n \qquad (1)$$

in integer unknowns x, y, n satisfying

$$x \in \mathbb{Z},\ \ y \ge 1 \text{and} 2 \le n \le 5 \qquad (2)$$

Theorem 1 Consider the equation (1) satisfying (2). Then the only solution of equation (2) is (x,y,n)=(± 4,3,3).

Auxiliary results

To prove theorem (1), we need the following result

Theorem 2 Let C be a positive integer satisfying $C\equiv 1$(mod 4), and write $C=cd^2$, where c is a square- free. Suppose that (x,y) is a solution of the equation

$$x^2+C=2y^p, \quad x,y \in \mathbb{Z}^+, \quad \gcd(x,y)=1,$$

Where $p \ge 5$ is a prime, then either

i) x=y=C=1, or

ii) p divides the class number of the quadratic field $\mathbb{Q}(\sqrt{-c})$ or

iii) p=5 and $(C,x,y)=(9,79,5),(125,19,3),(125,183,7),(2125,21417,47)$, or

iv) $p\,|\,(q-(-c\,|\,q))$, where q is some odd prime such that $q\,|\,d$ and $q \nmid c$. here $(c\,|\,q)$ denotes the Legendre symbol of the integer c with respect to the prime q.

Proof. See (3).

Proof of theorem 1:

We follow the notation from the statement of the theorem (1). We take n=p a prime, p does not divide the class number of the field $\mathbb{Q}(\sqrt{-c})$ considering equation (1) modulo 4 reveals that x is even and y odd. We work first in $\mathbb{Q}(\sqrt{-c})$. Since 5=1(mod4) this has ring of integers $\mathfrak{R}=\mathbb{Z}\left[\sqrt{-5}\right]$. Factoring the left hand of (1), we get

$$(\pm 1+x\sqrt{-5})(\pm 1-x\sqrt{-5})=3y^p \qquad (3)$$

multiplying both sides by 4, we obtain

$$(\pm 2+2x\sqrt{-5})(\pm 2-2x\sqrt{-5})=2(1+\sqrt{-5})(1-\sqrt{-5})y^p$$

and this equation becomes

$$\left(\frac{\pm 2+2x\sqrt{-5}}{1+\sqrt{-5}}\right)\left(\frac{\pm 2-2x\sqrt{-5}}{1-\sqrt{-5}}\right)=2y^p \qquad (4)$$

We put

$$\pi=\left(\frac{\pm 2+2x\sqrt{-5}}{1+\sqrt{-5}}\right)=\{.\left(\frac{5x+1}{3}\right)+\left(\frac{x-1}{3}\right)\sqrt{-5},\ \text{if} x\equiv 1(mod 3),\left(\frac{5x-1}{3}\right)+\left(\frac{x+1}{3}\right)\sqrt{-5},\text{if} x\equiv 2(mod 3)$$

It is clear that the π is a principal ideal in $\mathbb{Z}\left[\sqrt{-5}\right]$, then $\pi=\left(U+V\sqrt{-5}\right)$ for some odd coprime integers U, V. Then the equation (4) becomes

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$\pi.\overline{\pi} = 2y^p$

which implies

$U^2 + 5V^2 = 2y^p$

Using the theorem (2), with c=5 and d=V then this equation has solution if the statement (iv) holds if p ≥ 5, so $p \mid q \pm 1$ for some prime q such that $q \mid V$ and $q \nmid 5$. If q=3, 7 then p=2,3. Contradiction with the fact that p ≥ 5.

If q=11 then p=5, but it is easy to check modulo 11, that (1) has no solution. We conclude that (1) has no solution for all n=0(mod5).

Now we take n=p=3, the equation (1) becomes

$$1 + 5x^2 = 3y^3 \quad (5)$$

using the same argument in the proof for p ≥ 5, we get

$\pi.\overline{\pi} = 2y^3$

We have (2)=q^2 where q is a prime ideal of $\Re$. It is clear that the principal ideals $\pi, \overline{\pi}$ have q as their greatest common factor. From (5) we deduce that

$\pi.\Re = \mathbf{q.a}^3$

Where a is some ideal of $\Re$. Now multiply both sides by (2). We obtain

$2.\pi = (\mathbf{q.a})^3$

Since gcd(h,3)=1, where h is the class number of the field $\mathbb{Q}(\sqrt{-c})$ we see that q.a is a principal ideal. Moreover, the units of $\mathbb{Q}(\sqrt{-c})$ are ±1. Hence

$2.(U + V\sqrt{-5}) = (a + b\sqrt{-5})^3$

For some odd integers a, b. Moreover $y = (a^2 + 5b^2)/2$. From the coprimality of x and y, we see that a and 5b are coprime. Equating real and imaginary parts, we get

$$\{.2U = a(a^2 - 15b^2) 2V = b(3a^2 - 5b^2) \quad (6)$$

but $U = 5V \pm 2$, then (6) becomes

$a^3 - 15a^2b - 15ab^2 + 25b^3 = \pm 4$

which is a Thue type equation with only solutions (a,b)=(1,1), (-1,-1).

So U=±7 that means x= ± 4 and y=3. We conclude that (1) has no solution for all n=0(mod 3) and n>3.

Now, we take n=p=2, considering the equation (1) modulo 4, in one hand we get $1+5x^2 \equiv 1,2$ (mod 4), and in another we get $3y^2 \equiv 0,3$ (mod 4), we conclude that (1) has no solution for all n≡0 (mod 2).

Conjecture 1: We claim that the equation (1) has no solution for all n ≥ 7 when $n \not\equiv 0\,(mod 2), n \not\equiv 0\,(mod 3)$ and $n \not\equiv 0\,(mod 5)$.

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