On the Addition Modulus of the Aunu Pattern $\omega_i \in G_p$: An Investigation of Some Topological Properties

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Abstract

This paper investigates the topological properties on the structure generated by the addition modulus of the Aunu permutation pattern $\omega_i$. The pattern $\omega_i$ has the generating function defined as:

$$\omega_i = (1 (1+i) \pmod{p} (1+2i) \pmod{p} (1+3i) \pmod{p} \ldots \ldots (1+(p-1)i) \pmod{p})$$

For $i=1,2,\ldots,p-1$.

The numbers are arrangement on the structure $X=[1,\ldots,p]$ for primes $p \geq 5$.

It has been established in this paper that the generated set is a topology on $X$, it is not a convex set and finally, not a $\sigma$-algebra.

Keywords: Permutation pattern; Aunu numbers; Topology; Connectedness; Convexity; Sigma ($\sigma$) algebra

Introduction

Let $n$ be a positive integer. A permutation is a bijection from the set $[1,2,\ldots,n]$ to itself. It maps $i$ to $(i) \in [1,2,\ldots,n]$.

A pattern (classical permutation) on the other hand is a permutation $\omega \in \text{sk} [1]$. Pattern is governed by specific rules. It exists in different combinatorial objects.

The inception of permutation pattern have pave way for the discovery of several mathematical structures such as symmetric group, topological group, pattern avoidance, permutation polynomials and permutation statistics. Aunu pattern a permutation generated by a certain scheme.

Aunu pattern/permutation is a partial permutation in which the first entry of every permutation is a unity (one) and its length is prime $p$ [2]. Magami, Usman and Ibrahim represented the group theoretical approach for the Aunu numbers. Catalan numbers was used to scheme for prime numbers $p \geq 5$ and $X \subseteq \mathbb{N}$ which generate the cycles of the Aunu group $G_p$ [3,4];

$$G_p = \{w_1, w_2, \ldots, w_p\}$$

The group theoretical and topological properties of the Aunu numbers have also been studied, and in the Aunu pattern was represented as $\Gamma$ 1 non-deranged permutations. The work allows for further investigation into the behavior of the algebraic structure $G_p$ [5-7].

Definitions

Let $X=[1,2,\ldots,p]$ be a non-empty set of prime $p \geq 5$, and $G_p = \{\omega_1, \omega_2, \ldots, \omega_p\}$ a finite group formed using $\omega_i$ on an arbitrary set $X=[1,2,\ldots,p]$ using a permutation pattern $\omega(i)$. Then, each $\omega_i$ is called a cycle [5].

- Definition (power set): The power set of a set is the collection of all subsets of the set.
- Definition (topology): A topology $\tau$ on a set $X$ is the collection of subsets of $X$ such that:
  - $\emptyset$ and $X$ are open in $\tau$.
  - Finite union of open members of $\tau$ is open in $\tau$.
  - Intersection of open members of $\tau$ is open in $\tau$.
- Definition (open sets): let $(X, \tau)$ be a topological space. Then the members of $\tau$ are said to be open set.
- Definition (connectedness): A topological space $X$ is said to be disconnected if and only if there exist open subsets of $X$ say $U$ and $V$ such that:
  1. $U \neq \emptyset$ and $V \neq \emptyset$.
  2. $U \cap V = \emptyset$.
  3. $U \cup V = X$.

Otherwise $X$ is said to be connected.

- Definition (Cartesian product): If $A$ and $B$ are nonempty sets, then the Cartesian $A \times B$ is the set of all ordered pairs $(a, b)$ with $a \in A$ and $b \in B$.
- Definition (convex set): A set $C \subseteq R^n$ is said to be convex if and only if $x_1, x_2 \in C$ the line segment defined as:
  $$\lambda x_1 + (1-\lambda)x_2 \in C \text{ where } 0 \leq \lambda \leq 1 \text{ and } x_1 \neq x_2$$
- Definition ($\sigma$- algebra): Let $X$ be a set. A collection $\mu$ of subsets of $X$ is called a $\sigma$-algebra if satisfies the following:
  1. $X \in \mu$.
  2. $A \in \mu \Rightarrow X \setminus A \in \mu$.

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3. \( A_i, A_j, \ldots \in \mu \Rightarrow U_{i \neq j} A_i \in \mu \) (\( A_i, A_j, \ldots \), are pairwise disjoint).

**Method of Construction**

This research adopts the method reported in Garba and Ibrahim (2009), in construction of the group structure \( G_p \) using Aunu pattern of permutation. The research further provides an investigation of the topological properties of the derived set. Now let

\[
\omega = (1+1)^{m_0}(1+2)^{m_0} \cdots (1+(p-1))^{m_0}
\]

For all \( p \geq 5 \) and \( p \) a prime.

In constructing the subsets to be used for the study, the following definition is adopted.

\[ |\omega_i + \omega_j| \mod p \]

This shall be done for all prime greater than or equal to five (5). The resulting sets from the above definition shall be considered.

**Results**

**Theorem**

Let \( p \geq 5, X=\{1,2,\ldots,p\} \), \( \omega, \ast \in G_p \) such that \( i_j=\{1,\ldots,p\} \) and \( i \neq j \). Then \( |\omega_1 + \omega_j| \mod p = \{X\} \) if and only if \( \omega_1 = \omega_j \), else \( |\omega_1 + \omega_j| \mod p = \{2\} \).

