On irreducible weight representations of a new deformation $\mathfrak{U}_q(sl_2)$ of $U(sl_2)$

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Abstract

Starting from a Hecke R-matrix, Jing and Zhang constructed a new deformation $\mathfrak{U}_q(sl_2)$ of $U(sl_2)$ and studied its finite dimensional representations in [Pacific J. Math., **171** (1995), 437-454]. In this note, more irreducible representations for this algebra are constructed. At first, by using methods in noncommutative algebraic geometry the points of the spectrum of the category of representations over this new deformation are studied. The construction recovers all finite dimensional irreducible representations classified by Jing and Zhang, and yields new families of infinite dimensional irreducible weight representations.

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1 Introduction

Spectral theory of abelian categories was first initiated by Gabriel in [5]. In particular, Gabriel defined the injective spectrum of any noetherian Grothendieck category. The injective spectrum consists of isomorphism classes of indecomposable injective objects in the category endowed with the *Zariski* topology. If R is a commutative noetherian ring, then the injective spectrum of the category of all R-modules is homeomorphic to the prime spectrum of R. This homeomorphism is a part (and the main step in the argument) of the Gabriel's reconstruction Theorem [5], according to which any noetherian commutative scheme can be uniquely reconstructed up to isomorphism from the category of quasi-coherent sheaves on it.

The general spectrum of arbitrary abelian category was defined by Rosenberg [15]. Using this spectrum, one can reconstruct any quasi-separated and quasi-compact scheme from the category of quasi-coherent sheaves on the scheme.

Isomorphism classes of simple objects of any abelian category correspond to closed points of its spectrum, and, under some mild finiteness conditions, this correspondence is bijective. For instance, the correspondence is bijective for the category of modules over an associative ring, or, more generally, for the category of quasi-coherent sheaves on a noncommutative (that is not necessarily commutative) scheme.

Thus, in order to study irreducible representations, one can first study the spectrum of the category of all representations, then single out its closed points.

As a specific application of spectral theory to representation theory, points of the spectrum of the category of modules have been constructed for a large family of algebras, which are called Hyperbolic algebras in [15]. And it is a pure luck that a lot of important “small” algebras, including $U(sl_2)$ and its quantized versions, are Hyperbolic algebras.

Starting from a Hecke R-matrix, Jing and Zhang constructed a new deformation $\mathfrak{U}_q(sl_2)$ of $U(sl_2)$ (which is denoted by $U_q(sl_2)$ in [7]). This algebra shares quite a few properties with $U(sl_2)$, and all its finite dimensional irreducible representations are constructed explicitly in [7]. On the other hand, $\mathfrak{U}_q(sl_2)$ has a natural bialgebra structure, but, it is not a Hopf algebra. Besides, an example constructed in [7] shows that not all of its finite dimensional representations are completely reducible. So the representation theory of this new deformation differs from the representation theory of the standard quantized enveloping algebra of sl_2. Therefore, it seems to be an interesting problem to further study the irreducible representations of $\mathfrak{U}_q(sl_2)$.

Note that $\mathfrak{U}_q(sl_2)$ also belongs to a general class of algebras studied in [2], where the irreducible weight modules for these algebras are classified based on the study of certain dynamical system. However, this note studies representations from the perspective of noncommutative algebraic geometry, and serves the purpose of providing a more transparent construction of irreducible weight representations for $\mathfrak{U}_q(sl_2)$. Indeed, based on sufficient knowledge about the structure of $\mathfrak{U}_q(sl_2)$, we are able to carry out explicit calculations.

To solve the problem, we first construct families of points for the spectrum of the category of representations for this deformation. Applied to the study of representations, our construction recovers all finite dimensional irreducible representations of $\mathfrak{U}_q(sl_2)$ constructed in [7], and produces new families of infinite dimensional irreducible representations as well. This work can also be regarded as one more nice application of the methods in noncommutative algebraic geometry to representation theory. For more details about spectral theory, we refer the reader to [15].

The paper is organized as follows. In Section 2, we give a very brief review on the spectrum of an abelian category. In Section 3, we review the concept of Hyperbolic algebras. In Section 4, we review some basic facts about the new deformation $\mathfrak{U}_q(sl_2)$ introduced by Jing and Zhang, and prove some supplementary useful Lemmas. In Section 5, we construct families of points of the spectrum for $\mathfrak{U}_q(sl_2)$. Then we use them to construct irreducible representations for this new deformation $\mathfrak{U}_q(sl_2)$. We follow the notations in [7], but we will always denote the new deformation by $\mathfrak{U}_q(sl_2)$. The base field will be fixed to be the complex field $\mathbb{C}$, and q is not a root of unity.

