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On Discretizations of the Generalized Boole Type Transformations and their Ergodicity

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Abstract

There is studied an analytical discretization of the generalized Boole type transformations in Rn and their ergodicity properties.The fixed points of the corresponding finite-dimensional stochastic Frobenius-Perron operator discretization are constructed, the structure of the related invariant measures is analyzed.

Keywords: Frobenius-Perron operator; Discretizatrion; Invariant measure; Ergodic measure; Boole type transformations; Ergodicity and mixing

Introduction: The Frobenius-Perron Operator and Its Discretization

We consider an m-dimensional; not necessary compact; C^1-manifold M^m, endowed with a Lebesgue measure μ determined on the σ-algebra of Borel subsets of M^m andϕ: $M^m \to M^m$ being an almost everywhere smooth mapping. The related [1-5] Frobenius-Perron operator

$$\mathcal{P}_\varphi : L_{1,loc}(M^n;\mathbb{R}) \to L_{1,loc}(M^n;\mathbb{R}) \tag{1}$$

is defined by means of the integral relationship

$$\int_A \mathcal{P}_\varphi h d\mu := \int_{\varphi^{-1}(A)} h d\mu \tag{2}$$

for any $h \in L_{1,loc}(M^n;\mathbb{R})$ and all μ-measurable subsets $A \subset M^m$ Equivalently it can be defined as a mapping on the measure space $\mathcal{M}(M^m)$

$$\mathcal{P}_\varphi \boxplus \nu \boxtimes (A) := \nu(\varphi^{-1}(A)) \tag{3}$$

for any measure $\nu \in \mathcal{M}(M^m)$and all μ-measurable subsets $A \subset M^m$ In particular; if a measure $\nu \in (M^m)$ is absolutely continuous with respect to the measure μ on M^m then definitions (3) and (2) are equivalent. In the infinitesimal form the Frobenius-Perron operator (1) action is representable as

$$\mathcal{P}_\varphi h(x) = \sum_{y(x)\in\varphi^{-1}(x)} h(y(x)) \left|\frac{d\mu(y(x))}{d\mu(x)}\right| = \sum_{y(x)\in\varphi^{-1}(x)} h(y(x)) \left|\frac{d\mu(\varphi(y))}{d\mu(y)}\right|^{-1}_{y=y(x)} \tag{4}$$

for any $B_i \subset M^m, i=\overline{1,N},$ and $x \in M^m$, where $d\mu(\phi(y))/d\mu(y)$ means the usual Radon-Nikodym derivative [1,3] of the shifted measure $\mu.\phi$ with respect to the Lebesgue measure μ on M^m As we are mainly interested in studying the ergodic properties of the mapping $\phi: M^m \to M^m$ by means of the finite dimensional tools; we now proceed to a discretization approach [6-8] to the Frobenius-Perron operator (1) preliminarily choosing a partition $\mathcal{B}_N$ of the manifold M^m as $N \in Z_+$ boxes (or sells) $B_i \subset M^m, i=\overline{1,N},$ and introducing the space $\mathcal{L}^N$ of the step-functions on M^m with respect to the partition $\mathcal{B}_N$ which can be constructed using the projection operator $\Pi_N : L_{1,loc}(M^n;\mathbb{R}) \to \mathcal{L}^N \subset L_{1,loc}(M^n;\mathbb{R})$:

$$(\Pi_N h)(x) := \frac{\chi_{B_i}(x)}{\mu(B_i)} \int_{B_i} h d\mu \tag{5}$$

for any $h \in L_{1,loc}(M^n;\mathbb{R})$ and all $x \in M^m$ Then; by definition; one can define the Frobenius-Perron operator discretization as

$$\mathcal{P}_{\varphi,N} := \Pi_N \mathcal{P}_\varphi |_{\mathcal{L}^N}. \tag{6}$$

