Journal of Generalized Lie Theory and Applications

Ndoune and Bouetou, J Generalized Lie Theory Appl 2015, S2
DOI: 10.4172/1736-4337.S2-005

Research Article Open Access

On Representations of Bol Algebras

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Abstract

In this paper, we introduce the notion of representation of Bol algebra. We prove an analogue of the classical Engel's theorem and the extension of Ado-Iwasawa theorem for Bol Algebras. We study the category of representations of Bol algebras and show that it is a tensor category. In the case of regular representations of Bol algebras, we give a complete classification of them for all two-dimensional Bol algebras.

Keywords: Bol algebra; Lie triple System; Non-associative algebras; Jordan superalgebras; Nilpotent representation

Introduction

It is well known that the algebraic systems which characterize locally a totally geodesic subspace is a Lie triple system [1-3]. A Bol algebra is realized by equipping Lie triple System with an additional binary skew operation which satisfies a pseudo-differentiation property [4,5]. A morphism of Bol algebras is a linear map which preserves the ternary and the binary operations. More generally, the algebraic structures which characterize locally Bol loops are Bol algebras [6]. Until now, the representations of these algebras have not been studied. Since the representations of Lie algebras and Lie groups have natural connection with particulars physics, we claim that the representations of Bol algebras should lead with the physical applications. More precisely, in physics the representations of Bol algebras will be useful for the description of invariant properties of physical systems. and the concomitant conservation laws as a result. In literature of Mostovoy and Pérez-Izquierdo [7], it is shown that, Malcev algebras and Lie triple systems are particular subclasses of Bol algebras. The representations of Malcev algebras can be found studies of Kuz'min [8], and those of Lie triple systems were constructed by Hodge and Parshall [9], Bertrand, et al. [10]. Now, there already exists some representations of other classes of non-associative algebras; the case of alternative algebras was constructed by Schafer [11], the one of Leibniz algebras by Kolesnikov [12] and for Jordan superalgebras, the representations was given by Consuelo and Zelmanov [13].

Let $\mathfrak{B}$ be a Bol algebra over a field K of characteristic zero, a representation of Bol algebra $\mathfrak{B}$ on a K-vector space V is a triplet of maps (ρ,δ,Δ) which respect some conditions which will be given later in the paper.

Our first main result is the following.

Theorem 1.1. *Let $\mathfrak{B}$ be a finite dimensional Bol algebra over a field K and $\mathcal{R}$ consist of nilpotent representations of Bol algebra $\mathfrak{B}$ in a finite dimensional space V. Then there exists a vector $v \in V$, $v \neq 0$ such that $(\rho,\delta,\Delta)(v)=0$ for all $(\rho,\delta,\Delta)\in\mathcal{R}$.*

We agree that the image of any vector v of V by the operator (ρ,δ,Δ) is given by $(\rho,\delta,\Delta)(v)=(\rho(v_1),\delta(v_2),\Delta(v_3))$, where $v=(v_1,v_2,v_3)\in\mathfrak{B}^3$.

We define also the regular representations and the adjoint representations of Bol algebras. As an easy consequence, we show that if any representation of Bol algebra is nilpotent, then its adjoint representation is also nilpotent.

We are also interested by the question of the extension theorem of Ado-Iwasawa for Bol algebras. Pérez-Izquierdo established the existence of a Poincaré-Birkhoff-Witt type basis for a universal envelope of Bol algebra [5]. The above result allows us to interest ourselves to an extension of Ado-Iwasawa theorem for Bol algebra. let A be an alternative algebra, the the generalized right alternative nucleus is the algebra $RN_{alt}(A)$ defined by $RN_{alt}(A)=\{a\in A/(x,a,y)=-(x,y,a)\}$. We then give our second theorem.

Theorem 1.2. *Let $\mathfrak{B}$ be a finite-dimensional right Bol algebra over a field of characteristic different to 2 and 3. Then there exists a unital finite-dimensional algebra A and a monomorphism of Bol algebras $\mathfrak{B}\to RN_{alt}(A)$.*

The analogue of our second result above was established for Malcev algebras framed by Pérez-Izqquierdo and Shestakov [14]. The collection of all representations of Bol algebra and the morphisms between them form a category, named the category of representations of Bol algebras $Rep(\mathfrak{B})$. One can view a representation of Bol algebra as a B-module analogously as in literature of Consuelo and Zelmanov [13] in the case of Jordan superalgebras. One can understand also the representations of Bol algebras in term of matrices with sweet properties. The investigation between the category $Rep(\mathfrak{B})$ and the category of left $U(\mathfrak{B})$ -modules, where $U(\mathfrak{B})$ is the universal enveloping algebra of $\mathfrak{B}$, endowed with its bialgebra structure, leads us to our third main theorem.

Theorem 1.3. *The category of representations of Bol algebra $Rep(\mathfrak{B})$ is equivalent to the category of representations of its universal enveloping algebra $Rep(U(\mathfrak{B}))$.*

The paper is organized as follows: We introduce in section 2 the notion of representations of Bol algebra. In section 3 we establish the Engel's theorem for Bol algebras. In section 4 an extension of Ado-Iwasawa theorem to Bol algebras is proved. Finally in section 5, we present the category of representations of Bol algebras and show that

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Received December 02, 2015; **Accepted** December 13, 2015; **Published** December 15, 2015

Citation: Ndoune N, Bouetou TB (2015) On Representations of Bol Algebras. J Generalized Lie Theory Appl S2: 005. doi:10.4172/1736-4337.S2-005

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it is equivalent to the category of left modules under its universal enveloping algebra. As immediate consequence, we show the category $Rep(\mathfrak{B})$ is a tensor category. We end the section by given a complete classification of regular representations of two-dimensional Bol algebras.

Bol Algebras and their Representations

Bol algebras were introduced in differential geometry to study smooth Bol loops [6,15,16]. A *right loop* is a set $\mathcal{Q}$, together with a binary operation $(a,b)\mapsto a\cdot b$, such that for any b in $\mathcal{Q}$, the right multiplication operator $R_b : x\mapsto x\cdot b$ is bijective, and there exists an element $\varepsilon\in\mathcal{Q}$, such that $\varepsilon\cdot b=b$ for any b in $\mathcal{Q}$. The dual definition gives rise to a left Bol loop. In case that $\langle\mathcal{Q},\cdot,\varepsilon\rangle$ is both left and right loop then it is called a loop with identity element ε.