2 Basic facts about the spectrum of any abelian category

In this section, we are going to review some basic notions and facts about the spectrum of any abelian category for the purpose of understanding the rest of this work. First, we review the definition of the spectrum of any abelian category, then we explain its applications in representation theory.

Let C_X be an abelian category. Let $M, N \in C_X$ be any two objects in C_X; We say that $M \succ N$ if and only if N is a subquotient of the direct sum of finite copies of M. It is easy to verify that $\succ$ is a preorder. We say $M \approx N$ if and only if $M \succ N$ and $N \succ M$. It is obvious that $\approx$ is an equivalence. Let $Spec(X)$ be the family of all nonzero objects $M \in C_X$ such for all nonzero subobject N of M, $N \succ M$. The spectrum of the abelian category C_X is defined [15] by

$$\mathbf{Spec}(X) = Spec(X)/\approx$$

It is endowed with a natural analogue of the Zariski topology.

The spectrum of an abelian category is one of the fundamental notions of noncommutative algebraic geometry.

The spectrum also has important applications in representation theory. This is due to the fact that there is a natural embedding of the set of isomorphism classes of simple objects of the category C_X into the set of closed points of $\mathbf{Spec}(X)$. If every nonzero object of the category C_X has a simple subquotient, then the embedding is a bijection. In particular, if A is an algebra and C_X is the category of left A-modules, then the closed points of $\mathbf{Spec}(X)$ are in a bijective correspondence with isomorphism classes of irreducible A-modules. The spectrum $\mathbf{Spec}(X)$ has much better functorial properties than the set of its closed points, like in the case of commutative algebraic geometry. So one can study the spectrum via the methods in noncommutative algebraic geometry, then apply to representation theory ([15]).

2.1 The left spectrum of a ring

If C_X is the category $A - mod$ of left modules over a ring A, then it is sometimes convenient to express the points of $\mathbf{Spec}(X)$ in terms of left ideals of the ring A. In order to do it, the *left spectrum* $Spec_l(A)$ was defined in [15], which is by definition the set of all left ideals p of A such that A/p is an object of $Spec(X)$. The relation $\succ$ on $A - mod$ induces a *specialization* relation among left ideals, in particular, the specialization relation on $Spec_l(A)$. Namely, $A/m \succ A/n$ iff there exists a finite subset x of elements of A such that the ideal $(n : x) = \{a \in A \mid ax \subset n\}$ is contained in m. Following [15], we denote this by $n \leq m$. Note that the relation $\leq$ is just the inclusion if n is a two-sided ideal. In particular, it is the inclusion if the ring A is commutative. The map which assigns to an element of $Spec_l(A)$ induces a bijection of the quotient $Spec_l(A)/\approx$ of $Spec_l(A)$ by the equivalence relation associated with $\leq$ onto $\mathbf{Spec}(X)$. From now on, we will not distinguish $Spec_l(A)/\approx$ from $\mathbf{Spec}(X)$ and will express results in terms of the left spectrum.

The rest of this paper is a typical application of spectral theory to representation theory of "small" algebras.

3 Hyperbolic algebras $R\{\theta, \xi\}$ and points of the spectrum

Hyperbolic algebras are studied by Rosenberg in [15] and by Bavula under the name of Generalized Weyl algebras in [1]. Hyperbolic algebra structure is very convenient for the construction of points of the spectrum. And a lot of interesting algebras such as $U(sl_2)$ and its quantized versions have a Hyperbolic algebra structure. Points of the spectrum of the category of modules over these algebras have been constructed in [15]. In this section, we review some basic facts about Hyperbolic algebras and two important construction theorems due to Rosenberg ([15]). Let θ be an automorphism of a commutative algebra R; and let ξ be an element of R. Then we have the following definition from [15].

Definition 3.1. We denote by $R\{\theta, \xi\}$ the corresponding $R-$algebra generated by x, y subject to the following relations:

$$xy = \xi, \quad yx = \theta^{-1}(\xi), \quad xa = \theta(a)x, \quad ya = \theta^{-1}(a)y$$

for all $a \in R$. And $R\{\theta, \xi\}$ is called a Hyperbolic algebra over R.