As a consequence of the definitions above one obtains that the discretized Frobenius-Perron operator (6) can be represented with respect to the canonical basis in the finite-dimensional space $\mathcal{L}^N$ by means of the ($N \times N$) matrix

$$\mathcal{P}_{\varphi,N} = \{\mathcal{P}^{ij}_{\varphi,N} := \mu(\varphi^{-1}(B_i) \cap B_j)\mu(B_j)^{-1} : i,j = \overline{1,N}\}, \tag{7}$$

which is exactly a discretization of the infinitesimal expression (4). The matrix component $\mathcal{P}^{ij}_{\varphi,N}, i,j=\overline{1,N},$ can be; obviously; interpreted as a transition probability matrix for a point in B_j, being randomly chosen with respect to the measure μ to be mapped into the set B_i by the mapping $\phi: M^m \to M^m$. Thus; the obtained stochastic matrix $\mathcal{P}_{\varphi,N} : \mathcal{L}^N \to \mathcal{L}^N$ defines naturally a finite homogeneous Markov chain; and particularly a linear discrete dynamical system in the Euclidean space $\mathbb{E}^N \simeq \mathcal{L}^N$.

The described approach to study the dynamical properties of the mapping $\phi: M^m \to M^m$ by means of the discretized Frobenius-Perron operator (6) is widely used in the literature [6,8-11]. It was also effectively used S. Ulam for finding the approximation of the invariant measures for the mapping $\phi: M^m \to M^m$ which are related with nonnegative fixed points of the discretized Frobenius-Perron operator (6). In addition; the discretized Frobenius-Perron operator (6) appears to be very useful for analyzing the ergodicity and mixing properties

[2,4,5,7,8] of the mapping $\phi: M^m \to M^m$. Namely; the ergodicity of it with respect to the partition $\mathcal{B}_N$ is defined as the irreducibility of the discretized Frobenius-Perron operator (6); and the mixing with respect to the partition $\mathcal{B}_N$ is defined as its primitivity and ergodicity.

Discrete Ergodicity Analysis

As ergodicity of the mapping $\phi: M^m \to M^m$ is deeply connected with the suitably determined ergodic measure ν on M^m, which is a special invariant measure on M^m such that any ϕ-quasi-invariant function $\phi: M^m \to M^m$ is almost everywhere constant on M^m we will be mainly interested below in the invariant measure ν absolutely continuous with respect to the Lebesgue measure μ on M^m which is a fixed point

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of the Frobenius-Perron operator (1); defined by the mapping (3). In what follows there is accepted the next [12] definition of the discrete ergodicity.

Definition 2.1: *A measurable mapping $\phi{:}M^m \to M^m$ is called ergodic with respect to the partition $\mathcal{B}_N$ if the following discrete ergodic theorem holds:*

there exists a non-negative definite and normalized vector $H^{(0)} \in \mathbb{E}^N, H^{(0)} \geq 0, \| H^{(0)} \|_1 = 1,$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}_{\varphi,N}^k H = H^{(0)} \tag{8}$$

for any $H \in \mathbb{E}^N, H \geq 0, \| H \|_1 = 1.$

It is naturally to assume that the discrete ergodicity with respect to the partition $\mathcal{B}_N$ can happen to be persisting for almost all possible partitions of M^m and for arbitrary dimensions $N \in \mathbb{Z}_+$. In this case one can determine a set of functions $\{h_N^{(0)} : M^m \to \mathbb{R}_+ : N \in \mathbb{Z}_+\},$ where

$$h_N^{(0)}(x) := \sum_{j=1}^{N} H_j^{(0)} \chi_{B_j}(x) \tag{9}$$

for any $x \in M^m$ and next proceed to studying the existence of the pointwise limiting function

$$h^{(0)}(x) := \lim_{N\to\infty} h_N^{(0)}(x) \tag{10}$$

defining the corresponding absolutely continuous with respect to the measure μ on M^m and invariant with respect to the transformation $\phi{:}M^m \to M^m$ measure

$$\nu(A) := \int_A h^{(0)} d\mu \tag{11}$$

for any measurable subset $A \subset M^m$. If the constructed measure (11) proves to be finite; that is $\int_{M^m} h^{(0)} d\mu < \infty,$ then this invariant measure ν can be easily made probabilistic.

Taking into account the fact that the Frobenius-Perron matrix (7) is stochastic; one can recall the well known Frobenius-Perron theory [13] of non-negative stochastic matrices; in particular the following useful proposition.