A *right smooth loop* $\mathcal{M}$ is a right loop equipped with a structure of smooth manifold, that is the map $(a,b)\mapsto a\cdot b$ and R_b^{-1} are smooth, [15,16]. Since groups are particular loops, so the Lie groups are particular cases of smooth loops. In scientific literature, many classes of loops are known: homogeneous loops, Moufang loops, Bol loops, Kikkawa loops among others.

A *right Bol loop* $\langle\mathcal{Q},\cdot,\varepsilon\rangle$ is a right loop that satisfies the right Bol identity

$$x\cdot((a\cdot y)\cdot a)=((x\cdot a)\cdot y)\cdot a$$

for all a, x, y in $\mathcal{Q}$. Similarly, a *left Bol loop* satisfies the identity $a\cdot(x\cdot(a\cdot y))=(a\cdot(x\cdot a))\cdot y\cdot$

As in the case of Lie groups where the tangent space at each point is equipped with Lie algebra structure, the tangent space at each point of Bol loop is equipped with the structure of Bol algebra.

Definition 2.1. *A vector space $\mathfrak{B}$ over a field K is called Bol algebra if it is equipped with a trilinear operation $[-;-,-]$ and a skew-symmetric operation $x\cdot y$ satisfying the following identities:*

(i) $[x;x,y]=0$

(ii) $[x;y,z]+[z;x,y]+[y;z,x]=0.$

(iii) $[[x;y,z];\alpha,\beta]=[[x;\alpha,\beta];y,z]+[x;[y;\alpha,\beta],z]+[x;y,[z;\alpha,\beta]]$

(iv) $[x\cdot y;\alpha,\beta]=[x;\alpha,\beta]\cdot y+x\cdot[y;\alpha,\beta]+[\alpha\cdot\beta;x,y]+[x\cdot y]\cdot[\alpha\cdot\beta]$

for all x, y, z, α and β in $\mathfrak{B}$.

In other words, a Bol algebra is a Lie triple system $(\mathfrak{B},[-;-,-])$ with an additional bilinear skew-symmetric operation $x\cdot y$ such that, the derivation $D_{\alpha,\beta}: x\to[x;\alpha,\beta]$ on a ternary operation is a pseudo-differentiation with components α, β on a binary operation, that is; for all x, y and z in $\mathfrak{B}$, we have

$$D_{\alpha,\beta}(x\cdot y)=(D_{\alpha,\beta}(x))\cdot y+x\cdot(D_{\alpha,\beta}(y))+[\alpha\cdot\beta;x,y]+(x\cdot y)\cdot(\alpha\cdot\beta).$$

$D_{\alpha,\beta}$ is a differentiation on ternary operation $[-;-,-]$ that is;

$$D_{\alpha,\beta}[x;y,w]=[D_{\alpha,\beta}(x);y,w]+[x;D_{\alpha,\beta}(y),w]+[x;y,D_{\alpha,\beta}(w)].$$

In fact, the Bol algebra defined above is called right Bol algebra. In particular, any Lie triple system may be regarded as Bol algebra with the trivial multiplication $x\cdot y=0$, for all $x,y\in\mathfrak{B}$.

Bol algebras can be realized as the tangent algebras of Bol loops with the right Bol identity, and they allow embedding in Lie algebras [6,15].

Definition 2.2. *A linear map $\varphi:\mathfrak{B}_1\to\mathfrak{B}_2$ between two Bol algebras is called morphism of Bol algebras if it is preserve the ternary and the binary operations.*

The subspace S of Bol algebra $\mathfrak{B}$ is a sub-Bol algebra if the inclusion $j: S\hookrightarrow\mathfrak{B}$ is a morphism of Bol algebras.

Definition 2.3. *Let $(\mathfrak{B},[-;-,-],\cdot)$ be a Bol algebra over a field K, a pseudo-differentiation is a linear map $P:\mathfrak{B}\to\mathfrak{B}$ for which, there exists z ∈ B (a companion of D) such that $P(x\cdot y)=P(x)\cdot y+x\cdot P(y)+[z;x,y]+(x\cdot y)\cdot z;$ the companion is not necessarily unique.*

The set of all companions of D is denoted $Com(D)$. The map $D_{\alpha,\beta}: x\to[x;\alpha,\beta]$ is a pseudo-differentiation with companion α . β, called inner pseudo-differentiation of $\mathfrak{B}$. The pseudo-differentiations of B form a Lie algebra, denoted by pder $\mathfrak{B}$ under the natural product $[P,P']=PP'-P'P$. The algebra ipder $\mathfrak{B}$ generate by $\{D_{a,b}/a,b\in\mathfrak{B}\}$ is a Lie subalgebra of pder $\mathfrak{B}$, called the Lie algebra of inner pseudo-differentiations of B. The enlarged algebra of pseudo-differentiations of $\mathfrak{B}$ is defined as $Pder\,\mathfrak{B}=\{(D,z),D\in pder\,\mathfrak{B},z\in Com(D)\}$ and the enlarged algebra of inner pseudo-differentiation is defined as $Ipder\,\mathfrak{B}=\{(D,z),D\in ipder\,\mathfrak{B},z\in Com(D)\}$.

It is showed in [4,5] that, those algebras defined below are the Lie algebras with the brackets $[P,P']=PP'-P'P$

The direct sum $L=\mathfrak{B}\oplus Ipder\,\mathfrak{B}$ is a Lie algebra with the operation $[x,y]=D_{x,y}$, $[x,D_{a,b}]=D_{a,b}(x)$, for all x, y, a, b in B. The Lie algebra $(L,[,])$ is called the standard enveloping Lie algebra of Bol algebra $\mathfrak{B}$.

The map $\delta_a: x\mapsto x\cdot a$ is a linear map of B. We denote by $\overline{\mathfrak{B}}$ the Lie algebra generate by $\{\delta_a,a\in\mathfrak{B}\}$ with brackets $[\delta_a,\delta_b]=\delta_a\delta_b-\delta_b\delta_a$. We get an other Lie algebra $\overline{L}=\overline{\mathfrak{B}}\oplus Ipder\mathfrak{B}$ which is a subalgebra of the Lie algebra generated by linear maps of $\mathfrak{B}$.

