On Certain Analytic Functions

Janusz Sokol* and Mamoru Nunokawa

Introduction

For integer n ≥ 0, denote by Σn the class of meromorphic functions, defined in $\hat{D} = \{ z : 0 < |z| < 1 \}$, which are of the form

$$ F(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + ... $$

A function $F \in \Sigma_n$ is said to be starlike if it is univalent and the complement of $F(\hat{U})$ is starlike with respect to the origin. Denote by $\Sigma_n^*$ the class of such functions. If $F \in \Sigma_n^*$ if and only if

$$ |\text{Re} \left( \frac{z F'(z)}{F(z)} \right) | > 0 $$

for $z \in \hat{U}$. For $0 < \alpha < 1$, let

$$ \Sigma_n^\alpha = \left\{ F \in \Sigma_n : |\text{Re} \left( \frac{z F'(z)}{F(z)} \right) | > \alpha, z \in \hat{U} \right\}. $$

The class of meromorphic-starlike functions of order $\alpha$. For $0 < \alpha < 1$, let

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the class of meromorphic-strongly starlike functions of order $\alpha$.

Let p be positive integer and let $A(p)$ be the class of functions

$$ f(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n, $$

which are analytic in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$. Furthermore, denote by A the class of analytic functions in D and usually normalized, i.e. $A = \{ f \in H : f(0) = 0, f'(0) = 1 \}$. We say that $f \in H$ is subordinate to $g \in H$ in the unit disk $D$ if it is univalent and the complement of $F(\hat{U})$ is starlike with respect to the origin. Denote by $\Sigma_n^\alpha$ the class of such functions. If $F \in \Sigma_n^\alpha$ if and only if

$$ |\text{Re} \left( \frac{z F'(z)}{F(z)} \right) | > \alpha, z \in \hat{U}. $$

The subclass of $A(p)$ consisting of p-valently starlike functions is denoted by $S^*(p)$. An analytic description of $S^*(p)$ is given by

$$ S^*(p) = \left\{ f \in A(p) : \left| \arg \left( \frac{z f'(z)}{f(z)} \right) \right| < \frac{\pi}{2}, z \in D \right\}. $$

The subclass of $A(p)$ consisting of p-valently and strongly starlike functions of order $\alpha$, $0 < \alpha < 1$ is denoted by $S^{\alpha}(p)$. Its analytic description is given by

$$ S^{\alpha}(p) = \left\{ f \in A(p) : \left| \arg \left( \frac{z f'(z)}{f(z)} \right) \right| < \frac{\pi}{2}, z \in D \right\}. $$

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Main Result

To prove the main results, we also need the following generalization of Nunokawa’s lemma [1-12].

**Lemma 2.1**: [5] Let $p(z)$ be of the form

$$ p(z) = 1 + \sum_{n=m}^{\infty} a_n z^n, a_m \neq 0, (|z| < 1), $$

(2.1)

with $p(z) \neq 0$ in $|z| < 1$. If there exists a point $z_0$, $|z_0| < 1$, such that

$$ \arg \left( \frac{p(z)}{\alpha} \right) = \frac{\pi}{2} \text{ in } |z| < |z_0| $$

and

$$ \arg \left( p(z_0) \right) = \pi \alpha/2 $$

for some $\alpha > 0$, then we have

$$ \frac{z_0 p(z_0)}{p(z_0)} = i \alpha, $$

where

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k ≥ m(α²+1)/(2a) when \( \arg \{ p(z_0) \} = \pi \alpha / 2 \) \( \text{(2.2)} \)
and
k ≥ -m(α²+1)/(2a) when \( \arg \{ p(z_0) \} = -\pi \alpha / 2 \), \( \text{(2.3)} \)

Where
\[ \{ p(z_0) \} = \pm ia, \quad a > 0 \]

**Theorem 3.1:** Let \( p(z) \) be analytic in \( D \) with \( p(0) - 1 = p'(0) = 0 \). Assume that
\[ a \in [0, 1/2]. \text{ If for } z \in D \]
\[ \arg \{ p(z) - zp'(z) \} < \arctan(2\alpha) - \frac{\alpha \pi}{2}, \]
then
\[ \arg \{ p(z) \} < \frac{\alpha \pi}{2} \] \( \text{(3.1)} \)