Proposition 2.2: *The mapping $\phi{:}M^m \to M^m$ is with respect to the partition $\mathcal{B}_N$:*

ergodic iff the matrix $\mathcal{P}_{\phi,N}$ is irreducible; that is for every pair of states (i,j) it is possible to move from i to j and back again; in other words $\mathcal{P}_{\phi,N}$ is irreducible; if it is not block upper-triangular; up to reordering rows and columns;

mixing iff the matrix $\mathcal{P}_{\phi,N}$ is primitive; that is all its eigenvalues not equal to the unity have modulus less than unity;

ergodic; but not mixing; iff the matrix $\mathcal{P}_{\phi,N}$ is q-cycling with maximal $q>0$

Moreover; it is worthy of mentioning that the irreducibility and primitivity depend only on the structure of the directed graph G_N, naturally associated with the matrix $\mathcal{P}_{\phi,N}$. Concerning the effective studying of the sole ergodicity of the mapping $\phi_{:M}{}^m \to_M{}^m$ the following famous Frobenius-Perron theorem proves strongly important.

Proposition 2.3: *An irreducible stochastic matrix $\mathcal{P}_{\phi,N}$ is -cyclic with $q \in Z_+$ maximal iff one of the following equivalent conditions holds:*

a) There are q different eigenvalues of the matrix $\mathcal{P}_{\phi,N}$ of modulus one;

b) There are q symmetrically distributed and algebraically simple eigenvalues $exp(2\pi ik/q), k = \overline{0, q-1},$ of the matrix $\mathcal{P}_{\phi,N;}$

c) the whole spectrum of the matrix $\mathcal{P}_{\phi,N}$ is invariant under the rotation about the angle $2\pi/q$.

The Classical Boole Mapping and Its Ergodicity

The classical Boole transformation [14] ϕ: $\mathbb{R} \to \mathbb{R}$ is defined as the almost everywhere smooth mapping

$$\varphi(x) := x - 1/x, \tag{12}$$

defined for all $x \in \mathbb{R}\backslash\{0\}$ It was proved to be ergodic [1,15] with respect to the standard invariant infinite Lebesgue measure on $\mathbb{R}$. The corresponding fixed point equation for the Frobenius-Perron operator action (4) can be easily presented as

$$(\mathcal{P}_\varphi h^{(0)})(x) = \sum_{\pm} h^{(0)}(y_\pm(x)) y_\pm'(x) = h^{(0)}(x), \tag{13}$$

where; by construction; $\varphi(y_\pm(x)) = x,$ $y_\pm'(x) > 0,$ and $h^{(0)}(x) \geq 0$ for almost all $x \in \mathbb{R}$ Having assumed that there exists an meromorphic continuation $h^{(0)} : \mathbb{C} \to \mathbb{C}$ of the mapping $h^{(0)} : \mathbb{R} \to \mathbb{R}_+,$ such that $| h^{(0)}(z) - k^{(0)} | = O(1/|z^2|)$ for $|z| \to \infty$ and some $k^{(0)} \geq 0,$ the equality (13) can be rewritten as

$$\begin{aligned} &\sum_{\pm} [h^{(0)}(y_\pm(x)) - k^{(0)}] y_\pm'(x) = \\ &= -\lim_{r\to\infty} \frac{1}{2\pi i} \frac{d}{dx} \sum_{\pm} \int_{\partial\mathbb{C}_r} [\ln(z - y_\pm(x)][h^{(0)}(z) - k^{(0)}] dz + \\ &+ \frac{1}{2\pi i} \frac{d}{dx} \sum_{\{b=a,\bar{a}\}} \int_{\partial O_\varepsilon(b)} [\ln_\pm(z - y_\pm(x)][h^{(0)}(z) - k^{(0)}] dz = \\ &= -\lim_{r\to\infty} \frac{1}{2\pi i} \int_{\partial O_r(0)} [\frac{d}{dx} \ln(z^2 - zx - 1)][h^{(0)}(z) - k^{(0)}] dz + \\ &+ \sum_{\{a\}} \int_{\partial O_\varepsilon(a)} \frac{z[h^{(0)}(z) - k^{(0)}] dz}{(z^2 - zx - 1)} + \sum_{\{\bar{a}\}} \int_{\partial O_\varepsilon(\bar{a})} \frac{z[h^{(0)}(z) - k^{(0)}] dz}{(z^2 - zx - 1)} = \\ &= \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\partial O_r(0)} \frac{z[h^{(0)}(z) - k^{(0)}] dz}{(z^2 - zx - 1)} + \sum_{\{b=a,\bar{a}\}} \left. \frac{k_b^{(0)}(x - 2z)}{(z^2 - xz - 1)^2} \right|_{z=b} = \\ &+ \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\partial O_{1/r}(0)} \frac{[h^{(0)}(1/s) - k^{(0)}] ds}{s(1 - xs - s^2)} + \sum_{\{b=a,\bar{a}\}} \left. \frac{k_b^{(0)}(x - 2z)}{(z^2 - xz - 1)^2} \right|_{z=b} = \\ &= \sum_{\{b=a,\bar{a}\}} \left. \frac{k_b^{(0)}(x - 2z)}{(z^2 - xz - 1)^2} \right|_{z=b}, \end{aligned} \tag{14}$$