If the subspace $\mathcal{I}$ of $\mathfrak{B}$ satisfies the stronger condition $\mathcal{I}\cdot\mathfrak{B}+(\mathcal{I};\mathfrak{B},\mathfrak{B})\subset\mathcal{I}$, then $\mathcal{I}$ is an ideal of B. An ideal $\mathcal{I}$ of B automatically satisfies $(\mathfrak{B};\mathcal{I},\mathfrak{B})\subset\mathcal{I}$ and $(\mathfrak{B};\mathfrak{B},\mathcal{I})\subset\mathcal{I}$.

For more understanding of Bol algebras and Bol loops, it is important to investigate about their representations. We defined a representation of Bol algebra as follows.

Definition 2.4. *If $\mathfrak{B}$ is a Bol algebra over a field K and V a vector field over K, the pair (ρ,δ) with the skew-symmetric bilinear map $\rho:\mathfrak{B}^2\to EndV$ and the linear map $\delta:\mathfrak{B}\to EndV$ is said to be a representation of Bol algebra $\mathfrak{B}$ in V if there exists a bilinear operation $\Delta:\mathfrak{B}^2\to EndV$ such that the following statements are satisfied:*

(R1) $\rho(u,v)=\Delta(u,v)-\Delta(v,u)$

(R2) $[\rho(a,b),\rho(u,v)]=\rho([a,u,v],b)+\rho(a,[b,u,v])$

(R3) $[\rho(u,v),\delta(a)]=\delta([a,u,v])+\Delta(u\cdot v,a)+\delta(u\cdot v)\delta(a)$

for all x, y, a, b in $\mathfrak{B}$.

The operation Δ is called a companion of the representation (ρ,δ) of the Bol algebra $\mathfrak{B}$.

In this case we can denoted by (ρ,δ,Δ,V) or simply (ρ,δ,Δ), the representation (ρ,δ,V) with companion Δ. Following the approach of Consuelo and Zelmanov for the representations of Jordan Superalgebras [2], it is equivalent to say that the vector space V is a Bol module ($\mathfrak{B}$-module) i.e., $E_V=\mathfrak{B}\oplus V$ possesses the structure of Bol algebra such that:

(a) $\mathfrak{B}$ is a sub-Bol algebra of E_V,

(b) V is an ideal of Bol algebra E_V and

(c) $x \cdot y = 0$ if both $x, y \in V$ and $[x, y, z] = 0$ if any two of x, y, z lie in V.

A particular instance where $V = B$ and we set $D(u,v) = D_{u,v}, \delta(u) = \delta_u$ the pair (D,δ) is a representation of Bol algebra with companion $\Delta(u,v) = [u,-,v]$ called *regularrepresentation* of B.

Example 2.1. *Let* $(\mathfrak{B},[-;-,-],\cdot)$ *be the Bol algebra with basis* (e_1,e_2) *over a field of complex numbers, were* $[e_1,e_2,e_1] = e_1$, $[e_2,e_1,e_2] = e_2$ *and* $e_1 \cdot e_2 = e_2$. *We recall that* $det(u,v)$ *is the determinant of the pair of vectors* (u,v) *with* $u = u_1e_1 + u_2e_2$ *and* $u = u_1e_1 + u_2e_2$. *Note that this Bol algebra arise from the classification of two-dimensional Bol algebras obtained by Kuz'min and Zaidi* [4]. *We set*

$$D(u,v) = \begin{pmatrix} -det(u,v) & 0 \\ 0 & det(u,v) \end{pmatrix}$$

$$\delta(u) = \begin{pmatrix} 0 & 0 \\ u_2 & -u_1 \end{pmatrix}$$

$$\Delta(u,v) = \begin{pmatrix} -u_1v_2 & u_1v_1 \\ u_2v_2 & -u_2v_1 \end{pmatrix}.$$

It is clear that (D,δ,Δ) is a regular representation of $\mathfrak{B}$.

Now let (ρ,δ,Δ) and (ρ',δ',Δ') be two representations of Bol algebra $\mathfrak{B}$ on V. a morphism of the representation (ρ,δ,Δ) to a representation (ρ',δ',Δ') is a linear map $f: V \to V$ such that $\rho' = f\rho$, $\delta' = f\delta$ and $\Delta' = f\Delta$. Clearly the composition of morphisms of representations is a morphism of representations. The collection of all representations and their morphisms forms a K-linear category denoted by $Rep(\mathfrak{B})$ and called the category of representations of Bol algebra $\mathfrak{B}$.

We consider $Z_1(\mathfrak{B}) = \bigcap_{y\in\mathfrak{B}} ker(-\cdot y)$ and $Z_2(\mathfrak{B}) = \bigcap_{y,z\in\mathfrak{B}} ker[-;y,z]$, the center of Bol algebra is $Z(\mathfrak{B}) = Z_1(\mathfrak{B}) \cap Z_2(\mathfrak{B})$. It is simple to see that, the kernel of the operation $<\rho,\delta>$ given by $Ker<\rho,\delta> = \{x \in \mathfrak{B} / \rho(x,\mathfrak{B}) + \delta(x) = 0\}$ is the center of $\mathfrak{B}$.

Engel's Theorem for Bol Algebras

Before giving the Engel's theorem, we first need to define and characterize the nilpotent representations.

A representation (ρ,δ,Δ) of Bol algebra $\mathfrak{B}$ in V is nilpotent if for all $x,y,z \in \mathfrak{B}$, $\rho(x,y), \delta(x)$ and $\Delta(x,y)$ are nilpotent endomorphisms; that is if there is a positive integer n such that $(\rho,\delta,\Delta)^n = 0$. Let (ρ,δ,Δ) be a representation of $\mathfrak{B}$ in V. we define the triplet $(ad_\rho, ad_\delta, ad_\Delta)$ as follows: $ad_\rho(x,y) = [\rho(x,y),-]$, $ad_\delta(x,y) = [\delta(x),-]$ and $ad_\Delta(x,y) = [\Delta(x,y),-]$.

Proposition 3.1. *With the above notations, the pair* (ad_ρ, ad_δ) *is a representation of Bol algebra* $\mathfrak{B}$ *in a vector space V with companion* ad_Δ.