**Proof:** If there exists a point \( z_0, |z_0|<1 \), such that
\[ \frac{z_0p(z_0)}{p(z_0)} = ik\alpha \]
and
\[ \arg \{ p(z_0) \} = \frac{\alpha \pi}{2} \]
then from Nunokawa's lemma 2.1, with \( m=2 \), we have
\[ \frac{z_0p(z_0)}{p(z_0)} = \frac{k(a+1)}{a}, \quad \text{when } \arg \{ p(z) \} = \frac{\pi \alpha}{2} \]
And
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And where
\[ 1/0 \{ p(z) \} = ia, \quad a > 0 \]

**Theorem 3.5:** Let
\[ F(z) = \frac{1}{z} + a_2z^2 + a_3z^3 + \ldots, \quad z \in U \]
Assume that \( a \in [0, 1/2]. \text{ If for } z \in D \]
\[ \arg \{ -zF^\prime(z) \} < \arctan(2\alpha) - \frac{\alpha \pi}{2} \]
then
\[ \arg \{ zF(z) \} < \frac{\alpha \pi}{2} \quad (z \in D \text{ (3.16)}) \]

**Proof:** Let
\[ p(z) = zF(z) = 1 + a_1z^2 + a_3z^3 + \ldots, \quad p(0) = 1 \]
Then
\[ p(z) - zp'(z) = -z^2 F(z). \]
Applying Theorem 3.1 we obtain the result.

**Corollary 3.6:**

Let
\[ F(z) = \frac{1}{z} + a_1 z + a_2 z^2 + \ldots, \quad z \in \mathbb{U} \]
Assume that \( \alpha \in [0, 1/2] \). If for \( z \in \mathbb{D} \)
\[ \arg \left\{ -z^2 F'(z) \right\} < \arctan (2\alpha) - \frac{\alpha \pi}{2} \tag{3.17} \]
then \( F(z) \) is meromorphic-strongly starlike function of order \( \arctan (2\alpha) \).

**Proof:** For showing that \( F(z) \) is meromorphic-strongly starlike function we need to show (1.1). Applying theorem 3.5 and 3.17 we obtain
\[ \frac{\arg \left\{ -z^2 F'(z) \right\}}{F(z)} < \frac{\alpha \pi}{2}. \]
since this and since (3.17), we obtain
\[ \arg \left\{ -z^2 F'(z) \right\} < \arctan (2\alpha) - \frac{\alpha \pi}{2}. \]
Therefore, \( F(z) \) is meromorphic-strongly starlike function of order \( \arctan (2\alpha) \).

For \( \alpha \in [0, 1/2] \) we have
\[ 0 \leq \arctan (2\alpha) - \frac{\alpha \pi}{2} \leq \arctan (2\alpha_0) - \frac{\alpha_0 \pi}{2} \]
where
\[ \alpha_0 = \sqrt{\frac{4 - \pi}{4\pi}} = 0.26... \]

**Theorem 3.7:**

Let
\[ F(z) = \frac{1}{z} + a_1 z + a_2 z^2 + \ldots, \quad z \in \mathbb{U} \tag{3.18} \]
Assume that \( \alpha \in [0, 1/2] \). If for \( z \in \mathbb{D} \)
\[ \arg \left\{ \frac{z^2 F''(z) - (F'(z))^2}{F'(z)} \right\} < \arctan (2\alpha) - \frac{\alpha \pi}{2} \tag{3.19} \]
then
\[ \arg \left\{ \frac{z F'(z)}{F(z)} \right\} < \frac{\alpha \pi}{2}, \quad \text{if } z \in \mathbb{D} \tag{3.20} \]

**Proof:** Let \( p(z) = z F'(z)/F(z) \). By (3.18) we have that
\[ p(z) = l + p_2 z^2 + p_3 z^3 + \ldots. \]
Moreover
\[ p(z) - zp'(z) = z^2 \left( F''(z) - (F'(z))^2 \right) \]
Applying Theorem 3.1 we obtain the result.

**References**