First, we look at some basic examples of Hyperbolic algebras.

Example 3.1. The first Weyl algebra A_1 is a Hyperbolic algebra over $R = \mathbb{C}[xy]$ with $\theta(xy) = xy+1$; $U(sl_2)$ and its quantized versions are Hyperbolic algebras too. And the reader to referred to [15] for more details about Hyperbolic algebras.

Let $C_X = R\{\theta, \xi\}-mod$ be the category of left modules over $R\{\theta, \xi\}$. The Hyperbolic algebra structure is very convenient for the construction of points of the spectrum $\mathbf{Spec}(X)$. We replace the study of $\mathbf{Spec}(X)$ by the study of the left spectrum $Spec_l(R\{\theta, \xi\})$ of the hyperbolic algebra $R\{\theta, \xi\}$ (see 2.1 above).

For the left spectrum of the Hyperbolic algebra, we have the following two crucial construction theorems due to Rosenberg from [15].

Theorem 3.1 ([15], Theorem 3.2.2.)**.** *1. Let $P \in Spec(R)$, and the orbit of P under the action of the automorphism θ is infinite.*

(a) If $\theta^{-1}(\xi) \in P$, and $\xi \in P$, then the left ideal

$$P_{1,1} := P + R\{\theta, \xi\}x + R\{\theta, \xi\}y$$

is a two-sided ideal from $Spec_l(R\{\theta, \xi\})$.

(b) *If $\theta^{-1}(\xi) \in P$, $\theta^i(\xi) \notin P$ for $0 \leq i \leq n-1$, and $\theta^n(\xi) \in P$, then the left ideal*

$$P_{1,n+1} := R\{\theta,\xi\}P + R\{\theta,\xi\}x + R\{\theta,\xi\}y^{n+1}$$

belongs to $Spec_l(R\{\theta,\xi\})$.

(c) *If $\theta^i(\xi) \notin P$ for $i \geq 0$ and $\theta^{-1}(\xi) \in P$, then*

$$P_{1,\infty} := R\{\theta,\xi\}P + R\{\theta,\xi\}x$$

belongs to $Spec_l(R\{\theta,\xi\})$.

(d) *If $\xi \in P$ and $\theta^{-i}(\xi) \notin P$ for all $i \geq 1$, then the left ideal*

$$P_{\infty,1} := R\{\theta,\xi\}P + R\{\theta,\xi\}y$$

belongs to $Spec_l(R\{\theta,\xi\})$.

2. *If the ideal P in (b), (c) or (d) is maximal, then the corresponding left ideal of $Spec_l(R\{\theta,\xi\})$ is maximal.*
3. *Every left ideal $Q \in Spec_l(R\{\theta,\xi\})$ such that $\theta^\nu(\xi) \in Q$ for a $\nu \in \mathbb{Z}$ is equivalent to one left ideal as defined above uniquely from a prime ideal $P \in Spec(R)$. The latter means that if P and P' are two prime ideals of R and (α,β) and (ν,μ) take values $(1,\infty), (\infty,1), (\infty,\infty)$ or $(1,n)$, then $P_{\alpha,\beta}$ is equivalent to $P'_{\nu,\mu}$ if and only if $\alpha=\nu, \beta=\mu$ and $P=P'$.*

Theorem 3.2 ([15], Proposition 3.2.3.)**.** 1. *Let $P \in Spec(R)$ be a prime ideal of R such that $\theta^i(\xi) \notin P$ for $i \in \mathbb{Z}$ and $\theta^i(P) - P \neq 0$ for $i \neq 0$, then $P_{\infty,\infty} = R\{\theta,\xi\}P \in Spec_l(R\{\theta,\xi\})$.*

2. *Moreover, if $\mathbf{P}$ is a left ideal of $R\{\theta,\xi\}$ such that $\mathbf{P} \cap R = P$, then $\mathbf{P} = P_{\infty,\infty}$. In particular, if P is a maximal ideal, then $P_{\infty,\infty}$ is a maximal left ideal.*
3. *If a prime ideal $P' \subset R$ is such that $P_{\infty,\infty} = P'_{\infty,\infty}$, then $P' = \theta^n(P)$ for some integer n. Conversely, $\theta^n(P)_{\infty,\infty} = P_{\infty,\infty}$ for all $n \in \mathbb{Z}$.*