Proof. The objective is to show that (R_1), (R_2) and (R_3) are satisfied. Let $a,b,u,v \in \mathfrak{B}$ and $f \in EndV$. We have

$$\begin{aligned}
[ad_\rho(a,b), ad_\rho(u,v)](f) &= [ad_\rho(a,b),[ad_\rho(u,v),f]] \\
&= [\rho(a,b),[\rho(u,v),f]] - [\rho(u,v),[\rho(a,b),f]] \\
&= [[\rho(a,b),\rho(u,v)],f] \\
&= [\rho(a,b),\rho(u,v)]f - f[\rho(a,b),\rho(u,v)] \\
&= \rho([a;u,v],b)f + \rho(a,[b,u,v])f - f\rho([a;u,v],b) - f\rho(a,[b,u,v]) \\
&= (ad_\rho([a;u,v],b) + ad_\rho(a,[b,u,v]))(f)
\end{aligned}$$

Then (R_2) holds. In other hand we have

$$[ad_\rho(a,b), ad_\rho(u,v)] = ad_\rho([a;u,v],b) + ad_\rho(a,[b,u,v])$$

$$\begin{aligned}
lclad_\rho(a,b)(f) &= [\rho(a,b),f] = \rho(a,b)f - f\rho(a,b) \\
&= \Delta(a,b)f - \Delta(a,b)f - \Delta(b,a)f + f\Delta(b,a) \\
&= [\Delta(a,b),f] - [\Delta(b,a),f] \\
&= ((ad_\Delta(a,b) - ad_\Delta(b,a))(f)
\end{aligned}$$

Therefore we have the desire equality $ad_\rho(a,b) = ad_\Delta(a,b) - ad_\Delta(b,a)$. This shows that (R_1) is satisfied. Finally, we have for all $f \in EndV$,

Thus $[ad_\rho(a,b), ad_\delta(u)] = ad_\delta([u;a,b]) + ad_\delta(a\cdot b)ad_\delta(u) + ad_\Delta(a\cdot b,u)$ and the desire conclusion follows, that is (R_3) is verified.

Definition 3.1. *The representation* $(ad_\rho, ad_\delta, ad_\Delta)$ *is called the adjoint representation of* (ρ,δ,Δ).

Now we give the link between nilpotent representation and adjoint representation. The above result arises to the representations of Lie algebras.

lemma 3.1. *Let* (ρ,δ,Δ) *be a representation of Bol algebra on the vector space V. If* (ρ,δ,Δ) *is nilpotent, then its adjoint representation is also nilpotent.*

Proof. Let (ρ,δ,Δ) be a nilpotent representation of Bol algebra, and $(ad_\rho, ad_\delta, ad_\Delta)$ its adjoint representation. Then there exists a positive integer p such that $(\rho)^p = 0$, $(\delta)^p = 0$ and $(\Delta)^p = 0$. If σ is one of the map ρ, δ, or Δ it is clear that $ad_\sigma = l_\sigma + h_\sigma$ where l_σ and h_σ are nilpotent. we have $(ad_\sigma)^{2p-1} = (l_\sigma + h_\sigma)^{2p-1} = 0$. Hence the result.

Now we are in position to prove our first main theorem.

Theorem 3.1. *Let* $\mathfrak{B}$ *be a finite dimensional Bol algebra over a field K and* $\mathcal{R}$ *consists of nilpotent representations of Bol algebra* $\mathfrak{B}$ *in a finite dimensional space V. Then there exists a vector* $v \in V^3$, $v \neq 0$ *such that* $(\rho,\delta,\Delta)(v) = 0$ *for all* $(\rho,\delta,\Delta) \in \mathcal{R}$.

Proof. We agree that $(\rho,\delta,\Delta)(v) = (\rho(v_1),\delta(v_2),\Delta(_3))$, where $v = (v_1,v_2,v_3)$, that is we identify (ρ,δ,Δ) by $(\rho(a,b),\delta(a),\Delta(a,b))$ for all a, b in $\mathfrak{B}$. It is clear that $\mathcal{R}$ is a subspace of $(Env)^3$ and we can define on it the following bracket $[(f,g,h),(f',g',h')] = ([f,f'],[g,g'],[h,h'])$. $(\mathcal{R},[-,-])$ is a Lie algebra.

The proof of the theorem goes by induction on $dim\mathcal{R}$. When $dim\mathcal{R} = 1$, since $\mathcal{R}$ is generated by a single nilpotent representation then the claim is immediate.

Suppose now that the claim is true for all subalgebras of nilpotent representations spaces of dimension less than $dim\mathcal{R} \geq 1$.

Since, $dim\mathcal{R} \geq 1$, we have a proper Lie subalgebra $L \subseteq \mathcal{R}$. We can choose L to be a maximal subalgebra. We show before continuing that, L has a codimension one in $\mathcal{R}$ and L is an ideal.

L acts via the adjoint operator on $\mathcal{R}$ and L. In the latter case, since $dimL < dim\mathcal{R}$, we know by Engel's theorem apply for L, that there exists a nonzero element $\overline{r} \in \mathcal{R}/L$ such that $[l,\overline{r}] = 0$ $\overline{(\rho,\delta,\Delta)} \in \mathcal{R}/L$ and $[(l_1,l_2,l_3),\overline{(\rho,\delta,\Delta)}] = \overline{0}$ for $(l_1,l_2,l_3) \in L$. We know that $\overline{(\rho,\delta,\Delta)} = (\rho,\delta,\Delta) + L$; then $(\rho,\delta,\Delta) \in \mathcal{R} - L$. It follows that $[K(\rho,\delta,\Delta) + L, L] \subseteq L$. Moreover $[K(\rho,\delta,\Delta) + L, K(\rho,\delta,\Delta) + L] \subseteq L$. These imply that $K(\rho,\delta,\Delta) + L$ is a Lie subalgebra of $\mathcal{R}$, and contains L as an ideal. By maximality of L, it follows that $Kr + L = \mathcal{R}$, so we are done.