where $O_r(b) := \{| z - b | < r, b \in \mathbb{C}, \quad r > 0\}$ and $k_{\bar{a}}^{(0)} = \bar{k}_a^{(0)}, Rek_a^{(0)} \geq 0,$ are the corresponding residuum constants; related with the assumed finite second order pole set $\{a, \bar{a} \in \mathbb{C} \setminus \mathbb{R}$ of the function $h^{(0)} : \mathbb{C} \to \mathbb{C},$ satisfying some finite system of algebraic constraints; ensuring the positivity of the reduced function $h^{(0)} : \mathbb{R} \to \mathbb{R}.$ Based on simple enough yet cumbersome calculations one can get convinced that this system of constraints is compatible iff the constants $k_b^{(0)} = 0$ for all $b \in \{a, \bar{a}\}$. Then from (5) one easily derives that

$$\begin{aligned} h^{(0)}(x) &= k^{(0)} \sum_{\pm} y_\pm'(x) = k^{(0)} [\frac{y_+(x)}{2y_+(x) - x} + \frac{y_-(x)}{2y_-(x) - x}] = \\ &= k^{(0)} \frac{4y_+(x)y_-(x) - x[y_+(x) + y_-(x)]}{4y_+(x)y_-(x) + x^2 - 2x[y_+(x) + y_-(x)]} = k^{(0)}, \end{aligned} \tag{15}$$

where we made use of the obvious identities $y_+(x)y_-(x) = -1$ and

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$y_+(x)+y_-(x)=x$ for all $x \in$R Thus; the invariant infinitesimal measure with respect to the Boole mapping (12) equals

$$dv(x)=k^{(0)}dx, \tag{16}$$

being absolutely continuous subject to the standard Lebesgue measure dx on R Thus; one can formulate the following theorem.

Theorem 3.1: *Being unique; modulo the constant multiplier; the invariant with respect to the Boole mapping (12) measure expression (16) is ergodic on axis* $\mathbb{R}$

Having now constructed the uniformly discretized Frobenius-Perron operator matrix (7); one can check that the matrix $\mathcal{P}_{\phi,N}$ is reducible with respect to any partition $\mathcal{B}_N = \cup_{j=-N_-}^{N_+}[j/N,(j+1)/N] \subset \mathbb{R}$ for any its dimension $N:=(N_-+N_+)\to\infty$. Then; based on Proposition 2.2; one can claim that the Boole mapping (12) is ergodic with respect to any partition $\mathcal{B}_N$, $N\to\infty$ One can also verify that the positive definite vector $H^{(0)}=(1/N,1/N,...,1/N)\in\mathbb{E}^N$ solves the limiting condition (8); being its eigenvector for the unity eigenvalue:

$$\mathcal{P}_{\varphi,N}H^{(0)}=H^{(0)} \tag{17}$$

for any dimension $N\to\infty$ As a corollary of the claim above and the cycling properties of the Frobenius-perron matrix $\mathcal{P}_{\phi,N}$ one derives the next theorem; generalizing the one proved in [15] by means of different mostly qualitive tools.