4 A new deformation $\mathfrak{U}_q(sl_2)$ of $U(sl_2)$

Starting from an $R-$matrix, Jing and Zhang constructed a new deformation $\mathfrak{U}_q(sl_2)$ of $U(sl_2)$ (which is still denoted by $U_q(sl_2)$ in [7]). This new deformation is a bialgebra deformation of $U(sl_2)$ [7]. In this section, we first recall the definition of this new deformation $\mathfrak{U}_q(sl_2)$. Then we verify that $\mathfrak{U}_q(sl_2)$ has a Hyperbolic algebra structure over a polynomial ring in two variables. Finally, we will state and verify some supplementary useful formulas, which will be used in the next section.

Let $\mathbb{C}$ be the field of complex numbers and $0 \neq q$ be an element of $\mathbb{C}$. Let $\mathfrak{U}_q(sl_2)$ be the $\mathbb{C}-$algebra generated by e, f, h subject to following relations:

$$qhe - eh = 2e, \quad hf - qfh = -2f, \quad ef - qfe = h + \frac{1-q}{4}h^2$$

It is easy to see that this new deformation $\mathfrak{U}_q(sl_2)$ shares a lot of properties with $U(sl_2)$. However, this new deformation $\mathfrak{U}_q(sl_2)$ is just a bialgebra deformation of $U(sl_2)$ without having a Hopf algebra structure. The finite dimensional irreducible representations of this algebra were constructed in [7], and an example was constructed to show that not every finite dimensional representation is completely reducible.

In addition, it has a Casimir element which is defined as follows:

$$C = ef + fe + \frac{1+q}{4}h^2 = 2qfe + h + \frac{1}{2}h^2 = 2ef - h + \frac{q}{2}h^2$$

We have the following basic lemma about this Casmir element C from [7].

Lemma 4.1 ([7], Lemma 3.4)**.** *The Casimir element C $q-$commutes with generators of $U_q(sl_2)$ in the following sense:*

$$eC = qCe, \quad fC = q^{-1}Cf \quad hC = Ch$$

We have the following corollary of Lemma 4.1.

Corollary 4.1. *The element h commutes with ef.*

Proof. This follows directly from the definition of C and Lemma 4.1. □

Let us denote ef by ξ and e, f by x, y respectively. Let $R = \mathbb{C}[\xi, h]$ be the subalgebra of $\mathfrak{U}_q(sl_2)$ generated by ξ, h, then R is a polynomial ring in two variables ξ, h, which is thus commutative. We will verify that $\mathfrak{U}_q(sl_2)$ is a Hyperbolic algebra over R.

First of all, let us define an endomorphism θ of R by

$$\theta(h) = qh - 2, \quad \theta(\xi) = q\xi + q^2h + \frac{q^2 - q^3}{4}h^2 - (q+1)$$

It is obvious that θ is an algebra automorphism. In addition, we have the following basic

Lemma 4.2. *The following identities hold:*

1. $xh = \theta(h)x, \quad x\xi = \theta(\xi)x;$
2. $yh = \theta^{-1}(h)y, \quad y\xi = \theta^{-1}(\xi)y;$
3. $xy = \xi, \quad yx = \theta^{-1}(\xi).$

Proof. The verification of the above lemma is straightforward, and is left to the reader. □

From Lemma 4.2, we obtain the following

Proposition 4.1. *$\mathfrak{U}_q(sl_2)$ is a Hyperbolic algebra over R.*

Proof. This follows directly from the definition of Hyperbolic algebras and Lemma 4.2. □

Corollary 4.2. *The Gelfand-Kirillov dimension of $\mathfrak{U}_q(sl_2)$ is 3.*

Proof. This follows from the fact that $\mathfrak{U}_q(sl_2)$ is a Hyperbolic algebra over a polynomial algebra in two variables. Since the Gelfand-Kirillov dimension of the latter is 2, the Gelfand-Kirillov dimension of $\mathfrak{U}_q(sl_2)$ is 3. □

Before we finish this section, we would like to state another useful lemma, which will be needed in the next section.

Lemma 4.3. *One has*

$$\theta^n(h) = q^n h - 2\frac{q^n - 1}{q - 1}$$

and

$$\theta^n(\xi) = q^n\xi + \frac{q^{n+1}(1-q^n)}{4}h^2 + \frac{q^{n+1}(q^n - 1)}{q-1}h - \frac{(q^n-1)(q^{n+1}-1)}{(q-1)^2}$$

for all $n \in \mathbb{Z}$.