**Proof:** Let \( p \geq 5, \omega_1, \omega_j \in G_p \). Then by definition

\[
\omega_1 = (1+1)^{m_0}(1+2)^{m_0} \cdots (1+(p-1))^{m_0}
\]

Clearly,

\[ \omega_1 = \{a_1, \ldots, a_p\} \]

and each \( a_i \in \omega_1 \) is unique.

Also, let \( \omega_1 = \{b_1, \ldots, b_p\} \) with each \( b_i \in \omega_1 \),

\[ |\omega_1 + \omega_j| \mod p = \{c_1, \ldots, c_p\} \]

Is unique for each \( c_i \) and is the set is \( X \) itself.

Now, suppose \( \omega = \{1, \ldots, p\} \). By definition

\[ |\omega + \omega_j| \mod p = \{2\} \]

Then, for any \( \omega, \omega_j \in G_p \),

\[ |\omega + \omega_j| \mod p = \{X, \{2\}\} \]

**Proposition**

For any prime \( p \geq 5 \), and \( \omega, \omega_j \in G_p \) the set \( \{X, \{2\}\} \) is the power set of \( X \) defined by the restriction \( |\omega_1 + \omega_j| \mod p \).

**Proof:** The result is trivial by definition 1.2.1 and proves of theorem 2.1 above.

Remark: The numbers of subsets of \( X \) generated by \( |\omega_1 + \omega_j| \mod p \) is even.

**Proposition 5.3**

Let \( p \geq 5 \), and \( \omega_1, \omega \in G_p \) and \( \rho_{\omega, \omega_1} = \{\emptyset, X, \{2\}\} \). Then \( \rho_{\omega, \omega_1} \) is a topology on \( X \).

**Proof:** Let \( \tau = \rho_{\omega, \omega_1} = \{\emptyset, X, \{2\}\} \). From definition 1.2.2 we see that

1. \( \emptyset \in \tau \) and \( X \in \tau \).
2. \( X \cup \emptyset = X \in \tau \), \( X \cup \{2\} = X \in \tau \), and \( \emptyset \cup \{2\} = \{2\} \in \tau \).
3. \( X \cap \{2\} = \emptyset \in \tau \), \( X \cap \{2\} = \{2\} \in \tau \), and \( \emptyset \cap \{2\} = \emptyset \in \tau \).

All conditions are satisfied.

Therefore, \( \tau \) is a topology on \( X \) and the collection \( (\omega, \tau) \) is a topological space.

**Proposition**

The space \( X \) is a connected space.

**Proof:** Let \( \tau = \{X, \{2\}\} \), by definition 1.2.4 we note that,

1. \( X = \emptyset \) and \( \{2\} \neq \emptyset \)
2. \( X \cap \{2\} = \{2\} \neq \emptyset \)
3. \( X \cup \{2\} = X \neq \emptyset \)

Thus, no such open subsets in \( \tau \) such that all three conditions are satisfied.

**Proposition**

Let \( \tau = \{X, \{2\}\} \), and the Cartesian \( \Omega = X \times \Phi \), for \( \Phi = \{2\} \). Then \( \tau \) is not convex.

**Proof:** Let \( \Omega = X \times \Phi \) then,

\[ \Omega = (x, y_1), x \in X \text{ and } y_i \in \Phi, i = 1, 2, \ldots \]

So that,

\[ \lambda x + (1-\lambda) y_i \text{ for } 0<\lambda<1 \text{, and } i=1,2, \ldots \]

Clearly, by definition 3.1.6

\[ \lambda x + (1-\lambda) y_i \text{ for } x \neq y_i \in X \text{ is satisfied.} \]

On the other hand,

\[ \lambda y_i + (1-\lambda) y_i \text{ is not, since } y_i = y_i = \Phi \text{ the singleton.} \]

Therefore, \( \Omega \) is not convex which implies \( \tau \) is also not convex.

**Remark:** A set having a convex and a non-convex subsets is not convex.

**Proposition**

Let \( \tau = \{X, \{2\}\} \), by definition 1.2.7 we have;

1. \( X \in \tau \)
2. \( X \setminus \{2\} \notin \tau \)
3. \( X \cup \{2\} = X \neq \emptyset \)

Condition (2) and (3) are not satisfied.

**Conclusion**

The permutation \( o_{\omega} \) generate the set \( \tau = \{X, \{2\}\} \) under the definition \( |\omega_1 + \omega| \mod p \) where \( \omega, \omega_j \in G_p \). Then, \( \tau \) is not a - algebra.

**Proof:** Let \( \tau = \{X, \{2\}\} \), by definition 1.2.7 we have;

1. \( X \in \tau \)
2. \( X \setminus \{2\} \notin \tau \)
3. \( X \cup \{2\} = X \neq \emptyset \)

Condition (2) and (3) are not satisfied.

We therefore, recommend for further restrictions on the Aunu patterns to see if more subsets of \( X=\{1, \ldots, p\} \) for \( p \geq 5 \), can be generated.
References