Now we define the vector space $\mathbf{W} = \{w \in V^3 / Lw = 0\}$. Let $w = (w_1,w_2,w_3) \in \mathbf{W}$ and $(\rho,\delta,\Delta) \in L$, then $(l_1,l_2,l_3)(\rho,\delta,\Delta)(w) = 0$ for all $(l_1,l_2,l_3) \in L$. Other we have

$$\begin{aligned}
(l_1,l_2,l_3)(\rho,\delta,\Delta)(w) &= (\rho,\delta,\Delta)(l_1,l_2,l_3)(w) + [(l_1,l_2,l_3),(\rho,\delta,\Delta)](w) \\
&= [(l_1,l_2,l_3),(\rho,\delta,\Delta)](w)
\end{aligned}$$

and $[(l_1,l_2,l_3),(\rho,\delta,\Delta)]\in L$. Since L is an ideal, we have also $[(l_1,l_2,l_3),(\rho,\delta,\Delta)](w)=0$.

Now we have $\mathcal{R}=K(\rho,\delta,\Delta)+L$ for some $(\rho,\delta,\Delta)\in L$. We know that (ρ,δ,Δ) is a nilpotent operator on **W,** so $ker(\rho,\delta,\Delta)\cap\mathbf{W}\neq 0$. Let $v=(v_1,v_2,v_3)\in ker(\rho,\delta,\Delta)\cap\mathbf{W}$ such that $v\neq 0$; then any element of L and r annihilates v.

An Extension of Ado-Iwasawa Theorem to Bol Algebras

Let L be a finite-dimensional Lie algebra over a field K. The classical Ado-Iwasawa theorem asserts the existence of a finite-dimensional L-module which gives a faithful representation of L. However, Filippov proved [17] showed that this theorem does not hold for Malcev algebras, that is homogeneous Bol algebras. Thus it is not hold for general Bol algebras.

For the Lie algeras, the Poincaré-Birkhoff-Witt theorem says that any Lie algebra L is a subalgebra of A^- for some unital associative algebra A. In the case that L is finite dimensional, the Ado-Iwasawa theorem says that A can be taken finite dimensional too. This extension of Ado-Iwasawa theorem was established for the Malcev algebras by Pérez-Izqquierdo and Shestakov [14]. There is a version of the Poincaré-Birkhoff-Witt theorem for Bol algebra proved by Kuz'min and Zaidi [4]. Now let $\mathfrak{B}$ be a Bol algebra [14] that there is an alternative algebra A and an injective map $\mathfrak{B}\to RN_{alt}(A)$, where $RN_{alt}(A)=\{a\in A/(x,a,y)=-(x,y,a)\}$ is the generalized right alternative nucleus. In this section we prove that if $\mathfrak{B}$ is a finite-dimensional Bol algebra then A can be taken finite dimension too. Our second main result is the following.

Theorem 4.1. *Let $\mathfrak{B}$ be a finite-dimensional right Bol algebra over a field of characteristic $\neq 2,3$. Then there exists a unital finite-dimensional algebra A and a monomorphism of Bol algebra $j:\mathfrak{B}\to RN_{alt}(A)$.*

Proof. Let $\mathfrak{B}$ be a Bol algebra, according to Pérez-Izquierdo [5], there exists a linear map $j:\mathfrak{B}\to RN_{alt}(U(\mathfrak{B}))$, $a\mapsto a$ such that $j(a\cdot b)=ab-ba$ and $j(a,b,c)=(ab)c-(ac)b-[b,c]a$, where $U(\mathfrak{B})$ is the universal enveloping algebra of $\mathfrak{B}$. Since $RN_{alt}(U(\mathfrak{B}))$ is closed under the binary product $[-,-]$ given by the commutators and the ternary operation $[a,b,c]=(ab)c-(ac)b-[b,c]a$ for all a,b,c in $RN_{alt}(U(\mathfrak{B}))$. By the methods of Pérez-Izquierdo [5], $RN_{alt}(U(\mathfrak{B}))$ with the binary and ternary operations defined above has the structure of Bol algebra. Thus j is a monomorphism of Bol algebras. Let $E_{\mathfrak{B}}$ be the Lie enveloping algebra of $\mathfrak{B}$. Then $E_{\mathfrak{B}}=E_+\oplus E_-$ is the $\mathbb{Z}_2$-gradation and $E_-\approxeq\mathfrak{B}$ as vector space. According to Pérez-Izquierdo and Shestakov [14], there exists a two side ideal $\mathcal{I}\subseteq U(\mathfrak{B})$ of finite codimension. Then $A=U(\mathfrak{B})/\mathcal{I}$ is a unital finite-dimensional algebra and there exists an injective map $j:\mathfrak{B}\to U(\mathfrak{B})$. The injective map j induces a monomorphism of Bol algebras $j:\mathfrak{B}\to RN_{alt}(A)$.

The Category of Representations of Bol Algebra

We give a relation between the category of representation of Bol algebra $\mathfrak{B}$ and the category of representations of its universal enveloping algebra. As immediate consequence, we show that the representation category of a Bol algebra is monoidal, or tensor category. We recall that the category of representations of Bol algebras is $Rep(\mathfrak{B})$, and the one of finite dimensional representations of Bol algebra is $rep(\mathfrak{B})$. Let $A=(A,\cdot,\Delta,\varepsilon)$ be a bialgebra, Mod(A) means the category of left A-modules (ie., representations of A). If U, V are left A-modules, then the tensor product becomes a left A-module with multiplication rule $a\cdot(u\otimes v)=\Delta(a)\cdot(u\otimes v)$ for all $a\in A$, $u\in U$ and $v\in V$. The field K is also a left A-module by $a\cdot\varsigma=\varepsilon(a)\varsigma$. The category of left A-modules is equivalent to the category of (A, A)-bimodules. Any (A, A)-bimodule can be considered as left module over $A\otimes A^{op}$, where A^{op} is define on the same space as A, by new multiplication $x\cdot y=y\cdot x$. We know in virtue of Pérez-Izquierdo [5] that for a given Bol algebra $(\mathfrak{B},[-,-],[-,-,-])$ there exists a universal enveloping $U(\mathfrak{B})$ endowed with the structure of bialgebra, that is $(U(\mathfrak{B}),\cdot,\Delta,\varepsilon)$ is a bialgebra. Analogously we denote $Rep(U(\mathfrak{B}))$ the category of representation of the bialgebra $(U(\mathfrak{B}),\cdot,\Delta,\varepsilon)$. Now we state an equivalent characterization of the representation category $Rep(\mathfrak{B})$. We prove our third main result.