Proof. First of all, we prove the statement is true for $n \in \mathbb{Z}_{\geq 0}$. When $n = 0$, the statement is obviously true. And we have $\theta(h) = qh - 2$. Suppose that the statement is true for $n - 1$. Note that we have

$$\theta^n(h) = \theta(\theta^{n-1}(h)) = \theta\left(q^{n-1}h - 2\frac{q^{n-1}-1}{q-1}\right) = q^n h - 2\frac{q^n - 1}{q-1}$$

So we have proved the first statement for $n \geq 0$ by using induction. Similar argument shows that the statement is true for all $n \in \mathbb{Z}$.

Now we are going to prove the second statement. Since C is in R, then we have $xC = \theta(C)x$ and $xC = qCx$ by Lemma 4.1. So we have $\theta(C) = qC$. Hence $\theta^n(C) = q^n C$. Thus

$$2q^n\xi - q^n h + \frac{q^{n+1}}{2}h^2 = 2\theta^n(\xi) - \left(q^n h - 2\frac{q^n-1}{q-1}\right) + \frac{q}{2}\left(q^n h - 2\frac{q^n-1}{q-1}\right)^2$$

Therefore, we have

$$\theta^n(\xi) = q^n\xi + \frac{q^{n+1}(1-q^n)}{4}h^2 + \frac{q^{n+1}(q^n-1)}{q-1}h - \frac{(q^n-1)(q^{n+1}-1)}{(q-1)^2}$$

for all $n \in \mathbb{Z}$. □

5 Construction of points of the spectrum for $\mathfrak{U}_q(sl_2)$

In this section, we construct families of points of the spectrum of the category of representations of $\mathfrak{U}_q(sl_2)$ using the construction theorems quoted in Section 2 from [15]. As a result, we will obtain families of irreducible weight representations of $\mathfrak{U}_q(sl_2)$.

First of all, we have the following basic

Proposition 5.1. *Let $P = (\xi - \alpha, h - \beta)$ be a closed point of Spec(R). Then $\{\theta^n(p) \mid n \in \mathbb{Z}\}$ is a finite set if and only if $\alpha = -\frac{1}{(q-1)^2}$ and $\beta = \frac{2}{q-1}$.*

Proof. If

$$\alpha = -\frac{1}{(q-1)^2} \quad \text{and} \quad \beta = \frac{2}{q-1}$$

then

$$\theta^n(h-\beta) = q^n h - 2\frac{q^n-1}{q-1} - \beta = q^n(h-\beta) + (q^n-1)\beta - 2\frac{q^n-1}{q-1}$$

In addition,

$$\begin{aligned}\theta^n(\xi-\alpha) = {} & q^n(\xi-\alpha) + \left(\frac{q^{n+1}(1-q^n)}{4}h + \frac{q^{n+1}(q^n-1)}{q-1} + \frac{q^{n+1}(1-q^n)\beta}{4})(h-\beta\right) \\ & + \frac{q^{n+1}(1-q^n)\beta^2}{4} + \frac{q^{n+1}(q^n-1)\beta}{q-1} + (q^n-1)\alpha - \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2}\end{aligned}$$

So the orbit of P is finite if and only if

$$(q^n-1)\beta - 2\frac{q^n-1}{q-1} = 0$$

and

$$\frac{q^{n+1}(1-q^n)\beta^2}{4} + \frac{q^{n+1}(q^n-1)\beta}{q-1} + (q^n-1)\alpha - \frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2} = 0$$

hence if and only if

$$\alpha = -\frac{1}{(q-1)^2} \quad \text{and} \quad \beta = \frac{2}{q-1}$$

that completes the proof. □

For the rest of this section, we may assume that $\beta \neq \frac{2}{q-1}$. We have the following

Theorem 5.1. *Suppose that q is not a root of unity and $s \in \frac{\mathbb{Z}_{\geq 0}}{2}$ is a half non-negative integer. Let*

$$P = M_{\alpha,\beta} = (\xi - \alpha, h - \beta) = \left(\xi - \frac{q^{-2s}-1}{1-q^{2s}}, h - 2\frac{1 \mp q^{-s}}{1-q}\right)$$

be a maximal ideal of R, then the corresponding point

$$P_{1,n+1} = R\{\theta,\xi\}P + R\{\theta,\xi\}x + R\{\theta,\xi\}y^{n+1}$$

of the left spectrum $Spec_l(R\{\theta,\xi\})$ is a closed point. Hence the representation $R\{\theta,\xi\}/P_{1,n+1}$ corresponding to this point is a finite dimensional irreducible representation of $\mathfrak{U}_q(sl_2)$.

