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On Certain Analytic Functions

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Abstract

We apply Nunokawa's lemma, On Properties of Non-Carath´eodory Functions, Proc. Japan Acad. 68, Ser. A (1992) 152-153, to prove some new results.

Keywords: Analytic; Univalent; Convex; Starlike; Strongly starlike; Differential subordination

Introduction

For integer $n \geq 0$, denote by Σn the class of meromorphic functions, defined in $\dot{U} = \{z : 0 < |z| < 1\},$ which are of the form

$$F(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + \ldots$$

A function $F \in \Sigma_0$ is said to be starlike if it is univalent and the complement of $F(\dot{U})$ is starlike with respect to the origin. Denote by Σ^*_0 the class of such functions. If $F \in \Sigma_0$, then it is well-known that $F \in \Sigma^*_0$ if and only if

$$\Re e\left\{-\frac{zF'(z)}{F(z)}\right\} > 0$$

for $z \in \dot{U}$. For $\alpha<1$, let

$$\sum\nolimits^*_{n,\alpha} = \left\{F \in \sum\nolimits_n : \Re e\left\{\frac{zF'(z)}{F(z)}\right\} > \alpha,\ z \in \dot{U}\right\},$$

The class of meromorphic-starlike functions of order α. For $0<\alpha \leq 1$, let

$$\sum\nolimits^*_n(\alpha) = \left\{F \in \sum\nolimits_n : \left|\arg\left\{-\frac{zF'(z)}{F(z)}\right\}\right| < \frac{\alpha\pi}{2},\ z \in \dot{U}\right\}$$

the class of meromorphic-strongly starlike functions of order.

Let p be positive integer and let A(p) be the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n,$$

which are analytic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ Furthermore, denote by A the class of analytic functions in D and usually normalized, i.e. $A = \{f \in H : f(0), f'(0)\}$ We say that the $f \in H$ is subordinate to $g \in H$ in the unit disc D, written $f \prec g$ if and only if there exists an analytic function $w \in H$ such that and $f(z)=g[w(z)]$ for $z \in D$. Therefore $f \prec g$ in D implies $f(D) \subset g(D)$. In particular if g is univalent in D then the Subordination Principle says that $f \prec g$ if and only if $f(0)=g(0)$ and $f(|z|<r)$ $g(|z|<r)$, for all $r \in (0, 1]$.

The subclass of A(p) consisting of p-valently starlike functions is denoted by $S^*(p)$. An analytic description of $S^*(p)$ is given by

$$S^*(p) = \left\{f \in A(p) : \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2},\ z \in D\right\}$$

The subclass of A(p) consisting of p-valently and strongly starlike functions of order α, $0<\alpha \leq 1$ is denoted by $S^*\alpha(p)$. An analytic description of $S^*_\alpha(p)$ is given by

$$S^*_\alpha(p) = \left\{f \in A(p) : \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},\ z \in D\right\}$$

The subclass of A(p) consisting of p-valently convex functions and p-valently strongly convex functions of order α, $0<\alpha \leq 1$ are denoted by $C^*(p)$ and $C^*_\alpha(p)$ respectively. The analytic descriptions of $C^*(p)$ and $C^*_\alpha(p)$ are given by

$$C^*(p) = \left\{f \in A(p) : \left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2},\ z \in D\right\}$$

and

$$C^*_\alpha(p) = \left\{f \in A(p) : \left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\alpha\pi}{2},\ z \in D\right\}$$

Main Result

To prove the main results, we also need the following generalization of Nunokawa's lemma [1-12]. Lemma 2.1: [5] Let p(z) be of the form

$$p(z) = 1 + \sum_{n=m\geq 1}^{\infty} a_n z^n,\ a_m \neq 0,\ (|z| < 1), \tag{2.1}$$

with p(z) 6≠0 in $|z| < 1$. If there exists a point z_0, $|z_0|<1$, such that

$|\arg\{p(z)\}| < \pi\alpha/2$ in $|z| < |z_0|$

And

$\arg\{p(z_0)\}| = \pi\alpha/2$

for some $\alpha>0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

Where

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$$k \ge m(\alpha^2+1)/(2a) \quad \text{when} \quad \arg\{p(z_0)\} = \pi\alpha/2 \tag{2.2}$$

and

$$k \ge -m(\alpha^2+1)/(2a) \quad \text{when} \quad \arg\{p(z_0)\} = -\pi\alpha/2, \tag{2.3}$$