Theorem 5.1. *The category of representations of Bol algebra $Rep(\mathfrak{B})$ is equivalent to the category of representations of its universal enveloping algebra $Rep(U(\mathfrak{B}))$.*

Proof. We recall that $Rep(\mathfrak{B})$ is the category of modules over the Bol algebra $\mathfrak{B}$. Following the consideration of Consuelo and Zelmanov [13], apply for the modules over Bol algebras, every B-module has the form $E_V=\mathfrak{B}\oplus V$, where V is a vector space over a field K and E_V possesses the structure of Bol algebra such that:

(a) $\mathfrak{B}$ is a sub-Bol algebra of E_V,

(b) V is an ideal of Bol algebra E_V and

(c) x . y=0 if both x, $y\in V$ and $[x, y, z] = 0$ if any two of x, y, z lie in V.

We define the multiplication $U(\mathfrak{B})\times V\to V$ by $a\cdot x=\varepsilon(a)\cdot x$. We consider the following mapping defined from $Rep(\mathfrak{B})$ to Mod($U(\mathfrak{B})$) define on the objets by $F(E_V)=V$. The map F is naturally extended on the morphisms. If U and V are the images of E_U and E_V under F, in virtue of Pérez-Izqquierdo [5] there exits a map $\mu:\mathfrak{B}\to U(\mathfrak{B})\otimes U(\mathfrak{B})$ with $\mu(a)=a\otimes 1+1\otimes a$. This implies that $U\otimes V$ is a ($U(\mathfrak{B})$)-module.

Conversely, let V be a ($U(\mathfrak{B})$)-module, in virtue of Pérez-Izquierdo [5] there exist an injective map $\eta:\mathfrak{B}\to U(\mathfrak{B})$. We define the multiplication $\mathfrak{B}\times V\to V$ by $a\cdot x=\eta(a)\cdot x$. Then V has the structure of module. We set now the mapping G from Mod($U(\mathfrak{B})$ to $Rep(\mathfrak{B})$ by $G(V)=E_V$. It remains to define the image of $U\otimes V$. Let E_U and E_V be two modules over $\mathfrak{B}$, We set $E=\mathfrak{B}\oplus U\otimes V$. We define the binary operation by $[a,u\otimes v]_{\otimes}=[a,u]\otimes v$; $[a,u\otimes v]_{\otimes}=[a,u]\otimes v$ and a ternary by $[a,b,u\otimes v]_{\otimes}=[a,b,u]\otimes v$; $[a,u\otimes v,b]_{\otimes}=[a,u,b]\otimes v$ and $[a,b,u\otimes v]_{\otimes}=[a,b,u]\otimes v$ for all a, b in $\mathfrak{B}$, u in V and v in V. We assume also that the restrictions of $[-,-]_{\otimes}$ and $[-,-,-]_{\otimes}$ on $\mathfrak{B}$ correspond respectively to the binary and ternary operations of B; and $x\cdot y=0$ if both $x,y\in U\otimes V$ and $[x,y,z]=0$ if any two of x,y,z lie in $U\otimes V$.

It remains to show that $(E,[-,-]_{\otimes},[-,-,-]_{\otimes})$ is a Bol algebra, that is the conditions (i) - (iv) hold. By the definition, the condition (i) is satisfied. Now let x,y,z,α,β in $\mathfrak{B}$; u in U and v in V. We have

$$\begin{aligned}[x;y,u\otimes v]+[u\otimes v;x,y]+[y;u\otimes v,x] &= [x;y,u]\otimes v+[u;x,y]\otimes v+[y;z,u]\otimes v\\ &= ([x;y,u]+[u;x,y]+[y;z,u])\otimes v\\ &= 0,\end{aligned}$$

this shows that (ii) is true.

Now let us show that (iii) holds. We have

$$\begin{aligned}[[x;y,u\otimes v];\alpha,\beta] &= [[x;y,u]\otimes v;\alpha,\beta]\\ &= [[x;y,u];\alpha,\beta]\otimes v\\ &= ([[x;\alpha,\beta];y,u]+[x;[y;\alpha,\beta],u]+[x;y,[u;\alpha,\beta]])\otimes v\\ &= [[x;\alpha,\beta];y,u\otimes v]+[x;[y;\alpha,\beta],u\otimes v]+[x;y,[u\otimes v;\alpha,\beta]]\end{aligned}$$

One can show that the above equality holds for any x,y,α,β stands for $u\otimes v$. That is (iii) holds.

Citation: Ndoune N, Bouetou TB (2015) On Representations of Bol Algebras. J Generalized Lie Theory Appl S2: 005. doi:10.4172/1736-4337.S2-005

Finally, we have

$$\begin{aligned}[][u\otimes v;y];\alpha,\beta] &= [[u;y]\otimes v;\alpha,\beta] \\ &= [[u;y];\alpha,\beta]\otimes v \\ &= ([u;\alpha,\beta]\cdot y+[u,[y;\alpha,\beta]]+[[\alpha,\beta];u,y]+[[u,y],[\alpha,\beta]])\otimes v.\end{aligned}$$

Thus $[[u\otimes v;y];\alpha,\beta]=[u\otimes v;\alpha,\beta]\cdot y+[u\otimes v,[y;\alpha,\beta]]+[[\alpha,\beta];u\otimes v,y]+[[u\otimes v,y],[\alpha,\beta]]$. One can show this equality for any y, α, β stands for $u\otimes v$. This completes the proof.

Definition 5.1. *A monoidal (tensor) category* $(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda)$ *is a category* $\mathcal{C}$ *equipped with tensor functor* $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ *, with a fix objet* $\mathbf{1}$ *(called the unit of a tensor category),* $\alpha:\otimes\circ(\otimes\times Id)\to\otimes\circ(Id\times\otimes)$, $\lambda:\mathbf{1}\otimes-\to Id$, $-\otimes\mathbf{1}\to Id$ *are natural isomorphisms such that the associativity and unitary constraints hold, or equivalently the pentagon and the triangle diagrams are commutative* [18-20].

We can now give a special characterization of the category of representations of Bol algebra as a consequence of the above proposition.

