On Abstract Gronwall Lemmas

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In this paper, we present some problems related to Abstract Gronwall Lemmas. We begin our considerations with some notions from operatorial inequality [1-6].

Let $(X,\leq)$ be an ordered $L$-space [2-6], $A:X\rightarrow X$ an operator. We denote $F_A$ as the fixed points set of $A$. We consider that the equation $x=A(x)$ has a unique solution $x_A^*$. The operatorial inequality problem is the following [1-4]:

Find conditions under which

(i) $x \leq A(x) \Rightarrow x \leq x_A^*$;

(ii) $x \geq A(x) \Rightarrow x \geq x_A^*$.

To have a concrete result for this problem it is necessary to determine the $x_A^*$, or to find $y: z \in X$ such that $x_A^* \in [y,z]$.

Abstract Gronwall Lemma (AGL) [2,3,5,6].

Let $(X,\leq)$ be an ordered $L$-space and $A:X\rightarrow X$ an operator.

We suppose that

(i) $A$ is PO;

(ii) $A$ is increasing.

Then,

(i) $x \leq A(x) \Rightarrow x \leq x_A^*$;

(ii) $x \geq A(x) \Rightarrow x \geq x_A^*$.

Abstract Gronwall-Comparison Lemma (AGCL) [2,3,5,6]. Let $(X,\leq)$ be an ordered $L$-space and $A:B:X\rightarrow X$ two operators. We suppose that:

(i) $A$ and $B$ are POs;

(ii) $A$ is increasing;

(iii) $x \leq A(x), x \leq B(x)$.

Then,

$x \leq A(x) \Rightarrow x \leq x_A^*$.

(1)

We present two problems as follows. First is the simplest and most useful integral inequality.

Lemma Gronwall [7-9]. Let $y, g \in C([a,b,\mathbb{R}_+])$

Suppose, for $c \geq 0$, we have

$y(x) \leq c + \int_a^x y(s)g(s)ds\to 0$.

Then,

$y(x) \leq c \exp\left(\int_a^x g(s)ds\right)\to 0$.

(3)

In this case $(X,\leq)$ is $C([a,b,\mathbb{R}_+])$, where $\leq$ is the Bielecki norm on $C([a,b,\mathbb{R}_+])$. Let $A \in X$ and $A:X\rightarrow X$ is defined by

$A(y)(x) = c + \int_a^x y(s)g(s)ds.$

(4)

The fixed point of the operator $A$ is $x_A^* = c \exp\left(\int_a^x g(s)ds\right)$ and the inequality (3) derived from AGL.

The second problem is analogous to Gronwall lemma. Thus, we have:

Lemma Wendorff [1,7-9]. We consider that

(i) $u, v \in C([0,a] \times [0,b],\mathbb{R}_+), c \in \mathbb{R}_+$;

(ii) $v$ is increasing

If $u$ is a solution of inequality

$u(x,y) \leq c + \int_0^x u(s,t)u(s,t)dsdt, x[0,a], y \in [0,b]$ (5)

then

$u(x,y) \leq c \exp\left(\int_0^x u(s,t)dsdt\right)$. (6)

In this case, the operator $A$ is the second part of (5) inequality, but the function $u(x,y) \leq c \exp\left(\int_0^x u(s,t)dsdt\right)$ is not the fixed point of the operator $A$. This inequality is derived from AGCL.

For other applications see Lungu and Popa [10], Lungu and Rus [11], and Popa and Lungu [12].

References


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