Theorem 3.2: *The Boole transformation (12) is ergodic; yet not mixing.*

As it can be checked by means of direct computations; the Boole transformation (12) is ergodic yet not mixing; as the matrix $\mathcal{P}_{\phi,N}$ is q_N-cycling with maximal $q_N>0$ for any dimension $N\to\infty$

The generalized Boole Type Mapping and Its Ergodicity

In the present section; we will study the invariant measures and ergodicity properties for the generalized Boole type transformations of plane $\mathbb{R}^2$

$$\varphi_1(x_1,x_2):=(x_1-1/x_2,x_2+1/x_1),\quad \varphi_2(x_1,x_2):=(x_1+1/x_2,x_2-1/x_1), \tag{18}$$

where $(x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. The corresponding to the mapping $\varphi_1:\mathbb{R}^2\to\mathbb{R}^2$ local Frobenius-Perron operator $\mathcal{P}_\phi$ acts on a non-negative definite function $h^{(0)}:\mathbb{R}^2\to\mathbb{R}_+$ as

$$(\mathcal{P}_\varphi h^{(0)})(x_1,x_2)=\sum_{\pm}h^{(0)}(y_{1,\pm},y_{2,\pm})[1+y_{1,\pm}^{-2}\,y_{2,\pm}^{-2}], \tag{19}$$

where; by definition; $y_{1,\pm}:=y_{1,\pm}(x_1,x_2), y_{2,\pm}:=y_{2,\pm}(x_1,x_2), \varphi_1(y_{1,\pm},y_{2,\pm}):=(x_1,x_2)$, $y_{1,\pm}^2-x_1y_{1,\pm}+x_1/x_2=0$, $y_{2,\pm}=y_{1,\pm}x_2/x_1$ for any $(x_1,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. It is easy to check by means of direct and simple enough calculations that a positive constant function $h^{(0)}(x_1,x_2)=k^{(0)}\in\mathbb{R}_+$ is an eigenfunction of the mapping (19) with the unity eigenvalue:

$$\mathcal{P}_\varphi k^{(0)}=k^{(0)}. \tag{20}$$

This; in particular; means that the infinitesimal measure $dv(x_1,x_2)):=k^{(0)}dx_1dx_2$ on the plane R^2 is invariant with respect to the mapping $\phi_1:\mathbb{R}^2\to\mathbb{R}^2$. If to state now that this invariant measure is unique on the plane R^2 this will mean [2-5] that the mapping $\phi_1:\mathbb{R}^2\to\mathbb{R}^2$ is ergodic. To show this; we will make use of the uniform discretization of the Frobenius-Perron operator (19) and find by means of usual numerical calculations that the corresponding N-dimensional Frobenius-Perron matrix $\mathcal{P}_{\phi,N}$: $\mathbb{E}^N\to\mathbb{E}^N$ is irreducible for any dimension $N\to\infty$ This fact; owing to Proposition 2.2; makes it possible to formulate the following theorem.

Theorem 4.1: *The Boole type transformation* $\phi_1:\mathbb{R}^2\to\mathbb{R}^2$ *of (18) is ergodic.*

Concerning the mixing property of the mapping ϕ_2: $\mathbb{R}^2\to\mathbb{R}^2$ additional calculations still are needed to show; owing to Proposition 2.3; that the N-dimensional Frobenius-Perron matrix $\mathcal{P}_{\phi,N}$: $\mathbb{E}^N\to\mathbb{E}^N$ is q_N-maximal cycling for any dimension $N\to\infty$.

Remark 4.2: *Taking into account that the mapping* ϕ_2: $\mathbb{R}^2\to\mathbb{R}^2$ *is simply conjugated with the mapping* $\phi_1:\mathbb{R}^2\to\mathbb{R}^2$ *all statements above concerning its ergodicity also hold for the mapping* $\phi_2:\mathbb{R}^2\to\mathbb{R}^2$

The Boole type mappings (18) can be generalized on the three-dimensional space $\mathbb{R}^3$:

$$\begin{aligned}\varphi_1(x_1,x_2,x_3)&:=(x_1-1/x_2,x_2-1/x_3,x_3-1/x_1),\\ \varphi_2(x_1,x_2,x_3)&:=(x_1-1/x_3,x_2-1/x_1,x_3-1/x_2),\end{aligned} \tag{21}$$

defined for any $(x_1, x_2, x_3)\in \mathbb{R}^3\setminus\{(0,0,0)\}$ It was already proved in ref. [16] that these mapping are invariant with respect to the standard Lebesgue measure $dv(x_1, x_2, x_3)=dx_1\, dx_2\, dx_3$ on $\mathbb{R}^3$ yet their ergodicity is still under investigation.

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