Corollary 5.1. *Every category of representations of Bol algebras is a monoidal category.*

Proof. It was proved by Kassel [20] that $(A,\cdot,\Delta,\varepsilon)$ is bialgebra if and only if the category Mod(A) is monoidal category. In virtue of Theorem 5.0.6, the category of representations of Bol algebra is equivalent to the category of representations of its enveloping algebra endowed with bialgebra structure. Hence the category *Rep*(B) is monoidal.

More recently it was proved by Huang and Torecillas [21], that the path coalgebra *KQ* of a given quiver *Q* always admits a bialgebra structure. So the monoidal category arising from this quiver bialgebra is the category of representations of the bialgebra *KQ*. This leads to the following conjecture.

Conjecture 5.1. *Find necessary and sufficient conditions for the existence of quiver Q such that the monoidal category arising from quiver bialgebra KQ is the category of representations of a Bol algebra over algebraically closed field K.*

A monoidal category is said to be finite, if it is equivalent to the category of finite dimensional comodules over the finite dimensional coalgebra. Thus the category *Rep*(B) of finite dimensional representations is finite monoidal category. This is a particular case of tensor categories of Etingof et al. [19]. The particular case where *Q* is a quiver without loops and 2-cyles should leads to strong relation between Bol algebras and *cluster algebras* of Fomin and Zelevinsky [22,23] for more details. In the same vein, it has been shown in literature of Schauenburg [24] that if *A* is a finite dimensional bialgebra, then *A* is Hopf algebra if and only if the category of finitely generated *A*-modules is rigid, that is finitely generate modules admit dual objets. This allows us to the following conjecture.

Conjecture 5.2. *Find necessary and sufficient conditions for a finite dimensional Bol algebra to have Hopf algebra as universal enveloping algebra.*

Representations of Free Bol Algebra *Bol*[*X*] of Finite Dimension

Let $X=\{x_1,x_2,...,x_n\}$, we construct the set of binary-ternary monomials $BT[X]$, and we assume that $BT[X]$ is closed under $[-,-]$ and $[-,-,-]$. Let $BT[X]=\{\sum_{=1}\alpha_i x_i\mid\alpha_i\in\ \}$ be the space spanned by X. We define the multiplication by the following rules: if $f=\sum_{i=1}^{n}\alpha_i x_i$, $g=\sum_{j=1}^{n}\beta_j x_j$ and $h=\sum_{k=1}^{n}\gamma_k x_k$ in $BT[X]$, then $[f,g]=\sum_{i,j=1}^{n}\alpha_i\beta_j[x_i,x_j]$, $[f,g,h]=\sum_{i,j,k=1}^{n}\alpha_i\beta_j\gamma_k[x_i,x_j,x_k]$. The free Bol algebra $Bol[X]$ is the free binary-ternary algebra $BT[X]$ satisfying the identities (i) - (iv). The Bol types of degree m are always to construct a product of degree m in $Bol[X]$. For general construction and more details of the free Bol algebra $Bol[X]$ [25,26]. In studies of Peresi [26] it has been shown that any multilinear identity f of degree m can be written as a linear combination of multilinear monomials. We denote the Bol types of degree m by B_1, B_2, …, $B_{b(m)}$, that is $f=f_1+..+f_{b(m)}$, where f_k is a linear combination of polynomial having Bol type k. Therefore the author regards f as an element of $b(m)$ copies of $\mathbb{F}S_m$, where $\mathbb{F}S_m$ is group algebra of the group of permutation S_m. Applying the representation $\Phi_\sigma:\mathbb{F}S_m\to Md_\sigma(\mathbb{F})$, ($\sigma$ partition of m) of S_m to f we obtain the representation matrix of f in partition σ: $(\Phi_\sigma(f_1)\mid\Phi_\sigma(f_2)\mid...\mid\Phi_\sigma(f_{b(m)}))$.

Now let V be finite dimensional space, dim(V) = s and $\mathfrak{B}$ is a Bol algebras of dimension n. Give a representation (ρ,δ,Δ) of $\mathfrak{B}$ over the space V is equivalent to give the matrix $(D(u,v)\mid\delta(u)\mid\Delta(u,v))$, where $D(u,v)$, $\Delta(u,v)$ are $s\times n$ $s\times n$ matrices and $\delta(u)$ is also a $s\times n$ matrix. Hence the block matrix $(D(u,v)\mid\delta(u)\mid\Delta(u,v))$ is a $(3n)\times s$ matrix.

In the special case where $\mathfrak{B}=Bol[X]$, $K=\mathbb{F}$ and $V\quad S$, with Bol types B_1, B_2, …, $B_{b(m)}$ the representation matrix $(\Phi_\sigma(f_1)\mid\Phi_\sigma(f_2)\mid...\mid\Phi_\sigma(f_{b(m)}))$ of f corresponds to the matrix δ_f, that is the expression $\delta(f)=(\delta(f_1)\mid\delta(f_2)\mid...\mid\delta(f_{b(m)}))$. At this specific case mentioned by Peresi and Jacobson [26,27], the representation of element f is understood as a the representation of Bol algebra $Bol[X]$ given by the matrix $(D(f,0)\mid\delta(f)\mid\Delta(f,0))$.

Actually we recall the classification theorem of Kuz'min and Zaidi for two-dimensional Bol algebras [4] which states as follows.

Theorem 5.2. (Kuz'min-Zaidi). *Every Bol algebra* B *of dimension two over* $\mathcal{R}$ *has a canonical basis* (e_1,e_2) *in which its multiplication table is one of the following:*

I. $[e_1,e_2]=0$, $[e_2,e_1,e_2]=\varepsilon_1 e_1$, $[e_1,e_2,e_1]=\varepsilon_2 e_2$, where $(\varepsilon_1,\varepsilon_2)=(0,0)$, $(-1,0)$, $(1,0)$, $(1,-1)$, $(1,1)$, $(-1,-1)$

II. $[e_1,e_2]=e_2$, $[e_2,e_1,e_2]=\varepsilon e_1$, $[e_1,e_2,e_1]=\beta e_2$, where $\varepsilon=0,-1,1$; $[e_2,e_1,e_2]=e_2$, $[e_1,e_2,e_1]=e_1$.

Now we are in position to prove our classification result for regular representations of the two-dimensional Bol algebras.