Where

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \; a>0$$

Theorem 3.1: Let p(z) be analytic in D with p(0)−1=p'(0)=0. Assume that

α € [0, 1/2]. If for z € D

$$|\arg\{p(z) - zp'(z)\}| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.1}$$

then

$$|\arg\{p(z)\}| < \frac{\alpha\pi}{2} \quad (z \in D) \tag{3.2}$$

Proof: If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}| < \pi\alpha/2 \; (|z| \le |z_0|)$$

and

$$|\arg\{p(z)\}| = < \pi\alpha/2,$$

then from Nunokawa's lemma 2.1, with m=2, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where k is a real number

$$k \ge (a^2+1)/a, \text{ when } \arg\{p(z_0)\} = \pi\alpha/2$$

And

$$k \le -(a^2+1)/a, \text{ when } \arg\{p(z_0)\} = -\pi\alpha/2$$

And where $p(z_0)^{1/\alpha} = \pm ia$, $a>0$. For the case arg $\{p(z_0)\}$= πα/2, we have

$$p(z_0) - z_0 p'(z_0) = p(z_0)\left(1 - \frac{z_0 p'(z_0)}{p(z_0)}\right) = (ia)^\alpha (i - \alpha k) \tag{3.3}$$

where k ≥ $(a^2+1)/a$ and 0<a. Because, k ≥ $(a^2+1)/a$ ≥ 2, we have

$$-\frac{\pi}{2} < \arg(1 - i\alpha k) \le -\arctan(2\alpha) \tag{3.4}$$

It is easy to see that for α € [0, 1/2] we have

$$\arctan(2\alpha) - \frac{\alpha\pi}{2} \ge 0$$

Therefore, using (3.3) and (3.4), we obtain

$$\arg\{p(z_0) - z_0 p'(z_0)\} = \arg\{p(z_0)\} + \arg\left\{1 - \frac{z_0 p'(z_0)}{p(z_0)}\right\}$$

$$= \arg\{(ia)^\alpha\} + \arg\{1 - i\alpha k\}$$

$$\le -\left\{\arctan(2\alpha) - \frac{\alpha\pi}{2}\right\} \tag{3.5}$$

This is a contradiction with (3.1). For the case $\arg\{p(z_0)\} = -\pi\alpha/2$, we have

$$p(z_0) - z_0 p'(z_0) = p(z_0)\left(1 - \frac{z_0 p'(z_0)}{p(z_0)}\right)$$

$$= (-ia)^\alpha (1 - i\alpha k), \tag{3.6}$$

where $k \le -(a^2+1)/a \le -2$. We also have

$$\arctan(2\alpha) \le \arg(1 - i\alpha k) < \frac{\pi}{2} \tag{3.7}$$

Therefore, using (3.6) and (3.7) and applying the same method as above, we obtain

$$\arg\{p(z_0) - z_0 p'(z_0)\} = \arg\{p(z_0)\} + \arg\left\{1 - \frac{z_0 p'(z_0)}{p(z_0)}\right\}$$

$$= \arg\{(-ia)^\alpha\} + \arg(1 - i\alpha k) \ge \arctan(2\alpha) - \frac{\alpha\pi}{2} \tag{3.8}$$

This is also a contradiction with (3.1), and it completes the proof.

Let us put $p(z) = e^{-i\beta} q(e^{i\beta} z)$ in Theorem 3.1.

Corollary 3.2: Let $p(z) = e^{-i\beta} q(e^{i\beta} z), \beta \in \mathbb{R}$, be analytic in D with p(0) − 1 = p'(0) = 0. Assume

that α € [0, 1/2]. If for z € D

$$\left|\arg\left\{e^{-i\beta} q(e^{i\beta} z) - zq'(e^{i\beta} z)\right\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2}, \tag{3.9}$$

then

$$\left|\arg\left\{e^{-i\beta} q(e^{i\beta} z)\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in D) \tag{3.10}$$

Corollary 3.3: Let $p(z) = e^{-i\beta} q(e^{i\beta} z), \beta \in \mathbb{R}$, be analytic in D with p(0) − 1 = p'(0) = 0. Assume

that α € [0, 1/2]. If for z € D

$$\left|\arg\{q(e^{i\beta} z) - e^{i\beta} zq'(e^{i\beta} z)\} - \beta\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2} \tag{3.11}$$

$$\left|\arg\{q(e^{i\beta} z)\} - \beta\right| < \frac{\alpha\pi}{2} \quad (z \in D) \tag{3.12}$$

Corollary 3.4: Let q(z) be analytic in D with $q(0)=e^{i\beta}$, q'(0)=0, β € R,. Assume that α € [0, 1/2]. If for z € D

$$\left|\arg\{q(z) - zq'(z)\} - \beta\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2} \tag{3.13}$$

$$\left|\arg\{q(z) - \beta\right| < \frac{\alpha\pi}{2} \quad (z \in D) \tag{3.14}$$

Theorem 3.5: Let

$$F(z) = \frac{1}{z} + a_1 z + a_2 z^2 + \ldots, \quad z \in U$$

Assume that α € [0, 1/2]. If for z € D

$$\left|\arg\{-z^2 F'(z)\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2} \tag{3.15}$$

then

$$\left|\arg\{zF(z)\}\right| < \frac{\alpha\pi}{2} \quad (z \in D) \; (3.16)$$

Proof: Let

$$p(z) = zF(z) = 1 + a_1 z^2 + a_2 z^3 + \ldots, \quad p(0)=1$$

Then

$$p(z) - zp'(z) = -z^2F'(z).$$

Applying Theorem 3.1 we obtain the result.