Proof. If

$$P = (\xi - \alpha, h - \beta) = \left(\xi - \frac{q^{-2s}-1}{1-q^{2s}}, h - 2\frac{1 \mp q^{-s}}{1-q}\right)$$

then we have $\theta^{-1}(\xi) \in P$ and $\theta^{2s}(\xi) \in P$. Thus the statement follows from (b) of part (1) of the Theorem 3.1. □

Remark 5.1. The representations constructed above recover all finite dimensional irreducible representations as constructed in [7].

Now we are going to construct some new families of infinite dimensional irreducible weight representations. Suppose $P = M_{\alpha,\beta} = (\xi - \alpha, h - \beta)$ is a maximal ideal of R, then we have the following

Theorem 5.2. *1. If*

$$\alpha = \beta - \frac{q-1}{4}\beta^2 \quad \textit{and} \quad \beta \neq 2\frac{1 \mp q^{-s}}{1-q}$$

for all non-negative half integer s, then the corresponding point

$$P_{1,\infty} := R\{\theta,\xi\}P + R\{\theta,\xi\}x$$

of the spectrum is closed. And the corresponding representation $R\{\theta,\xi\}/P_{1,\infty}$ is an infinite dimensional irreducible highest weight representation.

2. If

$$\alpha = 0 \quad \textit{and} \quad \beta \neq 2\frac{1 \mp q^{-(s+\frac{1}{2})}}{1-q}$$

for all half positive integers s, then the corresponding point

$$P_{\infty,1} := R\{\theta,\xi\}P + R\{\theta,\xi\}y$$

of the spectrum is closed, and the corresponding representation $R\{\theta,\xi\}/P_{\infty,1}$ is an infinite dimensional irreducible lowest weight representation.

Proof. We will only verify the first part of this statement, and the rest is similar. According to Lemma 4.4, we have

$$\theta^{-1}(\xi) = q^{-1}\xi - q^{-1}h - \frac{1-q}{4q}h^2$$

If

$$\alpha = \beta - \frac{q-1}{4}\beta^2 \quad \text{and} \quad \beta \neq 2\frac{1 \mp q^{-s}}{1-q}$$

then we have $\theta^{-1}(\xi) \in P$ and $\theta^n(\xi) \notin P$ for all $n \geq 0$, so that $P_{1,\infty} := R\{\theta, \xi\}P + R\{\theta, \xi\}x$ is a closed point of the left spectrum $Spec_l(R\{\theta, \xi\})$ by Theorem 3.1, hence the corresponding representation $R\{\theta, \xi\}/P_{1,\infty}$ is an infinite dimensional highest weight irreducible representation. □

Theorem 5.3. *Let $M_{\alpha,\beta} = (\xi - \alpha, h - \beta)$ be a maximal ideal of R such that*

$$\alpha \neq \frac{q(q^n-1)}{4}\beta^2 - \frac{q(q^n-1)}{q-1}\beta + \frac{(q^n-1)(q^{n+1}-1)}{(q-1)^2 q^n}$$

for all $n \in \mathbb{Z}$, then the point $P_{\infty,\infty} = R\{\theta, \xi\}P \in Spec_l(R\{\theta, \xi\})$ is a closed point of the left spectrum, and the corresponding representation $R\{\theta, \xi\}/P_{\infty,\infty}$is an infinite dimensional irreducible weight representation.

Proof. The proof is a direct verification of the conditions in Theorem 3.2, and we will omit it here. □

Remark 5.2. It is tempting to construct some nonweight irreducible representations for $\mathfrak{U}_q(sl_2)$ [17, 18]. Unfortunately, the Whittaker model does not work here. The difficulty lies in that the algebra $\mathfrak{U}_q(sl_2)$ has a trivial center. So it would be an interesting problem to find a way of constructing nonweight irreducible representations for this new deformation $\mathfrak{U}_q(sl_2)$.

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