Theorem 5.3. *Every regular representation of two-dimensional Bol algebra* B *over K is up to equivalence of matrices given by one of the following matrices:*

$$(i)\quad R_1(u,v)=\begin{pmatrix}0 & \varepsilon_1 det(u,v) & 0 & 0 & u_2v_2\varepsilon_1 & -u_2v_1\varepsilon_1\\ -\varepsilon_2 det(u,v) & 0 & 0 & 0 & -u_1v_2\varepsilon_2 & u_1v_1\varepsilon_2\end{pmatrix}$$

$$(ii)\quad R_2(u,v)=\begin{pmatrix}0 & \varepsilon det(u,v) & 0 & 0 & u_2v_2\varepsilon & -u_2v_1\varepsilon\\ -\beta det(u,v) & 0 & u_2 & -v_1 & -u_1v_2\beta & u_1v_1\beta\end{pmatrix}$$

$$(iii)\quad R_3(u,v)=\begin{pmatrix}-det(u,v) & 0 & 0 & 0 & u_1v_2 & u_1v_1\\ 0 & det(u,v) & u_2 & -v_1 & u_2v_2 & -u_2v_1\end{pmatrix}$$

Proof. In virtue of classification theorem of Kuz'min and Zaidi [4], every Bol algebra of dimension two is of type (I) or of type (II) by using the items of their theorem.

We suppose in the first case that our Bol algebra is of type (I), that is B has a canonical basis (e_1,e_2) in which its multiplication table is given by $[e_1,e_2]=0$, $[e_2,e_1,e_2]=\varepsilon_1 e_1$, $[e_1,e_2,e_1]=\varepsilon_2 e_2$, where

$(\varepsilon_1,\varepsilon_2)=(0,0)$, $(-1,0)$, $(1,0)$, $(1,-1)$, $(1,1)$, $(-1,-1)$.

Let u and v be the two vectors of B, with $u=u_1e_1+u_2e_2$ and $u=u_1e_1+u_2e_2$. We have $D(u,v)(e_1)=u_1v_2[e_1,e_1,e_2]+u_2v_1[e_1,e_2,e_1]$. Since $[e_1,e_1,e_2]=-[e_1,e_2,e_1]$, we have

$$\begin{aligned} D(u,v)(e_1) &= -u_1v_2[e_1,e_2,e_1]+u_2v_1[e_1,e_2,e_1] \\ &= (-u_1v_2+u_2v_1)\varepsilon_2 e_2 \\ &= -det(u,v)\varepsilon_2 e_2, \end{aligned}$$

We have also

$$\begin{aligned} D(u,v)(e_2) &= u_1v_2[e_2,e_1,e_2]+u_2v_1[e_2,e_2,e_1] \\ &= (u_1v_2-u_2v_1)\varepsilon_1 e_1 \\ &= det(u,v)\varepsilon_1 e_1. \end{aligned}$$

Thus $D(u,v)=\begin{pmatrix} 0 & \varepsilon_1 det(u,v) \\ -\varepsilon_2 det(u,v) & 0 \end{pmatrix}$.

Now we compute the matrix of $\Delta(u,v)$ as follows. We have

$$\begin{aligned} \Delta(u,v)(e_1) &= u_1v_2[e_1,e_1,e_2]+u_2v_2[e_2,e_1,e_2] \\ &= u_2v_2\varepsilon_1 e_1 - u_1v_2\varepsilon_2 e_2, \end{aligned}$$

and

$$\begin{aligned} \Delta(u,v)(e_2) &= u_1v_1[e_1,e_2,e_1]+u_2v_1[e_2,e_2,e_1] \\ &= -u_2v_1\varepsilon_1 e_1 + u_1v_1\varepsilon_2 e_2, \end{aligned}$$

hence $\Delta(u,v)=\begin{pmatrix} u_2v_2\varepsilon_1 & -u_2v_1\varepsilon_1 \\ -u_1v_2\varepsilon_2 & u_1v_1\varepsilon_2 \end{pmatrix}$. Because $[e_1,e_2]=0$, we have $\delta(u)=0$. Therefore the bloc matrix $(D(u,v)\,|\,\delta(u)\,|\,\Delta(u,v))$ corresponds to the matrix $R_1(u,v)$.

The second case corresponds to Bol algebra of type (I), that is B has a canonical basis (e_1,e_2) in which its multiplication table is given by $[e_1,e_2]=e_2$, $[e_2,e_1,e_2]=\varepsilon e_1$, $[e_1,e_2,e_1]=\beta e_2$, where $\varepsilon=0,-1,1;$ $[e_2,e_1,e_2]=e_2$, $[e_1,e_2,e_1]=e_1$.

If $[e_1,e_2]=e_2$, $[e_2,e_1,e_2]=\varepsilon e_1$, $[e_1,e_2,e_1]=\beta e_2$, where $\varepsilon=0,-1,1;$ we use the analogous methods as at the first case to get $D(u,v)=\begin{pmatrix} 0 & \varepsilon det(u,v) \\ -\beta det(u,v) & 0, \end{pmatrix}$ $\delta(u)=\begin{pmatrix} 0 & 0 \\ u_2 & -u_1 \end{pmatrix}$ and $\Delta(u,v)=\begin{pmatrix} u_2v_2\varepsilon & -u_2v_1\varepsilon \\ -u_1v_2\beta & u_1v_1\beta \end{pmatrix}$. Hence the bloc matrix $(D(u,v)\,|\,\delta(u)\,|\,\Delta(u,v))$ corresponds to the matrix $R_2(u,v)$.

Finally, for $[e_1,e_2]=e_2$ and $[e_2,e_1,e_2]=e_2$, $[e_1,e_2,e_1]=e_1$, we have $(D(u,v)\,|\,\delta(u)\,|\,\Delta(u,v))=\begin{pmatrix} -det(u,v) & 0 & 0 & 0 & u_1v_2 & u_1v_1 \\ 0 & det(u,v) & u_2 & -v_1 & u_2v_2 & -u_2v_1 \end{pmatrix}$, this end the proof.

Acknowledgement

We aim at sending our sincere gratitude to the reviewers for their reports on our paper and their availability. The second author thanks the IHES for hospitality through the Launsbery Foundation and, the IMU for travel support throughout a grant from the Simons Foundation during the writing of this paper.

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