Corollary 3.6: Let

$$F(z) = \frac{1}{z} + a_1z + a_2z^2 + \ldots, \quad z \in U$$

Assume that α € [0, 1/2]. If for z € D

$$\left|\arg\{-z^2 F'(z)\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2} \tag{3.17}$$

then F(z) is meromorphic-strongly starlike function of order arctan (2α).

Proof: For showing that F(z) is meromorphic-strongly starlike function we need to show (1.1). Applying theorem 3.5 and 3.17 we obtain $\left|arg\{zF(z)\}\right| < \alpha\pi/2$, since this and since (3.17), we obtain

$$\left|arg\frac{\{-zF'(z)\}}{F(z)}\right|$$

$$= \left|arg\frac{\{-z^2F'(z)\}}{zF(z)}\right|$$

$$\le \left|arg\{-z^2F'(z)\}\right| + \left|\arg\{zF(z)\}\right|$$

$$< \arctan(2\alpha) - \frac{\alpha\pi}{2} + \frac{\alpha\pi}{2}$$

$$= \arctan(2\alpha)$$

Therefore, F(z) is meromorphic-strongly starlike function of order arctan (2α).

For α [0, 1/2] we have

$$0 \le \arctan(2\alpha) - \frac{\alpha\pi}{2} \le \arctan(2\alpha_0) - \frac{\alpha_0\pi}{2}$$

Where

$$\alpha_0 = \sqrt{\frac{4-\pi}{4\pi}} = 0.26\ldots$$

Theorem 3.7: Let

$$F(z) = \frac{1}{z} + a_1z + a_2z^2 + \ldots, \quad z \in U \tag{3.18}$$

Assume that α € [0, 1/2]. If for z € D

$$\left|\arg\left\{\frac{z^2(F''(z) - (F'(z))^2)}{F^2(z)}\right\}\right| < \arctan(2\alpha) - \frac{\alpha\pi}{2} \tag{3.19}$$

then

$$\left|\arg\left\{\frac{z\,F'(z)}{F(z)}\right\}\right| < \frac{\alpha\pi}{2} \quad (z \in D) \tag{3.20}$$

Proof: Let p(z) = zF'(z)/F(z). By (3.18) we have that

$$p(z) = 1 + p_2z^2 + p_3z^3 + \ldots.$$

Moreover

$$p(z) - zp'(z) = \frac{z^2(F''(z) - (F'(z))^2)}{F^2(z)}$$

Applying Theorem 3.1 we obtain the result.

References

1. Brannan DA, Kirwan WE (1969) On some classes of bounded univalent functions. J London Math Soc 1: 431-443.
2. Fukui S, Sakaguchi K (1980) An extension of a theorem of S Ruscheveyh. Bill Fac Edu Wakayama Univ Nat Sci 29: 1-3.
3. Nunokawa M (1992) On Properties of Non-Carath´eodory Functions. Proc Japan Acad 68 Ser A 152-153.
4. Nunokawa M (1993) On the Order of Strongly Starlikeness of Strongly Convex Functions. Proc Japan Acad 69 Ser: 234-237.
5. Nunokawa M, Sokol J New conditions for starlikeness and strongly starlikeness of order alpha.
6. Nunokawa M, Sokol J (2012) On some sufficient conditions for univalence and starlikeness. J Ineq Appl 282.
7. Nunokawa M, Sokol J (2013) On the order of strongly starlikeness of convex functions of order alpha. Mediterranean J Math.
8. Robertson MS (1936) On the theory of univalent functions. Ann Math 37: 374-408.
9. Sokol J, Trojnar-Spelina L (2013) On a sufficient condition for strongly starlikeness. J Ineq Appl 2013: 383.
10. Stankiewicz J (1966) Quelques probl`emes extr`emaux dans les classes des functions -angulairement `etoil`ees, Ann Univ Mariae Curie–Sk lodowska, Sect A 20: 59-75.
11. Strrohh¨acker E (1933) Beitrage z¨ur Theorie der schlichter Functinen. Math Z 37: 356-380.
12. Wilken DR, Feng J (1980) A remark on convex and starlike functions. J London Math Soc 21: 287-290.